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Green's Quasiorder on Menger Algebras of Terms

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Abstract : Two elements a, b of a monoid M are related with respect to Green's relation $\mathcal{L}$ if there are elements $c, d \in M$ such that $a = cb$ and $b = da$. The first equation $a = cb$ defines Green's quasiorder $\leq_{\mathcal{L}}$ on M. This quasiorder and Green's relations can also be defined for Menger algebras. After this definition we formulate some elementary propositions for Green's quasiorder $\leq_{\mathcal{L}}$ and then we consider $\leq_{\mathcal{L}}$ in several concrete Menger algebras: n-ary operations, terms and tree languages. In any case we give a characterization of $\mathcal{L}$ and of $\leq_{\mathcal{L}}$.

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1 Introduction

An algebra $\mathcal{M} = (M; S^n, e_1, \ldots, e_n)$ with an $(n+1)$-ary operation S^n and with n nullary operations $e_1, \ldots, e_n$, i.e. an algebra of type $\tau = (n+1, 0, \ldots, 0)$, satisfying the identities

(C1) $\tilde{S}^n(\tilde{Z}, \tilde{S}^n(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n), \ldots, \tilde{S}^n(\tilde{Y}_n, \tilde{X}_1, \ldots, \tilde{X}_n))$
$\approx \tilde{S}^n(\tilde{S}^n(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_n), \tilde{X}_1, \ldots, \tilde{X}_n)$,

(C2) $\tilde{S}^n(\lambda_i, \tilde{X}_1, \ldots, \tilde{X}_n) = \tilde{X}_i \quad$ for $1 \leq i \leq n$,

(C3) $\tilde{S}^n(\tilde{X}_i, \lambda_1, \ldots, \lambda_n) = \tilde{X}_i, \quad$ for $1 \leq i \leq n$,

where $\tilde{S}^n$ is an $(n+1)-$ary operation symbol, where $\lambda_1, \ldots, \lambda_n$ are nullary operation symbols and where $\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_n, \tilde{X}_1, \ldots, \tilde{X}_n$ are new variables, is called a *unitary Menger algebra of rank n*. Such algebras were introduced by K. Menger (see e.g. [9]) and considered by B. M. Schein, V. S. Trokhimenko (see [10], [12]) and other authors.

We want to mention some important examples of unitary Menger algebras of rank n.

1. Let $f : A^n \to A$ be an n-ary operation defined on the non-empty set A and let $O_A^{(n)}$ be the set of all n-ary operations defined on A. We define an $(n+1)$-ary

superposition operation $S^{n,A} : (O_A^{(n)})^{n+1} \to O_A^{(n)}$ as follows

$$S^{n,A}(g, f_1, \dots, f_n)(a_1, \dots, a_n) := g(f_1(a_1, \dots, a_n), \dots, f_n(a_1, \dots, a_n)).$$

We denote the $n-$ary projections defined on A by $e_1^{n,A}, \dots, e_n^{n,A}$, i.e. $e_i^{n,A}(a_1, \dots, a_n) := a_i$. Then $\mathcal{O}_{\mathcal{A}}^{(n)} := (O_A^{(n)}; S^{n,A}, e_1^{n,A}, \dots, e_n^{n,A})$ satisfies (C1), (C2), (C3) and is a unitary Menger algebra of rank n.

If $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ with n_i-ary operations $f_i^{\mathcal{A}} : A^{n_i} \to A$ is an algebra of type τ, then the set $T^{(n)}(\mathcal{A})$ of all n-ary operations generated from $\{f_i^{\mathcal{A}} \mid i \in I\}$ by using the superposition operations and the projections, is the universe of a subalgebra of $\mathcal{O}_{\mathcal{A}}^{(n)}$.

2. Let $X_n = \{x_1, \dots, x_n\}$ be an $n-$element alphabet of variables and let $(f_i)_{i \in I}$ be an indexed set of operation symbols. Each operation symbol f_i has arity n_i and let $\tau := (n_i)_{i \in I}$ be the sequence of all these arities. Then the set $W_\tau(X_n)$ of all $n-$ary terms of type τ is defined inductively in the following way:

(i) $x_i \in X_n$ are $n-$ary terms of type τ for all $1 \leq i \leq n$,

(ii) if $t_1, \dots, t_{n_i}$ are $n-$ary terms of type τ and if f_i is an n_i-ary operation symbol of type τ, then $f_i(t_1, \dots, t_{n_i})$ is an $n-$ary term of type τ.

On the set $W_\tau(X_n)$ we define an $(n+1)-$ary operation $S^{n,T} : W_\tau(X_n)^{n+1} \to W_\tau(X_n)$ as follows.

(i) $S^{n,T}(x_i, t_1, \dots, t_n) := t_i$ for $1 \leq i \leq n$,

(ii) $S^{n,T}(f_i(s_1, \dots, s_{n_i}), t_1, \dots, t_n)$
$:= f_i(S^{n,T}(s_1, t_1, \dots, t_n), \dots, S^{n,T}(s_{n_i}, t_1, \dots, t_n))$.

Then $n-$clone $\tau := (W_\tau(X_n); S^{n,T}, x_1, \dots, x_n)$ is a unitary Menger algebra of rank n. (see e.g. [6]).

3. Let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ be an algebra of type τ with fundamental operations $(f_i^{\mathcal{A}})_{i \in I}$ indexed by a set I where $f_i^{\mathcal{A}}$ is n_i-ary and $\tau = (n_i)_{i \in I}$ is the type of $\mathcal{A}$. Let V be a variety of algebras of type τ and let $Id_nV := IdV \bigcap W_\tau(X_n)^2$ be the set of all identities of V built up by $n-$ary terms. Clearly Id_nV is a congruence relation on the Menger algebra $n - clone\tau$ (see e.g. [6]) and therefore $n - cloneV := n - clone\tau / Id_nV$ is also a unitary Menger algebra of rank n.

