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On the Existence of Positive Solutions for a Class of Infinite Semipositone Systems with Singular Weights[1]

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Abstract : In this paper we consider the existence of positive solutions of infinite semipositone systems with singular weights of the form

$$\begin{cases} -div(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u) = \lambda\,|x|^{-(a+1)p+c_1}\,(f(v) - \frac{1}{u^{\alpha}}), & x \in \Omega, \\ -div(|x|^{-bq}\,|\nabla v|^{q-2}\,\nabla v) = \lambda\,|x|^{-(b+1)q+c_2}\,(g(u) - \frac{1}{v^{\beta}}), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $0 \in \Omega$, $1 < p, q < N$, $0 \leq a < \frac{N-p}{p}$, $0 \leq b < \frac{N-q}{q}$, $\alpha, \beta \in (0,1)$, and c_1, c_2, λ are positive parameters. Here $f, g : (0,\infty) \to (0,\infty)$ are C^2 functions. Our aim in this paper is to establish the existence of positive solution for λ large. We use the method of sub-super solutions to establish our existence result.

Keywords : positive solutions; infinite semipositone systems; singular weights.

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1 Introduction

We study the existence of positive solutions to infinite semipositone systems with singular weights

$$\begin{cases} -div(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u) = \lambda\,|x|^{-(a+1)p+c_1}\,(f(v) - \frac{1}{u^{\alpha}}), & x \in \Omega, \\ -div(|x|^{-bq}\,|\nabla v|^{q-2}\,\nabla v) = \lambda\,|x|^{-(b+1)q+c_2}\,(g(u) - \frac{1}{v^{\beta}}), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $0 \in \Omega$, $1 < p, q < N$, $0 \le a < \frac{N-p}{p}$, $0 \le b < \frac{N-q}{q}$, $\alpha, \beta \in (0,1)$, and c_1, c_2, λ are positive parameters. Here f and g are C^2 functions in $(0,\infty)$, $f(0) > 0$, $g(0) > 0$, $f' > 0$, and $g' > 0$.

Elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-div(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u)$, were motivated by the following Caaffarelli, Kohn and Nirenberg's inequality (see [1, 2]). The study of this type of problem is motivated by its various applications, for example, in fluid mechanics, in newtonian fluids, in flow through porous media and in glaciology (see [3, 4]). So, the study of positive solutions of singular elliptic problems has more practical meanings. We refer to [5–8] for additional results on elliptic problems.

Let $F(h,k) = f(k) - \frac{1}{h^{\alpha}}$, and $G(h,k) = g(h) - \frac{1}{k^{\beta}}$. Then $\lim_{(h,k)\to(0,0)} F(h,k) = -\infty = \lim_{(h,k)\to(0,0)} G(h,k)$, and hence we refer to (1) as an infinite semipositone problem. For the regular case, that is, when $a = b = 0$, $c_1 = p$ and $c_2 = q$, the quasilinear elliptic equation has been studied by several authors (see [9, 10]). For the single-equation case when $a = 0$, $c_1 = p = 2$, see [11]. In [9], the authors extended the study of [11], to the corresponding systems, including p-Laplacian. Here we focus on further extending the study in [9] for the quasilinear elliptic problem involving singularity. Due to this singularity in the weights, the extensions are challenging and nontrivial. Our approach is based on the method of sub-super solutions, see [12, 13].

2 Preliminaries and Existence Result

In this paper, we denote $W_0^{1,p}(\Omega, |x|^{-ap})$, the completion of $C_0^{\infty}(\Omega)$, with respect to the norm $\|u\| = (\int_{\Omega} |x|^{-ap}|\nabla u|^p dx)^{\frac{1}{p}}$. To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -div(|x|^{-sr}\,|\nabla \phi|^{r-2}\,\nabla \phi) = \lambda |x|^{-(s+1)r+t}\,|\phi|^{r-2}\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

