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Domain of Generalized Difference Matrix $B(r,s)$ on Some Maddox's Spaces

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Abstract : In the present paper, the sequence spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ of non-absolute type have been introduced and proved that the spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ are linearly isomorphic to the spaces $\ell_\infty(p)$, $c_0(p)$ and $c(p)$, respectively. The β- and γ-duals of the spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ have been computed and their basis have been constructed. Finally, some matrix mappings from $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ to the some sequence spaces of Maddox have been characterized and relationship between the modular σ_p and the Luxemburg norm on the sequence space $\widehat{\ell}_\infty(p)$ has been discussed.

Keywords : paranormed sequence space; β- and γ-duals; matrix transformations.
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1 Preliminaries, Background and Notation

By w, we shall denote the space of all real valued sequences. Any vector subspace of w is called as a *sequence space*. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p; we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively; where $1 < p < \infty$.

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Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1,\ H\}$. Then, the linear spaces $\ell_\infty(p),\ c(p),\ c_0(p)$ and $\ell(p)$ were defined by Maddox [1] (see also Simons [2] and Nakano [3]) as follows:

$$\ell_\infty(p) = \left\{ x = (x_k) \in w : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty \right\},$$

$$c(p) = \left\{ x = (x_k) \in w : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R} \right\},$$

$$c_0(p) = \left\{ x = (x_k) \in w : \lim_{k\to\infty} |x_k|^{p_k} = 0 \right\}$$

and

$$\ell(p) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty \right\}$$

which are the complete spaces paranormed by

$$g_1(x) = \sup_{k\in\mathbb{N}} |x_k|^{p_k/M} \text{ iff } \inf p_k > 0 \text{ and } g_2(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M}, \tag{1.1}$$

respectively. We shall assume throughout that $p_k^{-1} + (p'_k)^{-1} = 1$ provided $1 < \inf p_k \leq H < \infty$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$, where $\mathbb{N}$ is the set of natural numbers.

Let $\lambda,\ \mu$ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n,\ k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = ((Ax)_n)$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k,\ (n \in \mathbb{N}). \tag{1.2}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.2) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

The main purpose of this paper, which is a continuation of Kirişçi and Başar [4], is to introduce the sequence spaces $\widehat{\ell}_\infty(p),\ \widehat{c}_0(p)$ and $\widehat{c}(p)$ of non-absolute type which is the set of all sequences whose $B(r,s)$-transforms are in the spaces

$\ell_\infty(p)$, $c_0(p)$ and $c(p)$, respectively; where the generalized difference matrix $B(r,s) = (b_{nk})$ defined by

$$b_{nk} := \begin{cases} r, & k = n, \\ s, & k = n-1, \\ 0, & 0 \leq k < n-1 \text{ or } k > n, \end{cases}$$

for all $k,\ n \in \mathbb{N}$ with $r,\ s \in \mathbb{R}\backslash\{0\}$. Furthermore, the basis of the spaces $\widehat{c}_0(p)$ and $\widehat{c}(p)$ are constructed and the β- and γ-duals are computed for the space $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$. Besides this, the matrix transformations from the spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ to some other sequence spaces are characterized. Finally, some results related to the modular σ_p and the Luxemburg norm on the space $\widehat{\ell}_\infty(p)$ are derived.

2 The Sequence Spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ of Non-absolute Type

In this section, we define the sequence spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ and prove that $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ are the complete paranormed linear spaces. Later, we determine their β- and γ-duals.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

For a sequence space λ, the *matrix domain* λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}. \tag{2.1}$$

Choudhary and Mishra [5] have defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that S-transforms of them are in $\ell(p)$, where $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. Başar and Altay [6] have recently examined the space $bs(p)$ which is formerly defined by Başar in [7] as the set of all series whose sequences of partial sums are in $\ell_\infty(p)$. More recently, Aydın and Başar [8] have studied the space $a^r(u,p)$ which is derived from the sequence spaces $\ell(p)$, where A^r denotes the matrix $A^r = (a^r_{nk})$ defined by

$$a^r_{nk} = \begin{cases} \frac{1+r^k}{n+1} u_k, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$ and $0 < r < 1$. Also, Altay and Başar [9] have studied the sequence spaces $r^t(p)$, $r^t_\infty(p)$ which are derived from the sequence spaces $\ell(p)$ and $\ell_\infty(p)$ of Maddox by the Riesz means, respectively. Altay et al. [10], Mursaleen et al. [11] have studied the sequence spaces which are derived from the sequence spaces ℓ_p and ℓ_∞ by the Euler mean of order r. With the notation of (2.1), the spaces $\overline{\ell(p)}$, $bs(p)$, $a^r(u,p)$, $r^t(p)$, $r^t_\infty(p)$, e^p_r and e^∞_r can be redefined by

$$\overline{\ell(p)} = [\ell(p)]_S,\ bs(p) = [\ell_\infty(p)]_S,\ a^r(u,p) = [\ell_p]_{A^r},\ r^t(p) = [\ell(p)]_{R^t},$$

$$r^t_\infty(p) = [\ell_\infty(p)]_{R^t},\ e^p_r = (\ell_p)_{E_r},\ e^\infty_r = (\ell_\infty)_{E_r}.$$

