Thai Journal of Mathematics
Volume 11 (2013) Number 1 : 67–73
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Fixed Point Theorems in Generalized Types of Dislocated Metric Spaces and Its Applications

M.A. Ahmed, F.M. Zeyada and G.F. Hassan[1]

Department of Mathematics, Faculty of Science
Assiut University, Assiut 71516, Egypt
e-mail : mahmed68@yahoo.com (M.A. Ahmed)
zeyada1@hotmail.com (F.M. Zeyada)
gamal6@yahoo.com (G.F. Hassan)

Abstract : Two types of generalized dislocated metric spaces are introduced. Also, we prove some fixed point theorems in these spaces. These theorems are different generalizations of a fixed point theorem of Hitzler and Seda [P. Hitzler, A.K. Seda, Dislocated topologies, J. Electr. Engin. 51 (2000) 3–7]. Finally, we give applications on our results.

Keywords : d-metric; ld-metric; rd-metric; ld-continuous; rd-continuous; contraction.
2010 Mathematics Subject Classification : 47H10, 54H25.

1 Introduction

Fixed point theory has application in logic programming semantics, approximation theory, dynamic programming and integral-functional equations (see, e.g., [1–4]). Recently, some generalized types of metric spaces such as dislocated metric spaces and hyperbolic ordered metric spaces have appeared (see, for instance, [1, 5–7]). In 2000, Hitzler and Seda [6] proved a fixed point theorem in complete

[1]Current address: Department of Mathematics, Teachers College, P.O. Box 1457, Arar, Kingdom of Saudi Arabia.

dislocated metric spaces. Their theorem generalizes Banach contraction principle to dislocated metric spaces.

Following Waszkiewick [7], let X be a set. A *distance* on X is a map $d : X \times X \to [0, \infty)$. A pair (X, d) is called a *distance space*. If d satisfies the following conditions:

(DM_1) if $d(x, y) = d(y, x) = 0$, then $x = y$;

(DM_2) $d(x, y) = d(y, x)$;

(DM_3) $d(x, y) \leq d(x, z) + d(z, y)$

for all $x, y, z \in X$, then it is called *dislocated metric* (or simply d-metric) on X. It is obvious that if d satisfies $(DM_1) - (DM_3)$ and

(DM_4) $d(x, x) = 0 \ \forall x \in X$,

then d is a *metric* on X.

Definition 1.1 ([8])**.** Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is called *contraction* iff there exists $0 \leq \lambda < 1$ such that $d_2(f(x), f(y)) \leq \lambda d_1(x, y)$ for all $x, y \in X$.

Theorem 1.2 ([6, Theorem 2.2])**.** *Let (X, d) be a complete d-metric space and a function $f : X \to X$ be contraction. Then f has a unique fixed point.*

The plan of this paper is as follows. In Section 2, we introduce the concept of left dislocated metric spaces. Also, we establish a generalization of Theorem 1.2 to these spaces. In Section 3, we give the concept of right dislocated metric spaces and state another generalization of Theorem 1.2 to these spaces. Finally, in Section 4, we present some applications of our main theorems.

2 A Fixed Point Theorem in Left Dislocated Metric Spaces

In this section, we introduce the concept of left dislocated metric spaces (simply ld-metric spaces). Also, we state and prove a generalization of Theorem 1.2 to ld-metric spaces.

Definition 2.1. A distance function d is called *ld-metric* iff it satisfies (DM_1) and the condition

(LD) $d(x, y) \leq d(z, x) + d(z, y) \qquad \forall x, y, z \in X.$

It is obvious that any d-metric is a ld-metric but the converse may not be true.

Definition 2.2. A sequence $(x_n) \subseteq X$ is *ld-convergent* iff there exists a point $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. In this case, x is said to be *ld-limit* of (x_n).

Definition 2.3. A sequence (x_n) in a ld-metric space (X, d) is called *Cauchy* iff $\forall\ \epsilon > 0, \exists\ n_0 \in N$ ($N :=$ the set of all positive integers) such that $d(x_m, x_n) < \epsilon\ \forall m, n \geq n_0$.