For $r = p$, $s = a$ and $t = c_1$, let $\phi_{1,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of (2.1) such that $\phi_{1,p}(x) > 0$ in Ω, and $||\phi_{1,p}||_{\infty} = 1$ and for $r = q$, $s = b$ and $t = c_2$, let $\phi_{1,q}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,q}$ of (2.1) such that $\phi_{1,q}(x) > 0$ in Ω, and $||\phi_{1,q}||_{\infty} = 1$ (see [14, 15]). It can be shown that $\frac{\partial \phi_{1,r}}{\partial n} < 0$ on $\partial\Omega$ for $r = p, q$. Here n is the outward normal. This result is well known and hence, depending on Ω, there exist positive constants $\epsilon, \delta, \sigma_p, \sigma_q$ such that

$$|x|^{-sr}\,|\nabla\phi_{1,r}|^r \geq \epsilon, \qquad x \in \bar{\Omega}_\delta, \tag{2.2}$$

$$\phi_{1,r} \geq \sigma_r, \qquad x \in \Omega_0 = \Omega \setminus \bar{\Omega}_\delta, \tag{2.3}$$

with $r = p, q$; $s = a, b$; $t = c_1, c_2$ and $\bar{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$ (see [14]). We will also consider the unique solution $(\zeta_p(x), \zeta_q(x)) \in W_0(\Omega, |x|^{-ap}) \times W_0(\Omega, |x|^{-bq})$ for the system

$$\begin{cases} -div(|x|^{-ap}\,|\nabla\zeta_p|^{p-2}\,\nabla\zeta_p) = |x|^{-(a+1)p+c_1}, & x \in \Omega, \\ -div(|x|^{-bq}\,|\nabla\zeta_q|^{q-2}\,\nabla\zeta_p) = |x|^{-(b+1)q+c_2}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is known that $\zeta_r(x) > 0$ in Ω and $\frac{\partial\zeta_r(x)}{\partial n} < 0$ on $\partial\Omega$, for $r = p, q$ (see [14]).

A pair of nonnegative functions (ψ_1, ψ_2), (z_1, z_2) are called a subsolution and supersolution of (1) if they satisfy $(\psi_1, \psi_2) = (0, 0) = (z_1, z_2)$ on $\partial\Omega$ and

$$\int_\Omega |x|^{-ap}\,|\nabla\psi_1|^{p-2}\,\nabla\psi_1 \cdot \nabla w\,dx \leq \lambda \int_\Omega |x|^{-(a+1)p+c_1}\,(f(\psi_2) - \frac{1}{\psi_1^\alpha})\,w\,dx,$$
$$\int_\Omega |x|^{-bq}\,|\nabla\psi_2|^{q-2}\,\nabla\psi_2 \cdot \nabla w\,dx \leq \lambda \int_\Omega |x|^{-(b+1)q+c_2}\,(g(\psi_1) - \frac{1}{\psi_2^\beta})\,w\,dx,$$
$$\int_\Omega |x|^{-ap}\,|\nabla z_1|^{p-2}\,\nabla z_1 \cdot \nabla w\,dx \geq \lambda \int_\Omega |x|^{-(a+1)p+c_1}\,(f(z_2) - \frac{1}{z_1^\alpha})\,w\,dx,$$
$$\int_\Omega |x|^{-bq}\,|\nabla z_2|^{q-2}\,\nabla z_2 \cdot \nabla w\,dx \geq \lambda \int_\Omega |x|^{-(b+1)q+c_2}\,(g(z_1) - \frac{1}{z_2^\beta})\,w\,dx,$$

for all $w \in W = \{w \in C_0^\infty(\Omega) | w \geq 0, x \in \Omega\}$. Then the following result holds:

Lemma 2.1 (See [14]). *Suppose there exist sub and super- solutions* (ψ_1, ψ_2) *and* (z_1, z_2) *respectively of* (1.1) *such that* $(\psi_1, \psi_2) \leq (z_1, z_2)$. *Then* (1.1) *has a solution* (u, v) *such that* $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$.

We make the following assumptions:

(A.1) f and g are C^2 functions in $(0, \infty)$, $f(0) > 0$, $g(0) > 0$, $f' > 0$, and $g' > 0$.

(A.2) $\lim_{s\to\infty} g(s) = \infty$ and for all $M > 0$

$$\lim_{s\to\infty} \frac{f(Mg(s)^{\frac{1}{q-1}})}{s^{p-1}} = 0.$$

We establish:

Theorem 2.2. *Assume (H.1) and (H.2) hold. Then there exists positive constant* λ_* *such that* (1.1) *has a positive solution for* $\lambda > \lambda_*$.