Following Choudhary and Mishra [5], Başar and Altay [6], Altay and Başar [9, 12–14], Aydın and Başar [15–17], Mursaleen [18], Malkowsky et al. [19], we introduce the sequence spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$, as the set of all sequences such that $B(r,s)$-transforms of them are in the spaces $\ell_\infty(p)$, $c_0(p)$ and $c(p)$, respectively, that is

$$\widehat{\ell}_\infty(p) := \left\{x = (x_k) \in w : \sup_{k\in\mathbb{N}} |sx_{k-1} + rx_k|^{p_k} < \infty\right\},$$

$$\widehat{c}_0(p) := \left\{x = (x_k) \in w : \lim_{k\to\infty} |sx_{k-1} + rx_k|^{p_k} = 0\right\}$$

and

$$\widehat{c}(p) := \left\{x = (x_k) \in w : \lim_{k\to\infty} |sx_{k-1} + rx_k - l|^{p_k} = 0 \ \text{ for some } l \in \mathbb{R}\right\}.$$

With the notation of (2.1), we may redefine the spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ as follows:

$$\widehat{\ell}_\infty(p) := [\ell_\infty(p)]_{B(r,s)},\ \widehat{c}_0(p) := [c_0(p)]_{B(r,s)} \quad \text{and} \quad \widehat{c}(p) := [c(p)]_{B(r,s)}. \tag{2.2}$$

Define the sequence $y = (y_k)$, which will be frequently used, as the $B(r,s)$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k := sx_{k-1} + rx_k\ ; \quad (k \in \mathbb{N}). \tag{2.3}$$

Now, we may begin with the following theorem which is essential in the text:

Theorem 2.1. *$\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ are the complete linear metric spaces paranormed by g, defined by*

$$g(x) := \sup_{k\in\mathbb{N}} |sx_{k-1} + rx_k|^{p_k/M}.$$

Proof. We prove the theorem for the space $\widehat{c}_0(p)$. The linearity of $\widehat{c}_0(p)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $x \in \widehat{c}_0(p)$ (see Maddox [20, p. 30])

$$\sup_{k\in\mathbb{N}} |s(x_{k-1}+z_{k-1})+r(x_k+z_k)|^{p_k/M} \leq \sup_{k\in\mathbb{N}} |sx_{k-1}+rx_k|^{p_k/M} + \sup_{k\in\mathbb{N}} |sz_{k-1}+rz_k|^{p_k/M} \quad (2.4)$$

and for any $\alpha \in \mathbb{R}$ (see [21])

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}. \quad (2.5)$$

It is clear that $g(\theta)=0$ and $g(x)=g(-x)$ for all $x \in \widehat{c}_0(p)$. Again the inequalities (2.4) and (2.5) yield the subadditivity of g and

$$g(\alpha x) \leq \max\{1, |\alpha|\} g(x).$$

Let $\{x^n\}$ be any sequence of the points $\widehat{c}_0(p)$ such that $g(x^n - x) \to 0$ and (α_n) also be any sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality

$$g(x^n) \leq g(x) + g(x^n - x)$$

holds by subadditivity of g, $\{g(x^n)\}$ is bounded and we thus have

$$\begin{aligned} g(\alpha_n x^n - \alpha x) &= \sup_{k\in\mathbb{N}} \left|s(\alpha_n x^n_{k-1} - \alpha x_{k-1}) + r(\alpha_n x^n_k - \alpha x_k)\right|^{p_k/M} \\ &\leq |\alpha_n - \alpha|\, g(x^n) + |\alpha|\, g(x^n - x) \end{aligned}$$

which tends to zero as $n \to \infty$. That is to say that the scalar multiplication is continuous. Hence, g is a paranorm on the space $\widehat{c}_0(p)$.

