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On Certain New Difference Sequence Spaces Generated by Infinite Matrices

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Abstract : The main purpose of the present paper is to derive some inclusion relations and other interesting properties of certain difference sequence spaces $[C,1](\hat{A},\Delta)$, $(C,1)(\hat{A},\Delta), [V,\lambda](\hat{A},\Delta)$ and $(V,\lambda)(\hat{A},\Delta)$ generated by infinite matrices. Also we introduce the concept of λ_B-statistical convergence and derive many other results on these spaces by introducing Orlicz functions.

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1 Introduction

Let ω denote the space of all scalar sequences and any subspace of ω is called a sequence space. Let ℓ_∞, c and c_0 be the linear space of bounded, convergent and null sequences with complex terms respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in \mathbb{N} = \{1,2,3,...\}$ the set of positive intigers.

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The idea of difference sequence space was first introduced by Kizmaz [1] by defining the sequence space

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\} \tag{1.1}$$

for $X = \ell_\infty, c$ and c_0, where $\Delta x = (x_k - x_{k+1})$. Then Et and Colak [2] generalized the above sequence spaces to the sequence spaces

$$X(\Delta^r) = \{x = (x_k) : \Delta^r x \in X\} \tag{1.2}$$

where $r \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta^r x = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$ and

$$\Delta^r x_k = \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu} x_{k+\nu}.$$

The *generalized de la Vallee-Poussin mean* is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k \tag{1.3}$$

where $\lambda = (\lambda_k)$ is a non decreasing sequence of positive numbers satisfying $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if $t_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then (V, λ)-summability is reduced to $(C, 1)$-summability. The set of sequences $x = (x_k)$ which are strongly almost (V, λ)-summable was defined by Savas [3] such as

$$[V, \lambda] = \left\{ x = (x_k) : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0, \text{for some } L \right\}.$$

Then, Et and Bektas [4] defined the sequence spaces $(C, 1)(\Delta^m)$, $[C, 1](\Delta^m)$, $(V, \lambda)(\Delta^m)$ and $[V, \lambda](\Delta^m)$ and studied their various topological properties by using Orlicz functions. The spaces considered by them are as follows:

$$(C, 1)(\Delta^m) = \left\{ x \in \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (\Delta^m x_k - L) = 0, \text{for some } L \right\},$$

$$[C, 1](\Delta^m) = \left\{ x \in \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |\Delta^m x_k - L| = 0, \text{for some } L \right\},$$

$$(V, \lambda)(\Delta^m) = \left\{ x \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} (\Delta^m x_k - L) = 0, \text{for some } L \right\},$$

$$[V, \lambda](\Delta^m) = \left\{ x \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\Delta^m x_k - L| = 0, \text{for some } L \right\}.$$

Let $A = (a_{ij})$ be an infinite matrix of non negative real numbers with all rows are linearly independent for all $i, j = 1, 2, 3, ...,$ $B_{kn}(x) = \sum_{i=1}^{\infty} a_{ki} x_{n+i}$ if the series converges for each k and n. Now, we define

$$(C,1)(\hat{A},\Delta) = \left\{ x \in \omega : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} (\Delta B_{kn}(x) - L) = 0, \text{for some } L \right\},$$

$$[C,1](\hat{A},\Delta) = \left\{ x \in \omega : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |\Delta B_{kn}(x) - L| = 0, \text{for some } L \right\},$$

$$(V,\lambda)(\hat{A},\Delta) = \left\{ x \in \omega : \lim_{m\to\infty} \frac{1}{\lambda_m} \sum_{k\in I_m} (\Delta B_{kn}(x) - L) = 0, \text{for some } L \right\},$$

$$[V,\lambda](\hat{A},\Delta) = \left\{ x \in \omega : \lim_{m\to\infty} \frac{1}{\lambda_m} \sum_{k\in I_m} |\Delta B_{kn}(x) - L| = 0, \text{for some } L \right\}.$$

where $\Delta B_{kn}(x) = \sum_{i=1}^{\infty} (a_{ki} - a_{k+1,i}) x_{n+i}$. The concept of this infinite matrix has been introduced by Nanda [5] and Solak [6].