Definition 2.4. A ld-metric space (X, d) is called *complete* iff every Cauchy sequence in it is ld-convergent in X.

Definition 2.5. Let (X, d_1) and (Y, d_2) be ld-metric spaces. A function $f : X \to Y$ is said to be *ld-continuous* iff for each sequence (x_n) which is ld$_1$-convergent to x_0 in X, the sequence $(f(x_n))$ is ld$_2$-convergent to $f(x_0) \in Y$.

We state the following lemma without proof.

Lemma 2.6. *Every subsequence of ld-convergent sequence to x_0 is ld-convergent to x_0.*

It is obvious that the converse of Lemma 2.6 may not be true.

Lemma 2.7. *ld-limits in ld-metric spaces are unique.*

Proof. Let x and y be ld-limits of the sequence (x_n). By Property (LD), it follows that

$$d(x, y) \leq d(x_n, x) + d(x_n, y) \to 0 \ \text{as}\, n \to \infty.$$

Hence, $d(x, y) = 0$. In a similar way, one can deduce that $d(y, x) = 0$. So, we obtain from (DM$_1$) that $x = y$. □

Lemma 2.8. *Let (X, d) be an ld-metric space. If a function $f : X \to X$ is contraction, then $(f^n(x_0))$ is a Cauchy sequence for each $x_0 \in X$.*

Proof. Choose any $x_0 \in X$. We show that the sequence $(f^n(x_0))$ is Cauchy. Note that

$$\begin{aligned} d(f^{n+1}(x_0), f^n(x_0)) &\leq \lambda d(f^n(x_0), f^{n-1}(x_0)) \\ &\leq \lambda^2 d(f^{n-1}(x_0), f^{n-2}(x_0)) \\ &\leq \cdots \leq \lambda^n d(f(x_0), x_0). \end{aligned}$$

Now, for any integer $r \in N \cup \{0\}$, Property (LD) implies that

$$\begin{aligned} d(f^n(x_0), f^{n+r}(x_0)) &\leq d(f^{n+1}(x_0), f^n(x_0)) + d(f^{n+1}(x_0), f^{n+r}(x_0)) \\ &\leq d(f^{n+1}(x_0), f^n(x_0)) + d(f^{n+2}(x_0), f^{n+1}(x_0)) \\ &\quad\ + d(f^{n+2}(x_0), f^{n+r}(x_0)) \\ &\leq d(f^{n+1}(x_0), f^n(x_0)) + d(f^{n+2}(x_0), f^{n+1}(x_0)) \\ &\quad\ + \cdots + d(f^{n+r}(x_0), f^{n+r}(x_0)) \\ &\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+r-1}) d(f(x_0), x_0) + \lambda^{n+r} d(x_0, x_0) \\ &\leq \frac{\lambda^n}{1-\lambda} d(f(x_0), x_0) + \lambda^{n+r} d(x_0, x_0). \end{aligned}$$

The last term tends to zero as n tends to infinity. Also, we proceed similarly as above and obtain

$$d(f^{n+r}(x_0), f^n(x_0)) \leq \frac{\lambda^n}{1-\lambda} d(x_0, f(x_0)) + \lambda^{n+r} d(x_0, x_0).$$

The last term tends to zero as n tends to infinity. Thus $(f^n(x_0))$ is a Cauchy sequence. □

Lemma 2.9. *Let (X, d) be an ld-metric space. If a function $f : X \to X$ is contraction, then f is ld-continuous.*

Proof. Let $x_0 \in X$ be an arbitrary point in an ld- metric space (X, d) and let $\epsilon > 0$. Put $\delta \leq \epsilon$. Then,

$$d(x, x_0) < \delta \Longrightarrow d(f(x), f(x_0)) \leq \lambda d(x, x_0) < \lambda\delta \leq \lambda\epsilon < \epsilon.$$