It remains to prove the completeness of the space $\widehat{c}_0(p)$. Let $B = B(r,s)$ and $\{x^i\}$ be any Cauchy sequence in the space $\widehat{c}_0(p)$, where $x^i = \{x^{(i)}_0,\ x^{(i)}_1,\ x^{(i)}_2, \dots\}$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$g(x^i - x^j) < \varepsilon/2$$

for all $i, j > n_0(\varepsilon)$. Using definition of g we obtain for each fixed k that

$$\left|(Bx^i)_k - (Bx^j)_k\right| \leq \sup_{k\in\mathbb{N}} \left|(Bx^i)_k - (Bx^j)_k\right|^{p_k/M} < \varepsilon/2, \quad (i, j \geq n_0(\varepsilon)) \quad (2.6)$$

which leads us the fact that $\{(Bx^0)_k,\ (Bx^1)_k,\ (Bx^2)_k, \dots\}$ is a Cauchy sequence of real numbers for every fixed k. Since $\mathbb{R}$ is complete, it converges, say $(Bx^i)_k \to (Bx)_k$ as $i \to \infty$. Using these infinitely many limits $(Bx)_0,\ (Bx)_1,\ (Bx)_2, \dots$, we define the sequence $\{(Bx)_0,\ (Bx)_1,\ (Bx)_2, \dots\}$. From (2.6) with $j \to \infty$ we have

$$\left|(Bx^i)_k - (Bx)_k\right| \leq \varepsilon/2, \quad (i \geq n_0(\varepsilon)) \quad (2.7)$$

for every fixed k. Since $x^i = \{x_k^{(i)}\} \in \widehat{c}_0(p)$,

$$\left|(Bx^i)_k\right|^{p_k/M} < \varepsilon/2$$

for all k. Therefore, by (2.7) we obtain that

$$|(Bx)_k|^{p_k/M} \leq \left|(Bx)_k - (Bx^i)_k\right|^{p_k/M} + \left|(Bx^i)_k\right|^{p_k/M} < \varepsilon, \quad (i \geq n_0(\varepsilon)). \quad (2.8)$$

This shows that $Bx \to 0$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $\widehat{c}_0(p)$ is complete and this terminates the proof. □

Therefore, one can easily check that the absolute property does not hold on the spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ that is $g(x) \neq g(|x|)$, and this says that $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ are the sequence spaces of non-absolute type; where $|x| = (|x_k|)$.

Theorem 2.2. *The sequence spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ of non-absolute type are linearly isomorphic to the spaces $\ell_\infty(p)$, $c_0(p)$ and $c(p)$, respectively; where $0 < p_k \leq H < \infty$.*

Proof. We establish this for the space $\widehat{\ell}_\infty(p)$. To prove the theorem, we should show the existence of a linear bijection between the spaces $\widehat{\ell}_\infty(p)$ and $\ell_\infty(p)$ for $1 \leq p_k \leq H < \infty$. With the notation of (2.3), define the transformation T from $\widehat{\ell}_\infty(p)$ and $\ell_\infty(p)$ by $x \mapsto y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = 0$ whenever $Tx = 0$ and hence T is injective.

Let $y \in \ell_\infty(p)$ and define the sequence $x = \{x_k\}$ by

$$x_k = \sum_{j=0}^{k} \frac{1}{r}\left(\frac{-s}{r}\right)^{k-j} y_j \ ; \quad (k \in \mathbb{N}).$$

Then, we have

$$g(x) = \sup_{k\in\mathbb{N}} |sx_{k-1} + rx_k|^{p_k/M} = \sup_{k\in\mathbb{N}} |y_k|^{p_k/M} = g_1(y) < \infty$$

Thus, we have that $x \in \widehat{\ell}_\infty(p)$ and consequently T is surjective. Hence, T is a linear bijection and this says us that the spaces $\widehat{\ell}_\infty(p)$ and $\ell_\infty(p)$ are linearly isomorphic, as was desired.

It is clear here that if the spaces $\widehat{\ell}_\infty(p)$ and $\ell_\infty(p)$ are respectively change by the spaces $\widehat{c}_0(p)$ and $c_0(p)$, $\widehat{c}(p)$ and $c(p)$, then we obtain the fact that $\widehat{c}_0(p) \cong c_0(p)$ and $\widehat{c}(p) \cong c(p)$. This completes the proof. □

If a sequence space λ paranormed by h contains a sequence (b_k) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_k) such that

$$\lim_{n\to\infty} h\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) = 0$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for λ. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_k \alpha_k b_k$. Now, we may give the sequence of the points of the spaces $\widehat{c}_0(p)$ and $\widehat{c}(p)$ which form the Schauder bases for those spaces. Because of the isomorphism T, defined in the proof of Theorem 2.2, between the sequence spaces $\widehat{c}_0(p)$ and $c_0(p)$, $\widehat{c}(p)$ and $c(p)$ is onto, the inverse image of the bases of the spaces $c_0(p)$ and $c(p)$ are the bases of our new spaces $\widehat{c}_0(p)$ and $\widehat{c}(p)$, respectively. Therefore, we have:

Theorem 2.3. *Let $\lambda_k = (Bx)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \leq H < \infty$. Define the sequence $b^{(k)}(r,s) = \left\{b_n^{(k)}(r,s)\right\}_{n\in\mathbb{N}}$ by*

$$b_n^{(k)}(r,s) = \begin{cases} 0, & n < k, \\ \frac{1}{r}\left(-\frac{s}{r}\right)^n, & n \geq k, \end{cases} \tag{2.9}$$

for every fixed $k \in \mathbb{N}$. Then,

(a) The sequence $\{b^{(k)}(r,s)\}_{k\in\mathbb{N}}$ is a basis for the space $\widehat{c}_0(p)$ and any $x \in \widehat{c}_0(p)$ has a unique representation of the form

$$x = \sum_k \lambda_k b^{(k)}(r,s). \tag{2.10}$$

(b) The set $\left\{t,\ b^{(k)}(r,s)\right\}$ is a basis for the space $\widehat{c}(p)$ and any $x \in \widehat{c}(p)$ has a unique representation of the form

$$x = lt + \sum_k [\lambda_k - l] b^{(k)}(r,s); \tag{2.11}$$

where $t = \frac{1}{r}\sum_{k=0}^{n}\left(\frac{-s}{r}\right)^k$ for all $k \in \mathbb{N}$, and $l = \lim_{k\to\infty}\{B(r,s)x\}_k$.

For the sequence spaces λ and μ, define the set $S(\lambda,\ \mu)$ by

$$S(\lambda,\ \mu) = \left\{z = (z_k) \in w : xz = (x_k z_k) \in \mu \ \text{ for all } x \in \lambda\right\}. \tag{2.12}$$

With the notation of (2.12), the β- and γ-duals of a sequence space λ, which are respectively denoted by λ^β and λ^γ, are defined by

$$\lambda^\beta = S(\lambda,\ cs) \quad \text{and} \quad \lambda^\gamma = S(\lambda,\ bs).$$

Now, we determine the β- and γ-duals of the sequence spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$ of non-absolute type. We begin with quoting the lemmas.

Lemma 2.4 ([22, Corollary for Theorem 3])**.** *$A \in (\ell_\infty(p) : c(q))$ if and only if*

$$\forall L,\ \sup_{n\in\mathbb{N}} \sum_k |a_{nk}|\, L^{1/p_k} < \infty, \tag{2.13}$$

$$\exists(\alpha_k),\ \forall L,\ \lim_{n\to\infty}\left(\sum_k |a_{nk} - \alpha_k|\, L^{1/p_k}\right)^{q_n} = 0. \tag{2.14}$$

Lemma 2.5 ([22, Theorem 3]). *$A \in (\ell_\infty(p) : \ell_\infty(q))$ if and only if*

$$\forall L, \ \sup_{n\in\mathbb{N}} \left(\sum_k |a_{nk}| \, L^{1/p_k} \right)^{q_n} < \infty. \tag{2.15}$$

Lemma 2.6 ([23, Theorem 5.1.9]). *$A \in (c_0(p) : c(q))$ if and only if*

$$\exists M, \ \sup_n \sum_k |a_{nk}| \, M^{-1/p_k} < \infty, \tag{2.16}$$

$$\exists (\alpha_k) \subset \mathbb{R} \ \forall L, \ \exists M, \ \sup_n L^{1/q_n} \sum_k |a_{nk} - \alpha_k| \, M^{-1/p_k} < \infty, \tag{2.17}$$

$$\exists (\alpha_k) \subset \mathbb{R}, \ \lim_{n\to\infty} |a_{nk} - \alpha_k|^{q_n} = 0. \tag{2.18}$$

Lemma 2.7 ([23, Theorem 5.1.13]). *$A \in (c_0(p) : \ell_\infty(q))$ if and only if*

$$\exists M, \ \sup_n \left(\sum_k |a_{nk}| \, M^{-1/p_k} \right)^{q_n} < \infty. \tag{2.19}$$

Lemma 2.8 ([23, Theorem 5.1.10]). *$A \in (c(p) : c(q))$ if and only if (2.16), (2.17), (2.18) hold and*

$$\lim_{n\to\infty} \left| \sum_k a_{nk} - \alpha \right|^{q_n} = 0 \tag{2.20}$$

also holds.