Proof. Choose $\eta > 0$ such that $\eta \le \min\{|x|^{-(a+1)p+c_1}, |x|^{-(b+1)q+c_2}\}$, in $\bar{\Omega}_\delta$. For fixed $r_1 \in (\frac{1}{p-1+\alpha}, \frac{1}{p-1})$ and $r_2 \in (\frac{1}{q-1+\beta}, \frac{1}{q-1})$, we shall verify that

$$(\psi_{1,\lambda}, \psi_{2,\lambda}) = \left(\lambda^{r_1}\,\eta^{\frac{1}{p-1}}\,\left(\frac{p-1+\alpha}{p}\right)\phi_{1,p}^{\frac{p}{p-1+\alpha}},\,\lambda^{r_2}\,\eta^{\frac{1}{q-1}}\,\left(\frac{q-1+\beta}{q}\right)\phi_{1,q}^{\frac{q}{q-1+\beta}}\right),$$

is a sub-solution of (1.1). Let $w \in W$. Then a calculation shows that

$$\nabla\psi_{1,\lambda} = \lambda^{r_1}\,\eta^{\frac{1}{p-1}}\,\phi_{1,p}^{\frac{1-\alpha}{p-1+\alpha}}\,\nabla\phi_{1,p},\ \ \psi_{2,\lambda} = \lambda^{r_2}\,\eta^{\frac{1}{q-1}}\,\phi_{1,q}^{\frac{1-\beta}{q-1+\beta}}\,\nabla\phi_{1,q}$$

and

$$\begin{aligned}
&\int_\Omega |x|^{-ap}\,|\nabla\psi_{1,\lambda}|^{p-2}\,\nabla\psi_{1,\lambda}\,\nabla w\,dx\\
&= \lambda^{r_1(p-1)}\,\eta\int_\Omega |x|^{-ap}\,\phi_{1,p}^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}\,|\nabla\phi_{1,p}|^{p-2}\,\nabla\phi_{1,p}\,\nabla w\,dx\\
&= \lambda^{r_1(p-1)}\,\eta\int_\Omega |x|^{-ap}\,|\nabla\phi_{1,p}|^{p-2}\,\nabla\phi_{1,p}\left[\nabla\left(\phi_{1,p}^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}w\right)-\left(\nabla\phi_{1,p}^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}\right)w\right]dx\\
&= \lambda^{r_1(p-1)}\,\eta\int_\Omega\left[\lambda_{1,p}\,|x|^{-(a+1)p+c_1}\,\phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}} - |x|^{-ap}\,\frac{(1-\alpha)(p-1)}{p-1+\alpha}\,\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\alpha p}{p-1+\alpha}}}\right]w\,dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_\Omega |x|^{-bq}\,|\nabla\psi_{2,\lambda}|^{q-2}\,\nabla\psi_{2,\lambda}\,\nabla w\,dx\\
&= \lambda^{r_2(q-1)}\,\eta\int_\Omega\left[\lambda_{1,q}\,|x|^{-(b+1)q+c_2}\,\phi_{1,q}^{\frac{q(q-1)}{q-1+\beta}} - |x|^{-bq}\,\frac{(1-\beta)(q-1)}{q-1+\beta}\,\frac{|\nabla\phi_{1,q}|^q}{\phi_{1,q}^{\frac{\beta q}{q-1+\beta}}}\right]w\,dx.
\end{aligned}$$

First we consider the case when $x \in \bar{\Omega}_\delta$. We have $|x|^{-ap}\,|\nabla\phi_{1,p}|^p \ge \epsilon$ and $|x|^{-bq}\,|\nabla\phi_{1,q}|^q \ge \epsilon$ on $\bar{\Omega}_\delta$. Since $r_1(p-1) \ge 1-\alpha r_1$ and $r_2(q-1) \ge 1-\beta r_2$, we can find $\hat{\lambda} > 0$ such that