4. For every algebra $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ every term $t \in W_\tau(X_n)$ defines an n-ary operation $t^{\mathcal{A}} : A^n \to A$. These induced term operations are defined inductively in the following way:

If $t = x_i \in X_n$, then $x_i^{\mathcal{A}} := e_i^{n,A}$ is the $i-th$ $n-$ary projection. Assume that $t = f_i(t_1, \dots, t_{n_i})$ is a compound term and assume that $t_1^{\mathcal{A}}, \dots, t_{n_i}^{\mathcal{A}}$ are already defined. Then we define $t^{\mathcal{A}} := S^{n,A}(f^{\mathcal{A}}, t_1^{\mathcal{A}}, \dots, t_{n_i}^{\mathcal{A}})$. Let $W_\tau(X_n)^{\mathcal{A}}$ be the set of all these induced term operations. Then $n-clone\mathcal{A} := (W_\tau(X_n)^{\mathcal{A}}; S^{n,A}, e_1^{n,\mathcal{A}}, \dots, e_n^{n,\mathcal{A}})$ is a subalgebra of $\mathcal{O}_A^{(n)}$ and it is not difficult to see that $W_\tau(X_n)^{\mathcal{A}}$ agrees with $T^{(n)}(\mathcal{A})$. This means that the set of all induced n-ary term operations is equal to the set of all $n-$ary operations which can be produced from the fundamental operations $\{f_i^{\mathcal{A}} \mid i \in I\}$ of the algebra $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ together with all projections.

We recall that $s \approx t \in IdV$ iff $s^{\mathcal{A}} = t^{\mathcal{A}}$ for the induced term operations. If $V = V(\mathcal{A})$ is a variety generated by a single algebra $\mathcal{A}$, then $Id_nV = Id_n\mathcal{A}$ and $n - clone\ V(\mathcal{A})$ is isomorphic to n-clone $\mathcal{A}$. This isomorphism is given by the mapping

$$\varphi : W_\tau(X_n)/Id_nV \to W_\tau(X_n)^{\mathcal{A}}$$

defined by $[t]_{Id_n\mathcal{A}} \mapsto t^{\mathcal{A}}$ for every $t \in W_\tau(X_n)$.

Indeed, this mapping is well-defined since from $[s]_{Id_n\mathcal{A}} = [t]_{Id_n\mathcal{A}}$, i.e. from $s \approx t \in Id_n\mathcal{A}$ we get $s^{\mathcal{A}} = t^{\mathcal{A}}$. Conversely $s^{\mathcal{A}} = t^{\mathcal{A}}$ implies $[s]_{Id_n\mathcal{A}} = [t]_{Id_n\mathcal{A}}$. Thus φ is injective. Clearly, φ is also surjective. The mapping φ is compatible with the operations since

$$\begin{aligned}
\varphi(S^{n,T}([t]_{Id_n\mathcal{A}},&[t_1]_{Id_n\mathcal{A}},\ldots,[t_n]_{Id_n\mathcal{A}})) \\
&= \varphi([S^{n,T}(t,t_1,\ldots,t_n)]_{Id_n\mathcal{A}}) \\
&= S^{n,T}(t,t_1,\ldots,t_n)^{\mathcal{A}} \\
&= S^{n,\mathcal{A}}(t^{\mathcal{A}},t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}) \\
&= S^{n,\mathcal{A}}(\varphi([t]_{Id_n\mathcal{A}}),\varphi([t_1]_{Id_n\mathcal{A}}),\ldots,\varphi([t_n]_{Id_n\mathcal{A}})).
\end{aligned}$$

Moreover, we have $\varphi([x_i]_{Id_n\mathcal{A}}) = x_i^{\mathcal{A}} = e_i^{n,\mathcal{A}}$, for every $1 \le i \le n$.

5. In [2], the following $(n+1)-$ary superposition operation $\hat{S}^n$ on elements, $B, B_1, \ldots, B_n$ of the power set $\mathcal{P}(W_\tau(X_n))$ was inductively defined by

(i) If $B := \{x_i\}$ for $1 \le i \le n$, then $\hat{S}^n(\{x_i\}, B_1, \ldots, B_n) := B_i$ if $B_j \ne \emptyset$ for $1 \le j \le n$.

(ii) If $B = \{f_i(t_1,\ldots,t_{n_i})\}$ and if we assume that $\hat{S}^n(\{t_k\}, B_1, \ldots, B_n)$ for $1 \le k \le n_i$ are already defined, then $\hat{S}^n(\{f_i(t_1,\ldots,t_{n_i})\}, B_1, \ldots, B_n) := \{f_i(r_1,\ldots,r_{n_i}) \mid r_k \in \hat{S}^n(\{t_k\}, B_1, \ldots, B_n) \text{ for } 1 \le k \le n_i\}$ when $B_j \ne \emptyset$ for $1 \le j \le n$.

(iii) If B is an arbitrary non-empty subset of $W_\tau(X_n)$, then

$$\hat{S}^n(B, B_1, \ldots, B_n) := \bigcup_{b \in B} \hat{S}^n(\{b\}, B_1, \ldots, B_n).$$

(iv) If one of the sets $B, B_1, \ldots, B_n$ is empty, then

$$\hat{S}^n(B, B_1, \ldots, B_n) := \emptyset.$$

We notice that elements of $\mathcal{P}(W_\tau(X_n))$ are also called tree languages and therefore $\hat{S}^n$ is an operation on tree languages (see e.g. [4]). In [2] was proved that $\mathcal{P}_n - clone\ \tau := (\mathcal{P}(W_\tau(X_n)); \hat{S}^n, \{x_1\}, \ldots \{x_n\})$ is a unitary Menger algebra of rank n. Of course, similarly a superposition operation $\hat{S}^{n,A}$ on sets of n-ary operations defined on the same set A can be considered and we obtain one more example of a unitary Menger algebra of rank n.