2 Main Results

In this section, we discuss some topological properties and inclusion relations of the class of difference sequence spaces $[C,1](\hat{A},\Delta)$, $(C,1)(\hat{A},\Delta)$, $[V,\lambda](\hat{A},\Delta)$ and $(V,\lambda)(\hat{A},\Delta)$.

Theorem 2.1. *The spaces* $(C,1)(\hat{A},\Delta)$,$[C,1](\hat{A},\Delta)$, $(V,\lambda)(\hat{A},\Delta)$ *and* $[V,\lambda](\hat{A},\Delta)$ *are linear over* $\mathbb{C}$.

Proof. We give the proof only for the space$[C,1](\hat{A},\Delta)$ and for other spaces it will follow by applying similar arguments.

Let $x = (x_k)$ and $y = (y_k)$ be any two elements of $[C,1](\hat{A},\Delta)$, then there exist L and L' such that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |\Delta B_{kn}(x) - L| = 0 \quad \text{and} \quad \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |\Delta B_{kn}(y) - L'| = 0.$$

Now, for any scalar $\alpha, \beta \in \mathbb{C}$

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |\Delta B_{kn}(\alpha x + \beta y) - L''| \\
&\quad = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |\Delta B_{kn}(\alpha x + \beta y) - (\alpha L + \beta L')| \quad \text{by setting } L'' = \alpha L + \beta L' \\
&\quad \le |\alpha| \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |\Delta B_{kn}(x) - L| + |\beta| \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |\Delta B_{kn}(y) - L'| \\
&\quad \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

$\Rightarrow \alpha x + \beta y \in [C,1](\hat{A},\Delta)$. This completes the proof. □

Theorem 2.2. *The spaces* $(C,1)(\hat{A},\Delta)$,$[C,1](\hat{A},\Delta)$, $(V,\lambda)(\hat{A},\Delta)$ *and* $[V,\lambda](\hat{A},\Delta)$ *are normed linear spaces normed by*

$$\|x\| = \sum_{i=1}^{n} |x_i| + \sup_{m,n} \left| \frac{1}{m} \sum_{k=1}^{m} \Delta B_{kn}(x) \right|. \tag{2.1}$$

Proof. Suppose $x \in [C,1](\hat{A},\Delta)$, for $x = \theta, \|x\| = 0$. Conversely, if $\|x\| = 0$,

$$\begin{aligned} i.e. \quad & \sum_{i=1}^{n} |x_i| + \sup_{m,n} \left| \frac{1}{m} \sum_{k=1}^{m} \Delta B_{kn}(x) \right| = 0, \\ i.e. \quad & \sum_{i=1}^{n} |x_i| = 0 \ \text{ and } \ \sup_{m,n} \left| \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{\infty} (a_{ki} - a_{k+1,i}) x_{n+i} \right| = 0. \end{aligned}$$

Therefore, $x_1 = x_2 = \cdots = x_n = 0$. Since $A = (a_{nk})$ is a matrix with all linearly independent rows. So, for $i > n,\ x_i = 0$. Hence $x_i = 0$ for i=1,2,3,.... i.e. $x = \theta$.

Let $x, y \in [C,1](\hat{A},\Delta)$, so

$$\|x\| = \sum_{i=1}^{n} |x_i| + \sup_{m,n} \left| \frac{1}{m} \sum_{k=1}^{m} \Delta B_{kn}(x) \right|, \ \|y\| = \sum_{i=1}^{n} |y_i| + \sup_{m,n} \left| \frac{1}{m} \sum_{k=1}^{m} \Delta B_{kn}(y) \right|.$$

Now

$$\begin{aligned} \|x+y\| &= \sum_{i=1}^{n} |x_i + y_i| + \sup_{m,n} \left| \frac{1}{m} \sum_{k=1}^{m} \Delta B_{kn}(x+y) \right| \\ &\le \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| + \sup_{m,n} \left| \frac{1}{m} \sum_{k=1}^{m} \Delta B_{kn}(x) \right| + \sup_{m,n} \left| \frac{1}{m} \sum_{k=1}^{m} \Delta B_{kn}(y) \right| \\ &= \|x\| + \|y\|. \end{aligned}$$