$$\begin{aligned}
&-\lambda^{r_1(p-1)}\,\eta\,|x|^{-ap}\,\frac{(1-\alpha)(p-1)}{p-1+\alpha}\,\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\alpha p}{p-1+\alpha}}}\\
&\qquad\le \lambda\,|x|^{-(a+1)p+c_1}\left(-\frac{1}{[\lambda^{r_1}\,\eta^{\frac{1}{p-1}}(\frac{p-1+\alpha}{p})\,\phi_{1,p}^{\frac{p}{p-1+\alpha}}]^\alpha}\right),
\end{aligned}$$

and

$$\begin{aligned}
&-\lambda^{r_2(q-1)}\,\eta\,|x|^{-bq}\,\frac{(1-\beta)(q-1)}{q-1+\beta}\,\frac{|\nabla\phi_{1,q}|^q}{\phi_{1,q}^{\frac{\beta q}{q-1+\beta}}}\\
&\qquad\le \lambda\,|x|^{-(b+1)p+c_2}\left(-\frac{1}{[\lambda^{r_2}\,\eta^{\frac{1}{q-1}}(\frac{q-1+\beta}{q})\,\phi_{1,q}^{\frac{q}{q-1+\beta}}]^\beta}\right),
\end{aligned}$$

for all $x \in \bar{\Omega}_\delta$ and for all $\lambda \geq \hat{\lambda}$. Also since $r_1(p-1) < 1$ and $r_2(q-1) < 1$, we can choose $\check{\lambda} > 0$ so that

$$\begin{aligned}\lambda^{r_1(p-1)}\,\eta\,\lambda_{1,p}\,|x|^{-(a+1)p+c_1}\,\phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}} &\leq \lambda\,|x|^{-(a+1)p+c_1}\,f(0)\\ &\leq \lambda\,|x|^{-(a+1)p+c_1}\,f\left(\lambda^{r_2}\,\eta^{\frac{1}{q-1}}\left(\frac{q-1+\beta}{q}\right)\phi_{1,q}^{\frac{q}{q-1+\beta}}\right),\end{aligned}$$

and

$$\begin{aligned}\lambda^{r_2(q-1)}\,\eta\,\lambda_{1,q}\,|x|^{-(b+1)q+c_2}\,\phi_{1,q}^{\frac{q(q-1)}{q-1+\beta}} &\leq \lambda\,|x|^{-(b+1)q+c_2}\,g(0)\\ &\leq \lambda\,|x|^{-(b+1)q+c_2}\,g\left(\lambda^{r_1}\,\eta^{\frac{1}{p-1}}\left(\frac{p-1+\alpha}{p}\right)\phi_{1,p}^{\frac{p}{p-1+\alpha}}\right),\end{aligned}$$

for all $x \in \bar{\Omega}_\delta$ and for all $\lambda \geq \check{\lambda}$. Let $\lambda_0 = \max\{\hat{\lambda}, \check{\lambda}\}$. Hence, for all $x \in \bar{\Omega}_\delta$ and for all $\lambda \geq \lambda_0$,

$$\begin{aligned}&\int_{\bar{\Omega}_\delta} |x|^{-ap}\,|\nabla\psi_{1,\lambda}|^{p-2}\,\nabla\psi_{1,\lambda}\,\nabla w\,dx\\ &\leq \lambda\int_{\bar{\Omega}_\delta} |x|^{-(a+1)p+c_1}\\ &\qquad\times\left[f\left(\lambda^{r}\,\eta^{\frac{1}{q-1}}\left(\frac{q-1+\beta}{q}\right)\phi_{1,q}^{\frac{q}{q-1+\beta}}\right) - \frac{1}{\left(\lambda^{r_1}\,\eta^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{q})\,\phi_{1,p}^{\frac{p}{p-1+\alpha}}\right)^{\alpha}}\right] w\,dx\\ &= \lambda\int_{\bar{\Omega}_\delta} |x|^{-(a+1)p+c_1}\left(f(\psi_{2,\lambda}) - \frac{1}{\psi_{1,\lambda}^{\alpha}}\right) w\,dx,\end{aligned}$$

and similarly

$$\int_{\bar{\Omega}_\delta} |x|^{-bq}\,|\nabla\psi_{2,\lambda}|^{q-2}\,\nabla\psi_{2,\lambda}\cdot\nabla w\,dx \leq \lambda\int_{\bar{\Omega}_\delta} |x|^{-(b+1)q+c_2}\left(g(\psi_{1,\lambda}) - \frac{1}{\psi_{2,\lambda}^{\beta}}\right) w\,dx.$$