2 Green's Quasiorder

Green's relations are special equivalence relations which can be defined on any semigroup or monoid, using the idea of mutual divisibility of elements. But Green's relations can also be defined on Menger algebras (see e.g. [3]). We define Green's relation $\mathcal{L}$ using the following relation $\leq_{M'}$.

Definition 2.1 Let $(M; S^n, e_1, \ldots, e_n)$ be a unitary Menger algebra of rank n and let $M' \subseteq M$ be a subset. Then we define for $a, b \in M$

$$a \leq_{M'} b :\Longleftrightarrow \exists s_1, \ldots, s_n \in M' \ (a = S^n(b, s_1, \ldots, s_n)).$$

In the case $M' = M$ we write $\leq_{\mathcal{L}}$. Moreover we define $a\mathcal{L}b$ iff $a \leq_{\mathcal{L}} b$ and $b \leq_{\mathcal{L}} a$.

Then we have :

Proposition 2.2 *If M' is the universe of a subalgebra of $(M; S^n, e_1, \ldots, e_n)$, then $\leq_{M'}$ is a quasiorder on M, i.e. reflexive and transitive. Especially $\leq_{\mathcal{L}}$ is a quasiorder on M.*

Proof. From (C3), we have that $a = S^n(a, e_1, \ldots, e_n)$ for all $a \in M$. This means $\leq_{M'}$ is reflexive. If $a \leq_{M'} b$ and $b \leq_{M'} c$, then there are $s_1, \ldots, s_n, t_1, \ldots, t_n \in M'$ such that

$$a = S^n(b, s_1, \ldots, s_n) \ \ \textit{and} \ \ b = S^n(c, t_1, \ldots, t_n).$$

Therefore from (C1) we have

$$\begin{aligned} a &= S^n(b, s_1, \ldots, s_n) \\ &= S^n(S^n(c, t_1, \ldots, t_n), s_1, \ldots, s_n) \\ &= S^n(c, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)). \end{aligned}$$

Since M' is a subalgebra, then $a \leq_{M'} c$. Thus $\leq_{M'}$ is transitive. □

But the converse is also true and we have :

Theorem 2.3 *The set M' is the universe of a subalgebra of $(M; S^n, e_1, \ldots, e_n)$ if and only if $\leq_{M'}$ is a quasiorder on M.*

Proof. Because of Proposition 2.2 we have only to prove the opposite direction. We show at first that all nullary operations $e_1, \ldots, e_n$ belong to M'. Suppose that there exists $e_j \notin M'$ for some $1 \leq j \leq n$. Since $\leq_{M'}$ is reflexive, there exist elements $a_1, \ldots, a_n \in M'$ such that

$$e_j = S^n(e_j, a_1, \ldots, a_n) = a_j \in M', \quad \text{by (C2)}.$$

This gives a contradiction to $e_j \notin M'$. Thus $e_1, \ldots, e_n \in M'$. Now we prove that M' is closed under S^n. Let $a_1, a_2, \ldots, a_{n+1} \in M'$. Then by definition of $\leq_{M'}$, we have

$$S^n(a_1, a_2, \ldots, a_{n+1}) \leq_{M'} a_1 = S^n(e_1, a_1, \ldots, a_n) \leq_{M'} e_1.$$

From transitivity and the definition of $\leq_{M'}$, we obtain elements $m_1, \ldots, m_n \in M'$ such that

$$S^n(a_1, a_2, \ldots, a_{n+1}) = S^n(e_1, m_1, \ldots, m_n) = m_1 \in M'.$$

This means M' is closed under application of S^n. □

Further, it is easy to check that for subalgebras $\mathcal{M}_1, \mathcal{M}_2$ of a unitary Menger algebra $\mathcal{M}$ with $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}$ we have $\leq_{M_1} \subseteq \leq_{M_2}$.

Green's relation $\mathcal{R}$ (see [3]) is defined by $a\mathcal{R}b$ if there are elements $s, t \in M$ such that

$$S^n(s, a, \ldots, a) = b \ \ and \ \ S^n(t, b, \ldots, b) = a.$$

In a similar way as $\leq_{\mathcal{L}}$ we can define $\leq_{\mathcal{R}}$ by the first equation.

3 Menger Algebras of Operations

In [7], for a class $C \subseteq O_A$ of functions on A the author defined the concept of a $C-$subfunction of a given function as follows :

Definition 3.1 Let $f, g \in O_A^{(n)}$ and let $C \subseteq O_A^{(n)}$. Then f is called a *$C-$subfunction of* g if there are functions $h_1, \ldots, h_n \in C$ such that

$$f = S^{n,A}(g, h_1, \ldots, h_n).$$

In this case we write $f \leq_C g$.

Since $\mathcal{O}_{\mathcal{A}}^{(n)}$ is a unitary Menger algebra of rank n, we may apply Theorem 2.3 and obtain that C is a universe of a subalgebra of $\mathcal{O}_{\mathcal{A}}^{(n)}$ iff $\leq_C$ is a quasiorder on $O_A^{(n)}$.