Finally, for any scalar λ,

$$\begin{aligned} \|\lambda x\| &= \sum_{i=1}^{n} |\lambda x_i| + \sup_{m,n} |\frac{1}{m} \sum_{k=1}^{m} \Delta B_{kn}(\lambda x)| \\ &= |\lambda| \sum_{i=1}^{n} |x_i| + |\lambda| \sup_{m,n} \left| \frac{1}{m} \sum_{k=1}^{m} \Delta B_{kn}(x) \right| \\ &= |\lambda| \|x\|. \end{aligned}$$

This completes the proof. The proofs of $(C,1)(\hat{A},\Delta)$, $(V,\lambda)(\hat{A},\Delta)$ and $[V,\lambda](\hat{A},\Delta)$ are similar, hence omitted. □

Theorem 2.3. *The spaces* $(C,1)(\hat{A},\Delta)$ *and* $[C,1](\hat{A},\Delta)$ *are complete normed linear spaces under the norm* $\|x\|$*, defined by (2.1).*

Proof. Let x^N be a Cauchy sequence in $[C,1](\hat{A},\Delta)$, where

$$x^N=(x_k^N)\in[C,1](\hat{A},\Delta) \text{ and } x^M=(x_k^M)\in[C,1](\hat{A},\Delta).$$

Suppose $\|x^N-x^M\|\to 0$ as $M,N\to\infty$

$$\begin{aligned}
&i.e. \quad \sum_{i=1}^{n}|x_i^N-x_i^M|+\sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x^N-x^M)\right|\to 0 \text{ as } M,N\to\infty.\\
&i.e. \quad \sum_{i=1}^{n}|x_i^N-x_i^M|\to 0 \text{ and } \sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x^N-x^M)\right|\to 0 \text{ as } M,N\to\infty.
\end{aligned}$$

Thus, for k=1,2,3,...,n, (x_k^N) is a Cauchy sequence in $\mathbb{C}$ and

$$\begin{aligned}
&\sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x^N)-\Delta B_{kn}(x^M)\right|\to 0, \quad \text{as } M,N\to\infty\\
&\Rightarrow \sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{\infty}(a_{ki}-a_{k+1,i})(x_{n+i}^N-x_{n+i}^M)\right|\to 0, \text{ as } M,N\to\infty.\\
&\Rightarrow \frac{1}{m}\left|(a_{11}-a_{m+1,1})(x_{n+1}^N-x_{n+1}^M)+(a_{12}-a_{m+1,2})(x_{n+2}^N-x_{n+2}^M)+\cdots\right|\\
&\quad \to 0, \quad \text{as } M,N\to\infty.
\end{aligned}$$

As all the rows of A are linearly independent, so for $k=n+1,n+2,\ldots$, (x_k^N) is a Cauchy sequence in $\mathbb{C}$. Therefore (x_k^N) is a Cauchy sequence in $\mathbb{C}$ for all k. By the completeness of $\mathbb{C}$, $\lim_{N\to\infty}x_k^N=x_k$ *in* $\mathbb{C}$, for all k. For given $\epsilon>0$, there exists $N_0>M,N$ such that,

$$\lim_{M\to\infty}\sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x^N-x^M)\right|=\sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x^N-x)\right|<\epsilon. \qquad (2.2)$$

Now,

$$\begin{aligned}
\lim_{M\to\infty}\|x^N-x^M\| &= \lim_{M\to\infty}\sum_{i=1}^{n}|x_i^N-x_i^M|+\lim_{M\to\infty}\sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x^N)-\Delta B_{kn}(x^M)\right|\\
&\le \sum_{i=1}^{n}|x_i^N-x_i|+\sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x^N-x)\right|\\
&\le 2\epsilon.
\end{aligned}$$

Hence, $x^N \to x$, as $M \to \infty$. Now,

$$\begin{aligned}
&\sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x)-L\right| \\
&\qquad\qquad \le \sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x^N)-L\right|+\sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x^N-x)\right| \\
&\qquad\qquad\quad \to 0, \quad \text{as } N\to\infty.
\end{aligned}$$

Therefore, $x \in [C,1](\hat{A},\Delta)$. This completes the proof. □

Theorem 2.4.