Lemma 2.9 ([23, Theorem 5.1.14]). *$A \in (c(p) : \ell_\infty(q))$ if and only if (2.19) holds and*

$$\sup_n \left| \sum_k a_{nk} \right|^{q_n} < \infty. \tag{2.21}$$

also holds.

Lemma 2.10 ([12, Theorem 3.1]). *Let $C = (c_{nk})$ be defined via a sequence $a = (a_k) \in w$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $U = (u_{nk})$ by*

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} a_j v_{jk}, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \tag{2.22}$$

for all $k, \in \mathbb{N}$. Then,

$$\{\lambda_U\}^\gamma = \{a = (a_k) \in w : C \in (\lambda : \ell_\infty)\}$$

and

$$\{\lambda_U\}^\beta = \{a = (a_k) \in w : C \in (\lambda : c)\}.$$

Combining Lemmas 2.4-2.9 with $q_n = 1$ ($\forall n \in \mathbb{N}$) and Lemma 2.10, we have:

Corollary 2.11. *Define the sets* $e_1^{rs}(p)$, $e_2^{rs}(p)$, $e_3^{rs}(p)$, $e_4^{rs}(p)$, $e_5^{rs}(p)$, $e_6^{rs}(p)$ *and* $e_7^{rs}(p)$ *by*

$$e_1^{rs}(p) = \bigcap_{L>1} \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a_j \right| L^{1/p_k} < \infty \right\},$$

$$e_2^{rs}(p) = \bigcap_{L>1} \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_{k} \left| \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a_j - \alpha_k \right| L^{1/p_k} = 0 \right\},$$

$$e_3^{rs}(p) = \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k} \left| \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a_j \right| M^{-1/p_k} < \infty \right\},$$

$$e_4^{rs}(p) = \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k} \left| \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a_j - \alpha_k \right| M^{-1/p_k} < \infty \right\},$$

$$e_5^{rs}(p) = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \left| \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a_j - \alpha_k \right| = 0 \right\},$$

$$e_6^{rs}(p) = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_{k} \left| \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a_j - \alpha \right| = 0 \right\},$$

and

$$e_7^{rs}(p) = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \frac{1}{r} \sum_{j=k}^{n} \left(\frac{-s}{r} \right)^{j-k} a_j \right| < \infty \right\}.$$

Then,

(i) $\{\widehat{\ell}_\infty(p)\}^\beta = e_1^{rs}(p) \cap e_2^{rs}(p)$.

(ii) $\{\widehat{\ell}_\infty(p)\}^\gamma = e_1^{rs}(p)$.

(iii) $\{\widehat{c}_0(p)\}^\beta = e_3^{rs}(p) \cap e_4^{rs}(p) \cap e_5^{rs}(p)$.

(iv) $\{\widehat{c}_0(p)\}^\gamma = e_3^{rs}(p)$.

(v) $\{\widehat{c}(p)\}^\beta = e_3^{rs}(p) \cap e_4^{rs}(p) \cap e_5^{rs}(p) \cap e_6^{rs}(p)$.

(vi) $\{\widehat{c}(p)\}^\gamma = e_7(p)$.

3 Matrix Mappings on the Spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$

In this section, we characterize some matrix mappings on the spaces $\widehat{\ell}_\infty(p)$, $\widehat{c}_0(p)$ and $\widehat{c}(p)$. Firstly, we may give the following theorem which is useful for deriving the characterization of the certain matrix clases.

Theorem 3.1 ([4, Theorem 4.1]). *Let λ be an FK-space, U be a triangle, V be its inverse and μ be arbitrary subset of w. Then we have $A \in (\lambda_U : \mu)$ if and only if*

$$C^{(n)} = (c_{mk}^{(n)}) \in (\lambda : c) \ for\ all\ n \in \mathbb{N} \tag{3.1}$$

and

$$C = (c_{nk}) \in (\lambda : \mu), \tag{3.2}$$

where

$$c_{mk}^{(n)} = \begin{cases} \sum_{j=k}^{m} a_{nj} v_{jk}, & 0 \le k \le m, \\ 0, & k > m, \end{cases}$$

and

$$c_{nk} = \sum_{j=k}^{\infty} a_{nj} v_{jk} \quad for\ all\ k, m, n \in \mathbb{N}.$$