Using the relation $\leq_{\mathcal{C}}$ one can define an equivalence relation $\equiv_{\mathcal{C}}$ on $O_A^{(n)}$. For $\mathcal{C} = \mathcal{O}_A^{(n)}$ the relation $\equiv_{\mathcal{C}}$ agrees with Green's relation $\mathcal{L}$. It is clear that $Imf \subseteq Img$ for any $f \leq_{\mathcal{C}} g$ and any $\mathcal{C}$. Therefore, if $f \equiv_{\mathcal{C}} g$, then $Imf = Img$. For a semigroup $\mathcal{S}$ and for any $a, b \in S$ Green's relations are defined as follows

$$a\mathcal{L}b \Leftrightarrow \exists c, d \in S(a = cb \ and \ b = da)$$
$$a\mathcal{R}b \Leftrightarrow \exists c, d \in S(a = bc \ and \ b = ad).$$

Green's relation $\mathcal{R}$ can also be defined on $O_A^{(n)}$. Clearly, for Green's relation $\mathcal{R}$ we can also consider the quasiorder $a \leq_{\mathcal{R}} b$ corresponding to $a \leq_{\mathcal{L}} b$. It is well-known (see [5]) that for transformations $f, g \in O_A^{(1)}$ there holds

$$f\mathcal{L}g \Leftrightarrow Imf = Img$$
$$f\mathcal{R}g \Leftrightarrow Kerf = Kerg.$$

To transfer these results to $n-$ary operations for any $n-$ary operation $f : A^n \to A$ we consider the unary operation $f^{\otimes n} : A^n \to A^n$ defined by

$$f^{\otimes n}(a_1, \ldots, a_n) := (f(a_1, \ldots, a_n), f(a_1, \ldots, a_n), \ldots, f(a_1, \ldots, a_n))$$

for every $(a_1, \ldots, a_n) \in A^n$. Let $(O^{(1)}_{A^n}; \circ)$ be the semigroup of all unary operations (transformations) defined on A^n. (Remark that the composition of two operations $h_1, h_2 \in O^{(1)}_{A^n}$, i.e. $h_1 \circ h_2$ is equal to $S^{1,A^n}(h_2, h_1)$.) For different operations $f_1, \ldots, f_n \in O^{(n)}_A$ we define an operation $f_1 \otimes \cdots \otimes f_n \in O^{(1)}_{A^n}$ by

$$(f_1 \otimes \cdots \otimes f_n)(a_1, \ldots, a_n) := (f_1(a_1, \ldots, a_n), \ldots, f_n(a_1, \ldots, a_n)).$$

If conversely $h \in O^{(1)}_{A^n}$, then we can find n uniquely determined operations $h_1, \ldots, h_n$ such that $h = h_1 \otimes \cdots \otimes h_n$. If $\pi_j : A^n \to A, 1 \leq j \leq n$, are the canonical projections, then $h_j = \pi_j \circ h$ for $1 \leq j \leq n$, i.e. each element $f \in O^{(1)}_{A^n}$ can be represented as $(f_1 \otimes \cdots \otimes f_n)$ when $f_1, \ldots, f_n \in O^{(n)}_A$. Now we show that the composition in the semigroup $(O^{(1)}_{A^n}; \circ)$ corresponds to the operation $S^{n,A}$ in $(O^{(n)}_A; S^{n,A}, e^{n,A}_1, \ldots, e^{n,A}_n)$.

Lemma 3.2 *Let $f = f_1 \otimes \cdots \otimes f_n$ and $g = g_1 \otimes \cdots \otimes g_n$ be elements of $O^{(1)}_{A^n}$. Then*

$$(f_1 \otimes \cdots \otimes f_n) \circ (g_1 \otimes \cdots \otimes g_n) = S^{n,A}(g_1, f_1, \ldots, f_n) \otimes \cdots \otimes S^{n,A}(g_n, f_1, \ldots, f_n).$$

Proof. Let $\bar{a} := (a_1, \ldots, a_n) \in A^n$. Then

$$\begin{aligned}
&(f_1 \otimes \ldots \otimes f_n) \circ (g_1 \otimes \cdots \otimes g_n)(\bar{a}) \\
&\quad = (g_1 \otimes \cdots \otimes g_n)((f_1 \otimes \cdots \otimes f_n)(\bar{a})) \\
&\quad = (g_1 \otimes \cdots \otimes g_n)(f_1(\bar{a}), \ldots, f_n(\bar{a})) \\
&\quad = (g_1(f_1(\bar{a}), \ldots, f_n(\bar{a})), \ldots, g_n(f_1(\bar{a}), \ldots, f_n(\bar{a}))) \\
&\quad = (S^{n,A}(g_1, f_1, \ldots, f_n)(\bar{a}), \ldots, S^{n,A}(g_n, f_1, \ldots, f_n)(\bar{a})) \\
&\quad = (S^{n,A}(g_1, f_1, \ldots, f_n) \otimes \cdots \otimes S^{n,A}(g_n, f_1, \ldots, f_n))(\bar{a}).
\end{aligned}$$

□

In particular for $g_1 = \cdots = g_n$, i.e. if the second operation from $O^{(1)}_{A^n}$ has the form $g^{\otimes n}$, then $(f_1 \otimes \cdots \otimes f_n) \circ g^{\otimes n} = (S^{n,A}(g, f_1, \ldots f_n))^{\otimes n}$. Now we are able to prove :

Lemma 3.3 *Let $f, g \in O^{(n)}_A$. Then*

(i) *$f\mathcal{L}g$ if and only if $f^{\otimes n}\mathcal{L}g^{\otimes n}$ and,*

(ii) *$f\mathcal{R}g$ if and only if $f^{\otimes n}\mathcal{R}g^{\otimes n}$.*

Proof. (i) Let $f\mathcal{L}g$, then there are operations $f_1, \ldots, f_n, g_1, \ldots, g_n \in O_A^{(n)}$ such that

$$f = S^{n,A}(g, g_1, \ldots, g_n) \text{ and } g = S^{n,A}(f, f_1, \ldots, f_n).$$