(i) $[C,1](\hat{A},\Delta) \subset (C,1)(\hat{A},\Delta)$*;*

(ii) $[V,\lambda](\hat{A},\Delta) \subset (V,\lambda)(\hat{A},\Delta)$ *and the inclusions are strict.*

Proof. The proof of this theorem is trivial, so omitted. □

Theorem 2.5. *The spaces* $(C,1)(\hat{A},\Delta)$ *and* $[C,1](\hat{A},\Delta)$ *are isometrically isomorphic to* ℓ_∞.

Proof. We give the proof of the space $[C,1](\hat{A},\Delta)$ only and for the other spaces it will follow the similar technique. Let us consider the mapping

$$T : [C,1](\hat{A},\Delta) \to [C,1](\hat{A},\Delta)$$

defined by $Tx = y = (0,0,\dots,0,x_{n+1},x_{n+2},\dots)$. It is clear that T is bounded linear operator. Also $T([C,1](\hat{A},\Delta)) = \{x = (x_k) : x_1 = x_2 = \cdots = x_n = 0, x \in [C,1](\hat{A},\Delta)\}$ is a subset of $[C,1](\hat{A},\Delta)$ and $\|x\| = \|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x)\|_\infty$ in $T([C,1](\hat{A},\Delta))$.

On the other hand we can show that the mapping $\triangle_T : T([C,1](\hat{A},\Delta)) \to \ell_\infty$, defined by $\triangle_T(x) = (y_k) = (\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x))$ is a linear homomorphism.

Now,

$$\|\triangle_T(x)\| = \sup_{m,n}\left|\frac{1}{m}\sum_{k=1}^{m}\Delta B_{kn}(x)\right| = \|x\|.$$

Therefore, $\triangle_T$ is linear and bijective. Hence $(C,1)(\hat{A},\Delta)$ is isometrically isomorphic to ℓ_∞. □

Theorem 2.6. *The spaces* $(C,1)(\hat{A},\Delta)$ *and* $[C,1](\hat{A},\Delta)$ *are BK-spaces with the norm defined in (2.1).*

Proof. The proof follows immediately from Theorem 2.5. □

3 λ_B-Statistical Convergence

In this section, we establish the relationship of $S_\lambda(\hat{A},\Delta)$ with $[V,\lambda](\hat{A},\Delta)$ and $S(\hat{A},\Delta)$. The notion of statistical convergence was introduced by Fast [7] and studied by various authors [8–13].

Definition 3.1 ([13])**.** A sequence $x=(x_k)$ is said to be λ-*statistical convergent* or S_λ-*convergent* to L, if for every $\epsilon>0$

$$\lim_{m\to\infty}\frac{1}{\lambda_m}|\{k\in I_m : |x_k-L|\geq\epsilon\}|=0.$$

In this case, we write $S_\lambda-\lim x=L$ or $x_k\to L(S_\lambda)$ and

$$S_\lambda=\{x\in\omega : S_\lambda-\lim x=L, \text{for some } L\}.$$

Definition 3.2. A sequence $x=(x_k)$ is said to be λ_B-*statistical convergent* or $S_\lambda(\hat{A},\Delta)$-*convergent* to L, if for every $\epsilon>0$

$$\lim_{m\to\infty}\frac{1}{\lambda_m}|\{k\in I_m : |\Delta B_{kn}(x)-L|\geq\epsilon\}|=0.$$

In this case, we write $S_{\lambda_B}-\lim x=L$ or $x_k\to LS_\lambda(\hat{A},\Delta)$ and

$$S_\lambda(\hat{A},\Delta)=\{x\in\omega : S_{\lambda_B}-\lim x=L, \text{for some } L\}.$$

Theorem 3.1. *Let $\lambda=(\lambda_k)$ be same as above, then*

(i) If $x_k\to L[V,\lambda](\hat{A},\Delta)$, then $x_k\to LS_\lambda(\hat{A},\Delta)$.