Hence, f is ld-continuous at x_0. Therefore, f is ld-continuous. □

Theorem 2.10. *Let (X, d) be a complete ld-metric space. If a function $f : X \to X$ is contraction, then f has a unique fixed point.*

Proof. Existence: from Lemma 2.8, the sequence $(f^n(x_0))$ is Cauchy for each $x_0 \in X$. Since (X, d) is ld-complete, then $(f^n(x_0))$ is ld-convergent to $x \in X$, say. Using Lemma 2.6 and Lemma 2.9, we get that

$$f(x) = f(\lim_{n\to\infty} f^n(x_0)) = \lim_{n\to\infty} f^{n+1}(x_0) = x.$$

Uniqueness: suppose that there are two different fixed points x and y of f. Since f is contraction, then we find that

$$d(x, y) = d(f(x), f(y)) \leq \lambda d(x, y),$$

i.e., $(1 - \lambda)d(x, y) \leq 0$. Also, we have that

$$d(y, x) = d(f(y), f(x)) \leq \lambda d(y, x),$$

i.e., $(1 - \lambda)d(y, x) \leq 0$. Since $1 - \lambda < 1$, we obtain that $d(x, y) = d(y, x) = 0$. Hence, (DM_1) gives that $x = y$. □

In Theorem 2.10, if the mapping f is replaced by f_α, $\alpha \in \Lambda$, where Λ is an index set, we state the following theorem without proof.

Theorem 2.11. *Let (X, d) be a complete ld-metric space and a function $f_\alpha : X \to X$ be contraction for all $\alpha \in \Lambda$. Then, f_α has a unique fixed point for each $\alpha \in \Lambda$.*

3 A Fixed Point Theorem in Right Dislocated Metric Spaces

In this section, the concept of right dislocated metric spaces (simply rd-metric spaces) is given. Also, we establish another generalization of Theorem 1.2 to rd-metric spaces.

Definition 3.1. A distance function d is said to be *rd-metric* iff it satisfies (DM_1) and the condition

(RD) $d(x, y) \leq d(x, z) + d(y, z) \qquad \forall x, y, z \in X.$

It is clear that any d-metric is a rd-metric but the converse may not be true.

Definition 3.2. A sequence $(x_n) \subseteq X$ *rd-converges* to $x \in X$ iff $\lim_{n\to\infty} d(x, x_n) = 0$. In this case, x is called *rd-limit* of (x_n).

Definition 3.3. A sequence (x_n) in an rd-metric space (X, d) is called *Cauchy* iff $\forall\, \epsilon > 0, \exists\, n_0 \in N$ such that $d(x_m, x_n) < \epsilon \ \forall m, n \geq n_0$.

Definition 3.4. An rd-metric space (X, d) is called *complete* iff every Cauchy sequence in it is rd-convergent in X.

Definition 3.5. Let (X, d_1) and (Y, d_2) be rd-metric spaces. A function $f : X \to Y$ is said to be *rd-continuous* iff for each sequence (x_n) which is rd_1-convergent to $x_0 \in X$, the sequence $(f(x_n))$ is rd_2-convergent to $f(x_0) \in Y$.

We state the following lemma without proof.

Lemma 3.6. *Every subsequence of rd-convergent sequence to x_0 is rd-convergent to x_0.*

It is obvious that the converse of Lemma 3.6 may not be true.
Similarly, one can prove the following lemmas and theorems.

Lemma 3.7. *rd-limits in rd-metric spaces are unique.*

Lemma 3.8. *Let (X, d) be an rd-metric space. If a function $f : X \to X$ is contraction, then $(f^n(x_0))$ is a Cauchy sequence for each $x_0 \in X$.*

Lemma 3.9. *Let (X, d) be an rd-metric space. If a function $f : X \to X$ is contraction, then f is rd-continuous.*

Theorem 3.10. *Let (X, d) be a complete rd-metric space and a function $f : X \to X$ be contraction. Then f has a unique fixed point.*

In Theorem 3.10, if the mapping f is replaced by f_α, $\alpha \in \Lambda$, where Λ is an index set, we obtain the following theorem.