But then we have also $f^{\otimes n} = S^{n,A}(g, g_1, \ldots, g_n)^{\otimes n}$ and $g^{\otimes n} = S^{n,A}(f, f_1, \ldots, f_n)^{\otimes n}$ and by Lemma 3.2,

$$f^{\otimes n} = (g_1 \otimes \cdots \otimes g_n) \circ g^{\otimes n} \text{ and } g^{\otimes n} = (f_1 \otimes \cdots \otimes f_n) \circ f^{\otimes n}$$

and this means $f^{\otimes n}\mathcal{L}g^{\otimes n}$. If we conversely assume that $f^{\otimes n}\mathcal{L}g^{\otimes n}$ then there are operations $g_1 \otimes \cdots \otimes g_n, f_1 \otimes \cdots \otimes f_n \in O_{A^n}^{(1)}$ and $g_1, \ldots, g_n, f_1, \ldots, f_n \in O_A^{(n)}$ such that

$$f^{\otimes n} = (g_1 \otimes \cdots \otimes g_n) \circ g^{\otimes n} \text{ and } g^{\otimes n} = (f_1 \otimes \cdots \otimes f_n) \circ f^{\otimes n}$$

and by Lemma 3.2, $f^{\otimes n} = S^{n,A}(g, g_1, \ldots, g_n)^{\otimes n}$ and $g^{\otimes n} = S^{n,A}(f, f_1, \ldots, f_n)^{\otimes n}$. Application of one of the projections $\pi_j, 1 \leq j \leq n,$ gives $f = S^{n,A}(g, g_1, \ldots, g_n)$, $g = S^{n,A}(f, f_1, \ldots, f_n)$ and thus $f\mathcal{L}g$.

(ii) If $f\mathcal{R}g$, then there are operations $s, t \in O_A^{(n)}$ such that $f = S^{n,A}(t, g, \ldots, g)$ and $g = S^{n,A}(s, f, \ldots, f)$. From these equations we obtain

$$f^{\otimes n} = S^{n,A}(t, g, \ldots, g)^{\otimes n} \textit{ and } g^{\otimes n} = S^{n,A}(s, f, \ldots, f)^{\otimes n}$$

and by Lemma 3.2, $f^{\otimes n} = g^{\otimes n} \circ t^{\otimes n}$ and $g^{\otimes n} = f^{\otimes n} \circ s^{\otimes n}$; i.e. $f^{\otimes n}\mathcal{R}g^{\otimes n}$.

If conversely $f^{\otimes n}\mathcal{R}g^{\otimes n}$, then there are $h = h_1 \otimes \cdots \otimes h_n$ and $k = k_1 \otimes \cdots \otimes k_n$ and $h_1, \ldots, h_n, k_1, \ldots, k_n \in O_A^{(n)}$ such that

$$f^{\otimes n} = g^{\otimes n} \circ (h_1 \otimes \cdots \otimes h_n) \textit{ and } g^{\otimes n} = f^{\otimes n} \circ (k_1 \otimes \cdots \otimes k_n).$$

By Lemma 3.2, we get $f^{\otimes n} = S^{n,A}(h_1, g, \ldots, g) \otimes \cdots \otimes S^{n,A}(h_n, g, \ldots, g)$ and applying π_1 on both sides we have $f = S^{n,A}(h_1, g, \ldots, g)$. Similarly we get $g = S^{n,A}(k_1, f, \ldots, f)$. This shows $f\mathcal{R}g$. □

As a Corollary we get

Corollary 3.4 *Let $f, g \in O_A^{(n)}$. Then*

(i) $f\mathcal{L}g$ *if and only if* $Im\ f = Im\ g$ *and,*

(ii) $f\mathcal{R}g$ *if and only if* $Ker\ f = Ker\ g$.

If for $C \subseteq O_A^{(n)}$ we define $C^{\otimes n} := \{f_1 \otimes \ldots, \otimes f_n \mid f_j \in C\}$ and if for $h, k \in O_{A^n}^{(1)}$ we set $h \leq_{C^{\otimes n}} k$ iff there is a unary function $f \in C^{\otimes n} \subseteq O_{A^n}^{(1)}$ such that $h = f \circ g$, then we have already $f \leq_C g$ iff $f^{\otimes n} \leq_{C^{\otimes n}} g^{\otimes n}$.

4 Menger Algebras of Terms

Let $W_\tau(X_n)$ be the set of all $n-$ary terms of type τ and let $n - clone\ \tau = (W_\tau(X_n), S^{n,T}, x_1, \ldots, x_n)$ be the Menger algebra defined in section 1. Then we define :

Definition 4.1 Let $A \subseteq W_\tau(X_n)$ be a set of $n-$ary terms. Then for $f, g \in W_\tau(X_n)$ we define

$$f \leq_A g :\Leftrightarrow \exists\ t_1, \ldots, t_n \in A\ (f = S^{n,T}(g, t_1, \ldots, t_n)).$$

Since $n-clone\ \tau$ is a unitary Menger algebra of rank n, we may apply Theorem 2.3 and obtain that A is the universe of a subalgebra of $n - clone\ \tau$ iff $\leq_A$ is a quasiorder on $W_\tau(X_n)$.

For $A = W_\tau(X_n)$ we write again for short $\leq_{\mathcal{L}}$. There are different methods to measure the complexity of a term. The inductive definition of terms is based on the number $op(t)$ of occurrences of operation symbols in the term t. The operation symbol count $op(t)$ is inductively defined by

(i) $op(x_i) := 0$ if $x_i \in X_n$ and

(ii) $op(f_i(t_1, \ldots, t_{n_i})) := \sum\limits_{j=1}^{n_i} op(t_j) + 1.$

Let $vb_j(s)$ be the number of occurrences of the variable x_j in the term s. If $s, t_1, \ldots, t_n \in W_\tau(X_n)$, then from [1] we obtain that

$$op(S^{n,T}(s, t_1, \ldots, t_n)) = \sum_{j=1}^{n} op(s) + vb_j(s)op(t_j).$$

Therefore we have, if $s \leq_{\mathcal{L}} t$, then there are $t_1, \ldots, t_n \in W_\tau(X_n)$ such that $s = S^{n,T}(t, t_1, \ldots, t_n)$ and then $op(s) \geq op(t)$.