(ii) If $x\in\ell_\infty(\hat{A},\Delta)$ and $x_k\to LS_\lambda(\hat{A},\Delta)$, then $x_k\to L[V,\lambda](\hat{A},\Delta)$.

(iii) $S_\lambda(\hat{A},\Delta)\cap\ell_\infty(\hat{A},\Delta)=[V,\lambda](\hat{A},\Delta)\cap\ell_\infty(\hat{A},\Delta)$ where $\ell_\infty(\hat{A},\Delta)=\{x:\sup_{m,n}|\Delta B_{mn}(x)|<\infty\}$.

Proof. (i) Suppose $\epsilon>0$ and $\lim_k x_k=L[V,\lambda](\hat{A},\Delta)$, then we have

$$\begin{aligned}\sum_{k\in I_m}|\Delta B_{kn}(x)-L| &\geq \sum_{\substack{k\in I_m,\\ |\Delta B_{kn}(x)-L|\geq\epsilon}}|\Delta B_{kn}(x)-L|\\ &\geq\epsilon|\{k\in I_m : |\Delta B_{kn}(x)-L|\geq\epsilon\}|.\end{aligned}$$

Therefore, $x_k\to L[V,\lambda](\hat{A},\Delta), \ \Rightarrow\ x_k\to LS_\lambda(\hat{A},\Delta)$.

(ii) Suppose $x\in\ell_\infty(\hat{A},\Delta)$ and $x_k\to LS_\lambda(\hat{A},\Delta)$, i.e. for some K, $|\Delta B_{kn}(x)-L|\leq K$ for all k and n. Given $\epsilon>0$, we get

$$\begin{aligned}&\frac{1}{\lambda_m}\sum_{k\in I_m}|\Delta B_{kn}(x)-L|\\ &\qquad=\frac{1}{\lambda_m}\sum_{\substack{k\in I_m,\\ |\Delta B_{kn}(x)-L|\geq\epsilon}}|\Delta B_{kn}(x)-L|+\frac{1}{\lambda_m}\sum_{\substack{k\in I_m,\\ |\Delta B_{kn}(x)-L|<\epsilon}}|\Delta B_{kn}(x)-L|\\ &\qquad\leq\frac{K}{\lambda_m}|\{k\in I_m : |\Delta B_{kn}(x)-L|\geq\epsilon\}|+\epsilon.\end{aligned}$$

As $m \to \infty$, the right hand side goes to zero, which implies that $x_k \to L[V,\lambda](\hat{A},\Delta)$.

(iii) This immediately follows from (i) and (ii). $\square$

Theorem 3.2. *If* $\liminf_m \frac{\lambda_m}{m} > 0$, *then* $S(\hat{A},\Delta) \subset S_\lambda(\hat{A},\Delta)$.

Proof. Given $\epsilon > 0$, we have

$$\{k \in I_m : |\Delta B_{kn}(x) - L| \geq \epsilon\} \subset \{k \leq m : |\Delta B_{kn}(x) - L| \geq \epsilon\}.$$

Therefore,

$$\begin{aligned}\frac{1}{m}|\{k \leq m : |\Delta B_{kn}(x) - L| \geq \epsilon\}| &\geq \frac{1}{m}|\{k \in I_m : |\Delta B_{kn}(x) - L| \geq \epsilon\}| \\ &= \frac{\lambda_m}{m}.\frac{1}{\lambda_m}|\{k \in I_m : |\Delta B_{kn}(x) - L| \geq \epsilon\}|.\end{aligned}$$

Taking the limit as $m \to \infty$ we get $x_k \to LS(\hat{A},\Delta) \;\Rightarrow\; x_k \to LS_\lambda(\hat{A},\Delta)$. $\square$

4 Some New Sequence Spaces Defined by Orlicz Functions

In this part we introduce some new difference sequence spaces defined by Orlicz functions and examine their topological properties. Before giving certain results we give some required definitions.