Now, we may quote our theorems on the characterization of some matrix clases concerning with the sequence spaces $\widehat{\ell}(p)$ and $\widehat{\ell}_\infty(p)$. The necessary and sufficient conditions characterizing the matrix mappings between the sequence spaces $\ell(p)$ and $\ell_\infty(p)$ of Maddox are determined by Grosse-Erdmann [23]. Let N and K denote the finite subset of $\mathbb{N}$, L and M also denote the natural numbers and define the sets K_1 and K_2 by $K_1 = \{k \in \mathbb{N} : p_k \le 1\}$ $K_2 = \{k \in \mathbb{N} : p_k > 1\}$ and $p'_k = p_k/(p_k - 1)$ for $k \in K_2$. Prior to giving the theorems, let us suppose that (q_n) is a non-decreasing bounded sequence of positive numbers and consider the following conditions:

$$\lim_{m\to\infty} \frac{1}{r} \sum_{j=k}^{m} \left(\frac{-s}{r}\right)^{j-k} a_{nj} = c_{nk}, \tag{3.3}$$

$$\forall L,\ \lim_{m\to\infty} \sum_{k=0}^{m} \left| \frac{1}{r} \sum_{j=k}^{m} \left(\frac{-s}{r}\right)^{j-k} a_{nj} \right| L^{1/p_k} = \sum_k |c_{nk}| \, L^{1/p_k}, \tag{3.4}$$

$$\lim_{m\to\infty} \left| \frac{1}{r} \sum_{j=k}^{m} \left(\frac{-s}{r}\right)^{j-k} a_{nj} - c_{nk} \right| = 0, \quad \text{for all k}, \tag{3.5}$$

$$\exists M,\ \sup_{m\in\mathbb{N}}\sum_{k=0}^{m}\left|\frac{1}{r}\sum_{j=k}^{m}\left(\frac{-s}{r}\right)^{j-k}a_{nj}\right|M^{-1/p_k}<\infty, \tag{3.6}$$

$$\forall L,\ \exists M,\ \sup_{m\in\mathbb{N}}\sum_{k=0}^{m}\left|\frac{1}{r}\sum_{j=k}^{m}\left(\frac{-s}{r}\right)^{j-k}a_{nj}-c_{nk}\right|L^{1/q_n}M^{-1/p_k}<\infty, \tag{3.7}$$

$$\lim_{m\to\infty}\sum_{k}\left|\frac{1}{r}\sum_{j=k}^{m}\left(\frac{-s}{r}\right)^{j-k}a_{nj}-\alpha\right|=0, \tag{3.8}$$

$$\forall L,\ \sup_{n\in\mathbb{N}}\sum_{k}|c_{nk}|\,L^{1/p_k}<\infty, \tag{3.9}$$

$$\lim_{n\to\infty}c_{nk}=\alpha_k,\quad \text{for all k}, \tag{3.10}$$

$$\forall L,\ \lim_{n\to\infty}\sum_{k}|c_{nk}|\,L^{1/p_k}=\sum_{k}|\alpha_k|\,L^{1/p_k}, \tag{3.11}$$

$$\forall L,\ \lim_{n\to\infty}\sum_{k}|c_{nk}|\,L^{1/p_k}=0, \tag{3.12}$$

$$\exists M,\ \sup_{n\in\mathbb{N}}\left(\sum_{k\in K}|c_{nk}|\,M^{-1/p_k}\right)^{q_n}<\infty, \tag{3.13}$$

$$\lim_{n\to\infty}|c_{nk}|^{q_n}=0,\quad \text{for all k}, \tag{3.14}$$

$$\forall L,\ \exists M,\ \sup_{n\in\mathbb{N}}\sum_{k}|c_{nk}|\,L^{1/q_n}M^{-1/p_k}<\infty, \tag{3.15}$$

$$\lim_{n\to\infty}|c_{nk}-\alpha_k|^{q_n}=0,\quad \text{for all k}, \tag{3.16}$$

$$\exists M,\ \sup_{n\in\mathbb{N}}\sum_{k}|c_{nk}|\,M^{-1/p_k}<\infty, \tag{3.17}$$

$$\forall L,\ \exists M,\ \sup_{n\in\mathbb{N}}\sum_{k}|c_{nk}-\alpha_k|\,L^{1/q_n}M^{-1/p_k}<\infty, \tag{3.18}$$

$$\sup_{n\in\mathbb{N}}\left|\sum_k c_{nk}\right|^{q_n}<\infty, \tag{3.19}$$

$$\lim_{n\to\infty}\left|\sum_k c_{nk}\right|^{q_n}=0, \tag{3.20}$$

$$\lim_{n\to\infty}\left|\sum_k c_{nk}-\alpha\right|^{q_n}=0. \tag{3.21}$$

Theorem 3.2.