Using this fact, we prove

Proposition 4.2 *Assume that the type τ contains at least one at least unary operation symbol f. Then $W_\tau(X_n)$ contains an infinite descending chain with respect to the quasiorder $\leq_{\mathcal{L}}$.*

Proof. We assume at first that the type τ contains at least one at least unary operation symbol f. We consider the sequence

$$t_0 := f(x_1, \ldots, x_{n_i})$$
$$t_{k+1} := S^{n,T}(t_k, t_0, x_2, \ldots, x_n)$$

where $k \in \mathbb{N}$. This gives an infinite sequence in $W_\tau(X_n)$, since $t_{k+1} \leq_{\mathcal{L}} t_k$ for any $k \in \mathbb{N}$ and because of $op(t_k) \neq op(t_{k+1})$ we have $t_k \neq t_{k+1}$ for any $k \in \mathbb{N}$. $\square$

For any subclone $\mathcal{M} \subseteq n-clone\,\tau$ we have $t = S^{n,T}(x_i, t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n)$. Thus $t \leq_{\mathcal{M}} x_i$ for all $t \in M$. For $\mathcal{M} = n - clone\ \tau$ we have $t \leq_{\mathcal{L}} x_i$ for all $t \in W_\tau(X_n)$. Therefore every variable $x_i \in X_n$ is a maximal element.

Now we ask when $s \leq_{\mathcal{L}} t$ and $t \leq_{\mathcal{L}} s$, i.e. when $s\mathcal{L}t$. The answer is given in [3] Theorem 4.5.

Theorem 4.3 *Let $s, t \in W_\tau(X_n)$. Then $s\mathcal{L}t$ if and only if there exists a permutation r on the set $\{1, 2, \ldots, n\}$ such that $t = S^{n,T}(s, x_{r(1)}, \ldots, x_{r(n)})$.*

The relation $\leq_{\mathcal{M}}$ can also be applied to hypersubstitutions. A hypersubstitution σ of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which preserves the arity, i.e. such that $\sigma(f_i)$ is n_i-ary. Every hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ by the following inductive definition:

(i) $\hat{\sigma}[x] := x$ for variables

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i,T}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$ for compound terms.

Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. With the product $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and the identity hypersubstitution σ_{id} defined by $\sigma_{id}(f_i) = f_i(x_1, \ldots, x_{n_i})$ we get the monoid $(Hyp(\tau) : \circ_h, \sigma_{id})$.

Definition 4.4 Let M be a subset of $n - clone\ \tau$. Then we define the following binary relation on $Hyp(\tau)$:

$$\sigma_1 \leq_M \sigma_2 \ \text{ iff } \ \sigma_1(f_i) \leq_M \sigma_2(f_i) \ \text{ for all } i \in I.$$

Then we have:

Proposition 4.5 *Let $\mathcal{M}$ be a subalgebra of $n-clone\ \tau$. Then $\leq_{\mathcal{M}}$ is a quasiorder on $Hyp(\tau)$.*

Proof. Going back to the quasiorder $\leq_{\mathcal{M}}$ on $W_\tau(X_n)$ we prove reflexivity and transitivity of $\leq_{\mathcal{M}}$ on $Hyp(\tau)$. □

Clearly, by $\sigma_1 \leq_{\mathcal{M}} \sigma_2$ and $\sigma_2 \leq_{\mathcal{M}} \sigma_1$ we define an equivalence relation $\equiv_{\mathcal{M}}$ on $Hyp(\tau)$. The relation $\leq_{\mathcal{M}}$ is connected with the multiplication on $Hyp(\tau)$.

Proposition 4.6 *Let $\mathcal{M}$ be a subalgebra of $n - clone\ \tau$. Then $\leq_{\mathcal{M}}$ is left-compatible quasiorder on $Hyp(\tau)$.*

Proof. We show that the relation $\leq_{\mathcal{M}}$ is left compatible. Assume that $\sigma_1 \leq_{\mathcal{M}} \sigma_2$ and $\sigma \in Hyp(\tau)$. Then for all $i \in I$ we have $\sigma_1(f_i) \leq_{\mathcal{M}} \sigma_2(f_i)$. This means, there are terms $t_1, \ldots, t_n \in M$ such that $\sigma_1(f_i) = S^{n,T}(\sigma_2(f_i), t_1, \ldots, t_n)$. The extension of the hypersubstitution σ is an endomorphism of the Menger algebra $n - clone\ \tau$

and its restriction to M is an endomorphism of $\mathcal{M}$ ([6]). Therefore for each $i \in I$ we have

$$\begin{aligned}(\sigma \circ_h \sigma_1)(f_i) &= (\hat{\sigma} \circ \sigma_1)(f_i) = \hat{\sigma}(\sigma_1(f_i)) \\ &= \hat{\sigma}(S^{n,T}(\sigma_2(f_i), t_1, \ldots, t_n)) \\ &= S^{n,T}(\hat{\sigma}[\sigma_2(f_i)], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])\end{aligned}$$

and therefore $(\sigma \circ_h \sigma_1)(f_i) \leq_{\mathcal{M}} (\sigma \circ_h \sigma_2)(f_i)$. This means, $\sigma \circ_h \sigma_1 \leq_{\mathcal{M}} \sigma \circ_h \sigma_2$. □

Using the relation $\leq_{\mathcal{M}}$ for every subalgebra $\mathcal{M}$ of $n - clone\ \tau$ we may define an equivalence relation on $Hyp(\tau)$ by

$$\sigma_1 \equiv_{\mathcal{M}} \sigma_2 :\Leftrightarrow \sigma_1 \leq_{\mathcal{M}} \sigma_2 \text{ and } \sigma_2 \leq_{\mathcal{M}} \sigma_1.$$