Definition 4.1. A function $M : [0,\infty) \to [0,\infty)$ is said to be an *Orlicz function* if it is

(i) Continuous,

(ii) Non decreasing and

(iii) Convex with $M(0) = 0,\ M(x) > 0$ for $x > 0$ and $M(x) \to 0$ as $x \to \infty$.

Definition 4.2. Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. Now using this function we define

$$[V,\lambda](\hat{A},\Delta,M,p) = \left\{ \begin{array}{r} (x_k) : \lim\limits_{m\to\infty} \dfrac{1}{\lambda_m} \sum\limits_{k\in I_m} \left[M\left(\dfrac{|\Delta B_{kn}(x) - L|}{\rho} \right) \right]^{p_k} = 0, \\ \text{for some } L \text{ and } \rho > 0 \end{array} \right\},$$

$$[V,\lambda]_0(\hat{A},\Delta,M,p) = \left\{ \begin{array}{r} (x_k) : \lim\limits_{m\to\infty} \dfrac{1}{\lambda_m} \sum\limits_{k\in I_m} \left[M\left(\dfrac{|\Delta B_{kn}(x)|}{\rho} \right) \right]^{p_k} = 0, \\ \text{for some } \ \rho > 0 \end{array} \right\},$$

$$[V,\lambda]_\infty(\hat{A},\Delta,M,p) = \left\{ \begin{array}{r} (x_k) : \lim\limits_{m\to\infty} \dfrac{1}{\lambda_m} \sum\limits_{k\in I_m} \left[M\left(\dfrac{|\Delta B_{kn}(x)|}{\rho} \right) \right]^{p_k} < \infty, \\ \text{for some } \ \rho > 0 \end{array} \right\}.$$

For $p=(p_k)=(1,1,\dots)$, we denote $[V,\lambda](\hat{A},\Delta,M,p)$, $[V,\lambda]_0(\hat{A},\Delta,M,p)$ and $[V,\lambda]_\infty(\hat{A},\Delta,M,p)$ as $[V,\lambda](\hat{A},\Delta,M)$, $[V,\lambda]_0(\hat{A},\Delta,M)$ and $[V,\lambda]_\infty(\hat{A},\Delta,M)$ respectively.

Theorem 4.1. *For a bounded sequence $p=(p_k)$ of strictly positive real numbers, the the space $[V,\lambda](\hat{A},\Delta,M,p)$, $[V,\lambda]_0(\hat{A},\Delta,M,p)$ and $[V,\lambda]_\infty(\hat{A},\Delta,M,p)$ are linear over $\mathbb{C}$, the field of complex numbers.*

Proof. Let $x,y\in[V,\lambda]_0(\hat{A},\Delta,M,p)$ and $\alpha,\beta\in\mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{m\to\infty}\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1}\right)\right]^{p_k}=0$$

and

$$\lim_{m\to\infty}\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(y)|}{\rho_2}\right)\right]^{p_k}=0.$$

Define ρ_3 =max$\{2|\alpha|\rho_1,2|\alpha|\rho_2\}$. Since ΔB_{kn} is linear and M is non-decreasing and convex, we have

$$\begin{aligned}
&\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(\alpha x+\beta y)|}{\rho_3}\right)\right]^{p_k}\\
&\quad=\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\alpha\Delta B_{kn}(x)+\beta\Delta B_{kn}(y)|}{\rho_3}\right)\right]^{p_k}\\
&\quad\le\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\alpha\Delta B_{kn}(x)|}{\rho_3}+\frac{|\beta\Delta B_{kn}(y)|}{\rho_3}\right)\right]^{p_k}\\
&\quad\le\frac{1}{\lambda_m}\sum_{k\in I_m}\frac{1}{2^{p_k}}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1}\right)+M\left(\frac{|\Delta B_{kn}(y)|}{\rho_2}\right)\right]^{p_k}\\
&\quad\le\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1}\right)+M\left(\frac{|\Delta B_{kn}(y)|}{\rho_2}\right)\right]^{p_k}\\
&\quad\le C\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1}\right)\right]^{p_k}+C\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(y)|}{\rho_2}\right)\right]^{p_k}\to 0
\end{aligned}$$

where C =max$\{1,2^{N-1}\}$, $N=\max\{1,\sup_{k\in\mathbb{N}}p_k\}$; so that $\alpha x+\beta y\in[V,\lambda]_0(\hat{A},\Delta,M,p)$. Hence $[V,\lambda]_0(\hat{A},\Delta,M,p)$ is a linear space. The proof for the the spaces $[V,\lambda](\hat{A},\Delta,M,p)$ and $[V,\lambda]_\infty(\hat{A},\Delta,M,p)$ can be done in a similar way. $\square$