(i) $A\in(\widehat{\ell}_\infty(p):\ell_\infty)$ *if and only if (3.3), (3.4) and (3.9) hold.*

(ii) $A\in(\widehat{\ell}_\infty(p):c)$ *if and only if (3.3), (3.4), (3.10) and (3.11) hold.*

(iii) $A\in(\widehat{\ell}_\infty(p):c_0)$ *if and only if (3.3), (3.4) and (3.12) hold.*

Theorem 3.3.

(i) $A\in(\widehat{c}_0(p):\ell_\infty(q))$ *if and only if (3.5), (3.6), (3.7) and (3.13) hold.*

(ii) $A\in(\widehat{c}_0(p):c_0(q))$ *if and only if (3.5), (3.6), (3.7), (3.14) and (3.15) hold.*

(iii) $A\in(\widehat{c}_0(p):c(q))$ *if and only if (3.5), (3.6), (3.7), (3.16), (3.17) and (3.18) hold.*

Theorem 3.4.

(i) $A\in(\widehat{c}(p):\ell_\infty(q))$ *if and only if (3.5), (3.6), (3.7), (3.8), (3.13) and (3.19) hold.*

(ii) $A\in(\widehat{c}(p):c_0(q))$ *if and only if (3.5), (3.6), (3.7), (3.8), (3.14), (3.15) and (3.20) hold.*

(iii) $A\in(\widehat{c}(p):c(q))$ *if and only if (3.5), (3.6), (3.7), (3.8), (3.16), (3.17), (3.18) and (3.21) hold.*

4 The Modular σ_p and The Luxemburg Norm on The Sequence Space $\widehat{\ell}_\infty(p)$

Banach spaces have many geometric features. For details, the reader may refer to [24–26].

Let X be a real vector space. A functional $\sigma : X \to [0,\ \infty]$ is called a *modular* if

(i) $\sigma(x)=0$ if and only if $x=\theta$;

(ii) $\sigma(\alpha x)=\sigma(x)$ for all scalars α with $|\alpha|=1$;

(iii) $\sigma(\alpha x+\beta y)\le\sigma(x)+\sigma(y)$ for all $x,\ y\in X$ and $\alpha,\ \beta\ge 0$ with $\alpha+\beta=1$.

The modular σ is called *convex* if

(iv) $\sigma(\alpha x+\beta y)\le\alpha\sigma(x)+\beta\sigma(y)$ for all $x,\ y\in X$ and $\alpha,\ \beta>0$ with $\alpha+\beta=1$.

A modular σ on X is called

(a) *Right continuous* if $\lim_{\alpha\to 1^+}\ \sigma(\alpha x)=\sigma(x)$ for all $x\in X_\sigma$,

(b) *Left continuous* if $\lim_{\alpha\to 1^-}\ \sigma(\alpha x)=\sigma(x)$ for all $x\in X_\sigma$,

(c) *Continuous* if it is both right and left continuous;

where

$$X_\sigma=\left\{x\in X:\lim_{\alpha\to 0^+}\ \sigma(\alpha x)=0\right\}.$$

For $\widehat{\ell}_\infty(p)$, we define

$$\sigma_p(x)=\sup_{k\in\mathbb{N}}|sx_{k-1}+rx_k|^{p_k}.$$

If $p_k\ge 1$ for all $k\in\mathbb{N}$, by the convexity of the function $t\mapsto|t|^{p_k}$ for each $k\in\mathbb{N}$, we have that σ_p is a convex modular on the sequence space $\widehat{\ell}_\infty(p)$. We consider the sequence space $\widehat{\ell}_\infty(p)$ equipped with the Luxemburg norm given by

$$\|x\|=\inf\left\{\alpha>0:\sigma_p\left(\frac{x}{\alpha}\right)\le 1\right\}. \tag{4.1}$$

Now, we may establish some basic properties for modular σ_p.

Theorem 4.1. *The modular σ_p on the sequence space $\widehat{\ell}_\infty(p)$ satisfies the following properties:*

(i) If $0<\alpha\le 1$, then $\alpha^M\sigma_p(x/\alpha)\le\sigma_p(x)$ and $\sigma_p(\alpha x)\le\alpha\sigma_p(x)$.

(ii) If $\alpha\ge 1$, then $\sigma_p(x)\le\alpha^M\sigma_p(x/\alpha)$.

(iii) If $\alpha\ge 1$, then $\sigma_p(x)\ge\alpha\sigma_p(x/\alpha)$.

(iv) The modular σ_p is continuous on the sequence space $\widehat{\ell}_\infty(p)$.