On the other hand, on $\Omega\setminus\bar{\Omega}_\delta$, we have $\phi_{1,r} \geq \sigma_r$, for some $0 < \sigma_p < 1$, and for $r = p, q$. Since $r(p-1) < 1$ and $r(q-1) < 1$ we can find $\tilde{\lambda} > 0$ such that

$$\begin{aligned}&\lambda^{r(p-1)}\,\eta\,\lambda_{1,p}\,|x|^{-(a+1)p+c_1}\,\phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}}\\ &\leq \lambda\,|x|^{-(a+1)p+c_1}\left[f\left(\lambda^{r}\,\eta^{\frac{1}{q-1}}(\frac{q-1+\beta}{q})\,\sigma_q^{\frac{q}{q-1+\beta}}\right) - \frac{1}{\left(\lambda^{r}\,\eta^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{p})\,\sigma_p^{\frac{p}{p-1+\alpha}}\right)^{\alpha}}\right],\end{aligned}$$

and

$$\begin{aligned}&\lambda^{r(q-1)}\,\eta\,\lambda_{1,q}\,|x|^{-(b+1)p+c_2}\,\phi_{1,q}^{\frac{q(q-1)}{q-1+\beta}}\\ &\leq \lambda\,|x|^{-(b+1)p+c_2}\left[g\left(\lambda^{r}\,\eta^{\frac{1}{p-1}}(\frac{p-1+\alpha}{p})\,\sigma_p^{\frac{p}{p-1+\alpha}}\right) - \frac{1}{\left(\lambda^{r}\,\eta^{\frac{1}{q-1}}\left(\frac{q-1+\beta}{q}\right)\sigma_q^{\frac{q}{q-1+\beta}}\right)^{\beta}}\right],\end{aligned}$$

for all $x \in \Omega \setminus \bar{\Omega}_\delta$ and for all $\lambda \geq \tilde{\lambda}$. Hence

$$\lambda^{r(p-1)}\eta \int_{\Omega\setminus\bar{\Omega}_\delta} \left[\lambda_{1,p}\, |x|^{-(a+1)p+c_1}\, \phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}} - |x|^{-ap}\, \frac{(1-\alpha)(p-1)}{p-1+\alpha}\, \frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\alpha\, p}{p-1+\alpha}}}\right] w\, dx$$
$$\leq \lambda^{r(p-1)}\eta \int_{\Omega\setminus\bar{\Omega}_\delta} \lambda_{1,p}\, |x|^{-(a+1)p+c_1}\, \phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}}\, w\, dx$$
$$\leq \lambda \int_{\Omega\setminus\bar{\Omega}_\delta} |x|^{-(a+1)p+c_1}$$
$$\times \left[f\left(\lambda^r\, \eta^{\frac{1}{q-1}} \left(\frac{q-1+\beta}{q}\right) \sigma_q^{\frac{q}{q-1+\beta}}\right) - \frac{1}{\left(\lambda^r\, \eta^{\frac{1}{p-1}} (\frac{p-1+\alpha}{p})\, \sigma_p^{\frac{p}{p-1+\alpha}}\right)^\alpha}\right] w\, dx$$
$$\leq \lambda \int_{\Omega\setminus\bar{\Omega}_\delta} |x|^{-(a+1)p+c_1} \left(f(\psi_{2,\lambda}) - \frac{1}{\psi_{1,\lambda}^\alpha}\right) w\, dx.$$

and similarly

$$\lambda^{r(q-1)}\eta \int_{\Omega\setminus\bar{\Omega}_\delta} \left[\lambda_{1,q}\, |x|^{-(b+1)q+c_2}\, \phi_{1,q}^{\frac{q(q-1)}{q-1+\beta}} - |x|^{-bq}\, \frac{(1-\beta)(q-1)}{q-1+\beta}\, \frac{|\nabla\phi_{1,q}|^q}{\phi_{1,q}^{\frac{\beta\, q}{q-1+\beta}}}\right] w\, dx$$
$$\leq \lambda \int_{\Omega\setminus\bar{\Omega}_\delta} |x|^{-(b+1)q+c_2} \left(g(\psi_{1,\lambda}) - \frac{1}{\psi_{2,\lambda}^\beta}\right) w\, dx.$$