Theorem 4.2. *For any Orlicz function M and a bounded sequence $p=(p_k)$ of strictly positive real numbers the space $[V,\lambda]_0(\hat{A},\Delta,M,p)$ is a paranormed space* *with the paranorm*

$$g(x)=\inf\left\{\rho^{p_m/N}:\left\{\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\right\}^{1/N}\leq 1, m=1,2,3,\ldots\right\}. \tag{4.1}$$

Proof. The subadditivity of g follows from the Theorem 4.1 by taking $\alpha=1$ and $\beta=1$ and it is clear that $g(x)=g(-x)$. Since $M(0)=0$, we get $\inf\{\rho^{p_m/N}\}=0$, for $x=0$. Now we prove that scalar multiplication is continuous. Let α be any complex number, by definition

$$\begin{aligned}
g(\alpha x)&=\inf\left\{\rho^{p_m/N}:\left\{\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(\alpha x)|}{\rho}\right)\right]^{p_k}\right\}^{1/N}\leq 1, m=1,2,3,\ldots\right\}\\
&=\inf\left\{\rho^{p_m/N}:\left\{\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\alpha\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\right\}^{1/N}\leq 1, m=1,2,3,\ldots\right\}.
\end{aligned}$$

Suppose $s=\rho/|\alpha|$, then $\rho=s|\alpha|$ and since $|\alpha|^{p_m}\leq max\{1,|\alpha|^{\sup p_m}\}$ then we get

$$\begin{aligned}
&g(\alpha x)\\
&=\inf\left\{(s|\alpha|)^{p_m/N}:\left\{\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\alpha\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\right\}^{1/N}\leq 1, m=1,2,3,\ldots\right\}\\
&\leq(\max\{1,|\alpha|^{\sup p_m}\})^{1/N}\\
&\qquad\times\inf\left\{\rho^{p_m/N}:\left\{\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(\alpha x)|}{\rho}\right)\right]^{p_k}\right\}^{1/N}\leq 1, m=1,2,3,\ldots\right\}\\
&\qquad\quad\to 0 \text{ as } \; g(x)\to 0 \; \text{ in } \; [V,\lambda]_0(\hat{A},\Delta,M,p).
\end{aligned}$$

Now suppose that $\alpha_i\to 0$ as $i\to\infty$ and x is fixed in $[V,\lambda]_0(\hat{A},\Delta,M,p)$. For arbitrary $\epsilon>0$ and let N_0 be a positive integer such that

$$\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\leq\left(\frac{\epsilon}{2}\right)^N.$$

For some $\rho>0$ and all $m>N_0$, we have

$$\left\{\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\right\}^{1/N}\leq\frac{\epsilon}{2}.$$

Let $0<|\alpha|<1$, using the convexity of M for all $m>N_0$, we get

$$\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\alpha\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\leq\sum_{k\in I_m}|\alpha|\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\leq\left(\frac{\epsilon}{2}\right)^N.$$

Since M is continuous everywhere in $[0,\infty)$, then we consider the function

$$f(r)=\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|r.\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}.$$

For each $m\leq N_0$, f is continuous at 0. So there is a $\delta\in(0,1)$ such that $|f(r)|<\left(\frac{\epsilon}{2}\right)^N$ for $0<r<\delta$, this implies that there exists K such that $|\alpha_i|<\delta$ for $i>K$ and $m\leq N_0$

$$\left\{\frac{1}{\lambda_m}\sum_{k\in I_m}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\right\}^{1/N}\leq\frac{\epsilon}{2}.$$

This completes the proof. □

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