5 Menger Algebras of Tree Languages

In section 1, we introduced the unitary Menger algebra $\mathcal{P}_n - clone\ \tau = (\mathcal{P}(W_\tau(X_n)); \hat{S}^n, \{x_1\}, \ldots, \{x_n\})$. Subsets of $W_\tau(X_n)$ are also called tree languages. Therefore we can speak of a Menger algebra of tree languages. Now we define

Definition 5.1 Let $\mathcal{T} \subseteq \mathcal{P}(W_\tau(X_n))$ be a collection of sets of $n-$ary terms of type τ. For $B_1, B_2 \subseteq W_\tau(X_n)$ we define

$$B_1 \leq_{\mathcal{T}} B_2 :\Leftrightarrow \exists A_1, \ldots, A_n \in \mathcal{T}\ (B_1 = \hat{S}^n(B_2, A_1, \ldots, A_n)).$$

The relation $\leq_{\mathcal{T}}$ is a quasiorder and we may define the equivalence relation $\equiv_{\mathcal{T}}$ by $B_1 \equiv_{\mathcal{T}} B_2 :\Leftrightarrow B_1 \leq_{\mathcal{T}} B_2$ and $B_2 \leq_{\mathcal{T}} B_1$. For $\mathcal{T} = \mathcal{P}(W_\tau(X_n))$ we write $B_1 \leq_{\mathcal{L}} B_2$ and obtain Green's relation $\mathcal{L}$ on $\mathcal{P}_n - clone\ \tau$. To characterize Green's relation $\mathcal{L}$ on $(\mathcal{P}(W_\tau(X_n)); \hat{S}^n)$ we define the concept of a near homomorphism.

Definition 5.2 A mapping $\alpha : \mathcal{P}(W_\tau(X_n)) \to \mathcal{P}(W_\tau(X_n))$ is called a *near homomorphism* of $(\mathcal{P}(W_\tau(X_n)); \hat{S}^n)$ if

$$\alpha(\hat{S}^n(A, B_1, \ldots, B_n)) = \hat{S}^n(A, \alpha(B_1), \ldots, \alpha(B_n))$$

for all $A, B_1, \ldots, B_n \subseteq W_\tau(X_n)$.

Now we consider a mapping $\alpha : \{\{x_i\} \mid x_i \in X_n\} \to \mathcal{P}(W_\tau(X_n)) \setminus \{\emptyset\}$ and extend this mapping to a mapping $\bar{\alpha} : \mathcal{P}(W_\tau(X_n)) \to \mathcal{P}(W_\tau(X_n))$ using the following inductive definition :

Definition 5.3

(i) $\bar{\alpha}(\{x_i\}) := \alpha(\{x_i\})$ for all $x_i \in X_n$.

(ii) $\bar{\alpha}(\{f_i(t_1,\ldots,t_{n_i})\}) := \{f_i(r_1,\ldots,r_{n_i}) \mid r_j \in \bar{\alpha}(\{t_j\})\ for\ 1 \le j \le n_i\}$ assuming that $\bar{\alpha}(\{t_j\})$ for $1 \le j \le n_i$ are already defined.

(iii) For any nonempty set $A \in \mathcal{P}(W_\tau(X_n))$ we set $\bar{\alpha}(A) := \bigcup\limits_{a\in A} \bar{\alpha}(\{a\})$ and if A is empty, we define $\bar{\alpha}(A) := \emptyset$.

Lemma 5.4 *Let $\alpha : \{\{x_i\} \mid x_i \in X_n\} \to \mathcal{P}(W_\tau(X_n)) \setminus \{\emptyset\}$ be a mapping. Then the extension $\bar{\alpha}$ of α is a near homomorphism.*

Proof. We shall prove by induction on the complexity of terms in A that

$$\bar{\alpha}(\hat{S}^n(A, B_1,\ldots,B_n)) = \hat{S}^n(A, \bar{\alpha}(B_1),\ldots,\bar{\alpha}(B_n)).$$

Let $A = \{x_i\}$ then we have that

$$\bar{\alpha}(\hat{S}^n(\{x_i\}, B_1,\ldots,B_n)) = \bar{\alpha}(B_i) = \hat{S}^n(\{x_i\}, \bar{\alpha}(B_1),\ldots,\bar{\alpha}(B_n)).$$

If $a = f_i(t_1,\ldots,t_{n_i})$ and if we assume that $\bar{\alpha}(\hat{S}^n(\{t_j\}, B_1,\ldots,B_n)) = \hat{S}^n(\{t_j\}, \bar{\alpha}(B_1),\ldots,\bar{\alpha}(B_n))$ for all $1 \le j \le n$, then

$$\begin{aligned}
&\bar{\alpha}(\hat{S}^n(\{f_i(t_1,\ldots,t_{n_i})\}, B_1,\ldots,B_n))\\
&\quad = \bar{\alpha}(\{f_i(r_1,\ldots,r_{n_i}) \mid r_j \in \hat{S}^n(\{t_j\}, B_1,\ldots,B_n)\ for\ 1 \le j \le n_i\})\\
&\quad = \bigcup\{\bar{\alpha}(f_i(r_1,\ldots,r_{n_i})) \mid r_j \in \hat{S}^n(\{t_j\}, B_1,\ldots,B_n)\ for\ 1 \le j \le n_i\}\\
&\quad = \bigcup\{f_i(u_1,\ldots,u_{n_i}) \mid u_j \in \bar{\alpha}(r_j)\ and\ r_j \in \hat{S}^n(\{t_j\}, B_1,\ldots,B_n)\ for\ 1 \le j \le n_i\}\\
&\quad = \bigcup\{f_i(u_1,\ldots,u_{n_i}) \mid u_j \in \bar{\alpha}(\hat{S}^n(\{t_j\}, B_1,\ldots,B_n)\ for\ 1 \le j \le n_i\}\\
&\quad = \bigcup\{f_i(u_1,\ldots,u_{n_i}) \mid u_j \in \hat{S}^n(\{t_j\}, \bar{\alpha}(B_1),\ldots,\bar{\alpha}(B_n))\ for\ 1 \le j \le n_i\}\\
&\quad = \hat{S}^n(\{f_i(t_1,\ldots,t_{n_i})\}, \bar{\alpha}(B_1),\ldots,\bar{\alpha}(B_n)).
\end{aligned}$$