Let $\lambda_* = \max\{\lambda_0, \tilde{\lambda}\}$. Hence

$$\int_\Omega |x|^{-ap}\, |\nabla\psi_{1,\lambda}|^{p-2}\, \nabla\psi_{1,\lambda} \cdot \nabla w\, dx \leq \int_\Omega |x|^{-(a+1)p+c_1} \left(f(\psi_{2,\lambda}) - \frac{1}{\psi_{1,\alpha}^\alpha}\right) w\, dx,$$

and

$$\int_\Omega |x|^{-bq}\, |\nabla\psi_{2,\lambda}|^{q-2}\, \nabla\psi_{2,\lambda} \cdot \nabla w\, dx \leq \int_\Omega |x|^{-(b+1)p+c_2} \left(g(\psi_{1,\lambda}) - \frac{1}{\psi_{2,\lambda}^\beta}\right) w\, dx,$$

i.e., $(\psi_{1,\lambda}, \psi_{2,\lambda})$ is a sub-solution of (1) for all $\lambda \geq \lambda_*$.

Now, we will prove there exists a N large enough so that

$$(z_1, z_2) = \left(N\, \zeta_p(x), [\lambda\, g(N\, l_p)]^{\frac{1}{q-1}}\, \zeta_q(x)\right),$$

is a super-solution of (1), where $l_r = \|\zeta_r\|_\infty; r = p, q$. A calculation shows that:

$$\int_\Omega |x|^{-ap}\, |\nabla z_1|^{p-2}\, \nabla z_1\, \nabla w\, dx = N^{p-1} \int_\Omega |x|^{-ap}\, |\nabla\zeta_p|^{p-2}\, \nabla\zeta_p\, \nabla w\, dx$$
$$= N^{p-1} \int_\Omega |x|^{-(a+1)p+c_1}\, w\, dx.$$

By monotonicity condition on f and **(A.2)**, we can choose N large enough so that

$$\begin{aligned}
N^{p-1} &\geq \lambda f\left([\lambda\, g(N\, l_p)]^{\frac{1}{q-1}}\, l_q\right)\\
&\geq \lambda f\left([\lambda\, g(N\, l_p)]^{\frac{1}{q-1}}\, \zeta_q(x)\right)\\
&= \lambda f(z_2)\\
&\geq \lambda\left(f(z_2) - \frac{1}{z^{\alpha}}\right).
\end{aligned}$$

Hence

$$\int_{\Omega} |x|^{-ap}\, |\nabla\, z_1|^{p-2}\, |\nabla\, z_1| \cdot\, \nabla w\, dx \geq \lambda \int_{\Omega} |x|^{-(a+1)p+c_1} \left(f(z_2) - \frac{1}{z^{\alpha}}\right) w\, dx.$$

Next, we have

$$\begin{aligned}
\int_{\Omega} |x|^{-bq} |\nabla z_2|^{q-2} \nabla z_2 \nabla w\, dx &= \lambda\, g(N l_p) \int_{\Omega} |x|^{-bq}\, |\nabla \zeta_q|^{q-2}\, \nabla \zeta_q\, \nabla w\, dx\\
&= \lambda\, g(N l_p) \int_{\Omega} |x|^{-(b+1)q+c_2}\, w\, dx\\
&\geq \lambda \int_{\Omega} |x|^{-(b+1)q+c_2} g(N\, \zeta_p(x)) w\, dx\\
&= \lambda \int_{\Omega} |x|^{-(b+1)q+c_2} g(z_1) w\, dx\\
&\geq \lambda \int_{\Omega} |x|^{-(b+1)q+c_2} \left(g(z_1) - \frac{1}{z_2^{\beta}}\right) w\, dx
\end{aligned}$$

i.e. (z_1, z_2) is a super-solution of (1) with $z_i \geq \psi_i$ for M large, $i = 1, 2$. Thus, by [14] there exists a positive solution (u, v) of (1) such that $(\psi, \psi) \leq (u, v) \leq (z_1, z_2)$. This completes the proof. □

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