Theorem 3.11. *Let (X, d) be a complete rd-metric space and a function $f_\alpha : X \to X$ be contraction for all $\alpha \in \Lambda$. Then f_α has a unique fixed point for each $\alpha \in \Lambda$.*

4 Some Applications

In this section, we give some applications of our main results. First we introduce an application of Theorem 2.10 to f^r where $r \in N$ as follows.

Theorem 4.1. *Let (X, d) be a complete ld-metric space. If a function $f : X \to X$ satisfies that f^r is contraction for some $r \in N$, then f has a unique fixed point.*

Proof. Fix r such that f^r is contraction. Then we obtain from Theorem 2.10 that f^r has a unique fixed point $x \in X$, say. Then, $f(x) = f(f^r(x)) = f^r(f(x))$ implies that $f(x)$ is a fixed point of f^r. By uniqueness of the fixed point for f^r, we have that $x = f(x)$. To prove the uniqueness of a fixed point of f, assume that y is another fixed point of f. Then,

$$d(x, y) = d(f^r(x), f^r(y)) \leq \lambda d(x, y) \text{ and } d(y, x) = d(f^r(y), f^r(x)) \leq \lambda d(y, x).$$

So, $(1 - \lambda)d(x, y) \leq 0$ and $(1 - \lambda)d(y, x) \leq 0$. Since $1 - \lambda < 1$, we obtain that $d(x, y) = d(y, x) = 0$. Hence, (DM_1) leads to $x = y$. □

In Theorem 4.1, if the mapping f is replaced by f_α, $\alpha \in \Lambda$, where Λ is an index set, we state an application of Theorem 2.11 to f^r where $r \in N$ without proof.

Theorem 4.2. *Let (X, d) be a complete ld-metric space. If a function $f_\alpha : X \to X$ satisfies that f_α^r is a contraction for some $r \in N$ and for all $\alpha \in \Lambda$. then f_α has a unique fixed point for each $\alpha \in \Lambda$.*

Remark 4.3.

(I) If we replace a complete ld-metric space by a complete rd-metric space, then the conclusion of Theorems 4.1 and 4.2 remains valid. In this case, the deduced results are applications of Theorems 3.10 and 3.11 to f^r where $r \in N$, respectively.

(II) Since d-metric is ld-metric and rd-metric, then the conclusion of Theorems 4.1 and 4.2 holds in d-metric spaces.

Acknowledgements : The authors would like to thank Professor P. Hitzler for his valuable comments and numerous discussions on this manuscript. Also, the authors wish to thank the referees for their remarks on the manuscript.

References

[1] P. Hitzler, Generalized metrics and topology in logic programming semantics, Ph. D. Thesis, National University of Ireland (University College, Cork), 2001.

[2] N. Hussain, A.R. Khan, common fixed-point results in best approximation theory, Appl. Math. Lett. 16 (2003) 575–580.

[3] H.K. Pathak, B. Fisher, Common fixed point theorems with applications in dynamic programming, Glasnik Math. 31 (1996) 321–328.

[4] M. Zima, A certain fixed point theorem and its applications to integral-functional equations, Bull. Austral. Math. Soc. 46 (1992) 179–186.

[5] M. Abbas, M.A. Khamsi, A.R. Khan, Common fixed point and invariant approximation in hyperbolic ordered metric spaces, Fixed Point Theory and Applications, 2011, 2011:25.

[6] P. Hitzler, A.K. Seda, Dislocated topologies, J. Electr. Engin. 51 (2000) 3–7.

[7] P. Waszkiewicz, The local triangle axioms in topology and domain theory, Applied General Topology 4 (1) (2003) 47–70.

[8] E. Zeidler, Nonlinear functional analysis and its applications, I: Fixed point theorems, Springer-Verlag, New york Inc., 1986.

(Received 11 August 2011)
(Accepted 13 February 2012)