Proof. (i) We have for any $x\in\widehat{\ell}_\infty(p)$ and $\alpha\in(0,\ 1]$ that

$$\begin{aligned}\sigma_p(x)&=\sup_{k\in\mathbb{N}}|sx_{k-1}+rx_k|^{p_k}\\&=\sup_{k\in\mathbb{N}}\left|\frac{\alpha(sx_{k-1}+rx_k)}{\alpha}\right|^{p_k}\\&\ge\alpha^M\sup_{k\in\mathbb{N}}\left|\frac{(sx_{k-1}+rx_k)}{\alpha}\right|^{p_k}=\alpha^M\sigma_p\left(\frac{x}{\alpha}\right).\end{aligned}$$

Since $p_k \geq 1$ for all k and $0 < \alpha \leq 1$, we have $\alpha^{p_k} \leq \alpha$ for all k, hence $\sigma_p(\alpha x) \leq \alpha\sigma_p(x)$.

(ii) If $\alpha \geq 1$, then $1/\alpha \leq 1$. From (i), we have

$$\left(\frac{1}{\alpha}\right)^M \sigma_p(x) = \left(\frac{1}{\alpha}\right)^M \sigma_p\left(\frac{x/\alpha}{1/\alpha}\right) \leq \sigma_p\left(\frac{x}{\alpha}\right)$$

and hence $\sigma_p(x) \leq \alpha^M \sigma_p(x/\alpha)$.

(iii) If we apply the second part of (i) with $\beta = 1/\alpha \leq 1$, then it is immediate that

$$\alpha\sigma_p\left(\frac{x}{\alpha}\right) = \alpha\sigma_p(\beta x) \leq \alpha\beta\sigma_p(x) = \sigma_p(x),$$

as expected.

(iv) By (ii) and (iii) of the present theorem, we have for $\alpha > 1$ that

$$\sigma_p(x) \leq \alpha\sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha^M \sigma_p(x). \tag{4.2}$$

By passing to limit as $\alpha \to 1^+$ in (4.2), we have $\lim_{\alpha\to 1^+} \sigma_p(\alpha x) = \sigma_p(x)$. Hence, σ_p is right continuous. If $0 < \alpha < 1$, by (i) of the present theorem, we have

$$\alpha^M \sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha\sigma_p(x). \tag{4.3}$$

Also by letting $\alpha \to 1^-$ in (4.3), we observe that $\lim_{\alpha\to 1^-} \sigma_p(\alpha x) = \sigma_p(x)$ and hence σ_p is left continuous. These two consequences give us the desired fact that σ_p is continuous. □

Now, we may give some relationship between the modular σ_p and the Luxemburg norm on the sequence space $\widehat{\ell}_\infty(p)$.

Theorem 4.2. *Let $x \in \widehat{\ell}_\infty(p)$. Then, the following statements hold:*

(i) If $\|x\| < 1$, then $\sigma_p(x) \leq \|x\|$.

(ii) If $\|x\| > 1$, then $\sigma_p(x) \geq \|x\|$.

(iii) $\|x\| = 1$ if and only if $\sigma_p(x) = 1$.

(iv) $\|x\| < 1$ if and only if $\sigma_p(x) < 1$.

(v) $\|x\| > 1$ if and only if $\sigma_p(x) > 1$.

Proof. (i) Let $\varepsilon > 0$ such that $0 < \varepsilon < 1 - \|x\|$. By the definition of $\|\cdot\|$, there exists an $\alpha > 0$ such that $\|x\| + \varepsilon > \alpha$ and $\sigma_p(x) \leq 1$. From Theorem 4.1 (i) and (ii), we have

$$\sigma_p(x) \leq \sigma_p\left[(\|x\| + \varepsilon)\frac{x}{\alpha}\right] \leq (\|x\| + \varepsilon)\sigma_p\left(\frac{x}{\alpha}\right) \leq \|x\| + \varepsilon.$$

Since ε is arbitrary, we have (i).

(ii) If we choose $\varepsilon > 0$ such that $0 < \varepsilon < 1 - 1/\|x\|$, then $1 < (1-\varepsilon)\|x\| < \|x\|$. Combining the definition of the Luxemburg norm given by (4.1) and Theorem 4.1(i), we have

$$1 < \sigma_p \left[\frac{x}{(1-\varepsilon)\|x\|} \right] \leq \frac{1}{(1-\varepsilon)\|x\|} \sigma_p(x),$$

so $(1-\varepsilon)\|x\| < \sigma_p(x)$ for all $\varepsilon \in (0, 1 - 1/\|x\|)$. This implies that $\|x\| < \sigma_p(x)$.

Since σ_p is continuous, (iii) directly follows from Theorem 1.4 of [26].

(iv) follows from (i) and (iii).

(v) follows from (iii) and (iv). □

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