If A is an arbitrary non-empty subset of $W_\tau(X_n)$ then

$$\begin{aligned}
\bar{\alpha}(\hat{S}^n(A, B_1,\ldots,B_n)) &= \bar{\alpha}(\bigcup_{a\in A} \hat{S}^n(\{a\}, B_1,\ldots,B_n))\\
&= \bigcup_{a\in A} \bar{\alpha}(\hat{S}^n(\{a\}, B_1,\ldots,B_n))\\
&= \bigcup_{a\in A} S^n(\{a\}, \bar{\alpha}(B_1),\ldots,\bar{\alpha}(B_n))\\
&= \hat{S}^n(A, \bar{\alpha}(B_1),\ldots,\bar{\alpha}(B_n)).
\end{aligned}$$

If A is the empty set, then

$$\bar{\alpha}(\hat{S}^n(A, B_1,\ldots,B_n)) = \emptyset = \hat{S}^n(A, \bar{\alpha}(B_1),\ldots,\bar{\alpha}(B_n)).$$

Thus $\bar{\alpha}$ is a near homomorphism. □

Theorem 5.5 *Let $A, B \in \mathcal{P}(W_\tau(X_n))$ be nonempty. Then $A \leq_{\mathcal{L}} B$ if and only if there is near homomorphism $\alpha : \mathcal{P}(W_\tau(X_n)) \to \mathcal{P}(W_\tau(X_n))$ such that $\alpha(B) = A$.*

Proof. Assume that $A \leq_{\mathcal{L}} B$, then there are $B_1, \ldots, B_n \in \mathcal{P}(W_\tau(X_n))$ such that

$$A = \hat{S}^n(B, B_1, \ldots, B_n).$$

We consider the mapping $\alpha : \{\{x_i\} \mid x_i \in X_n\} \to \mathcal{P}(W_\tau(X_n))$ defined by $\alpha : \{x_i\} \mapsto B_i$ for $x_i \in X_n$. Thus by Lemma 5.4 this mapping can be extended to a near homomorphism $\bar{\alpha} : \mathcal{P}(W_\tau(X_n)) \to \mathcal{P}(W_\tau(X_n))$ such that

$$A = \hat{S}^n(B, B_1, \ldots, B_n) = \bar{\alpha}(B).$$

Conversely, let $\alpha : \mathcal{P}(W_\tau(X_n)) \to \mathcal{P}(W_\tau(X_n))$ be a near homomorphism such that $\alpha(B) = A$. Since A,B are non-empty, thus $\alpha(\{x_i\}) \neq \emptyset$. We put $B_i = \alpha(\{x_i\})$ for all $1 \leq i \leq n$, then

$$\begin{aligned} \hat{S}^n(B, B_1, \ldots, B_n) &= \hat{S}^n(B, \alpha(\{x_1\}), \ldots, \alpha(\{x_1\})) \\ &= \alpha(\hat{S}^n(B, \{x_1\}, \ldots, \{x_n\})) = \alpha(B) = A. \end{aligned}$$

This means $A \leq_{\mathcal{L}} B$. □

¿From the relation $\leq_{\mathcal{L}}$, Green's relation $\mathcal{L}$ can be obtained by $A \equiv_{\mathcal{L}} B$ iff $A \leq_{\mathcal{L}} B$ and $B \leq_{\mathcal{L}} A$. Thus by Theorem 5.5 $A \equiv_{\mathcal{L}} B$ if and only if there are near homomorphisms $\alpha, \beta : \mathcal{P}(W_\tau(X_n)) \to \mathcal{P}(W_\tau(X_n))$ such that $\alpha(B) = A$ and $\beta(A) = B$.

Remark 5.6 (i) Let $A \subseteq W_\tau(X_n)$ be a subset, let $B \subseteq X_n$ and let

$$\alpha : \{\{x_i\} \mid x_i \in X_n\} \to \mathcal{P}(W_\tau(X_n))$$

be a mapping defined by $\alpha(\{x_i\}) = A$ for all $x_i \in X_n$. Then by Lemma 5.4 we can extend α to a near homomorphism $\bar{\alpha} : \mathcal{P}(W_\tau(X_n)) \to \mathcal{P}(W_\tau(X_n))$. Then $\bar{\alpha}$ maps B to A because of $\bar{\alpha}(B) = \bigcup_{x_i \in B} \bar{\alpha}(\{x_i\}) = \bigcup_{x_i \in B} A = A$. Thus $A \leq_{\mathcal{L}} B$.

(ii) If $A, B \subseteq X_n$, then $A \leq_{\mathcal{L}} B$ and $B \leq_{\mathcal{L}} A$ and so $A\mathcal{L}B$.

(iii) Let $A = \{a\}$ and $B = \{b\}$ be singleton sets. Then $A\mathcal{L}B$ if and only if the mapping $\bar{\alpha}$ from Theorem 5.5 is the extension of a bijective mapping $\alpha : \{\{x_i\} \mid x_i \in var(a)\} \to \{\{x_i\} \mid x_i \in var(b)\}$ (see [3], Theorem 4.5)

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