Thai Journal of Mathematics
Volume 11 (2013) Number 1 : 59–66

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Matrix Mappings on Multiplier Sequence Spaces

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Abstract : In this article we focus on the multiplier sequence spaces which are Banach spaces. We show that the characterization of a random matrix operator $A = (a_{nk}) \in (E(\lambda), F(\mu))$, where $E(\lambda)$ and $F(\mu)$ are multiplier sequence spaces with multiplier sequences λ and μ, depends on the characterization of the matrix $B = (b_{nk}) \in (E, F)$, with $b_{nk} = \mu_n a_{nk} \lambda_k^{-1}$. By this way, the necessary and sufficient conditions for the matrix operators between multipliers of the classical sequence spaces can be found. We also give some applications of these results.

Keywords : multiplier sequence spaces; BK spaces; matrix mappings.
2010 Mathematics Subject Classification : 46A45; 46B45; 40C05.

1 Introduction and Preliminaries

Let ω denote the set of all complex sequences. Any subspace of ω is called as a sequence space. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. By ℓ_p, we denote the space of all p-absolutely summable sequences, where $1 \leq p < \infty$.

The studies on sequence spaces was extended by using the notion of associated multiplier sequences. Goes et al. [1] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E, using the multiplier sequences (k^{-1}) and (k) respectively. Kamthan [2] used the multiplier sequence $(k!)$. Recently, Tripathy and Sen [3] examined some vector valued paranormed multiplier sequence spaces.

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Let $\lambda = (\lambda_k)_{k\in\mathbb{N}}$ be a sequence of nonzero scalars, where $\mathbb{N} = \{1, 2, \ldots\}$. For any $z = (z_k) \in \omega$ let $\lambda z = (\lambda_k z_k)$. Then, for a sequence space E, the multiplier sequence space $E(\lambda)$, associated with the multiplier sequence λ is defined as

$$E(\lambda) = \{z \in \omega : \lambda z \in E\}.$$

For example $c_0((1/k)_{k=1}^{\infty})$ is a sequence space which includes ℓ_∞ and includes some unbounded sequences such as $(\sqrt{k})$.

Let E and F be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk} , where $n,\ k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from E into F, and we denote it by writing $A : E \to F$, if for every sequence $x = (x_k) \in E$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in F; where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

By (E, F), we denote the class of all matrices A such that $A : E \to F$. Thus, $A \in (E, F)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in E$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in F$ for all $x \in E$.

Let X and Y be Banach spaces. Then $B(X, Y)$ is the set of all continuous linear operators $L : X \to Y$, a Banach space with the operator norm defined by $\|L\| = \sup\{\frac{\|L(x)\|}{\|x\|} : 0 \neq x \in X\} \quad (L \in B(X, Y))$.

A BK space is a Banach sequence space with continuous coordinates. The sequence spaces $c_0, c, \ell_p, \ell_\infty$ are the well-known examples of BK spaces. A BK space with uniformly continuous coordinates will be called as a UBK space. It can be seen that the sequence spaces $c_0, c, \ell_p, \ell_\infty$ are also UBK spaces.

Theorem 1.1 ([4, Theorem 4.2.8])**.** *Matrix operators between BK spaces are continuous.*

Theorem 1.2. *If E is a normed sequence space with the norm $\|\cdot\|$, then $E(\lambda)$ is a normed sequence space with norm $\|z\|_\lambda = \|\lambda z\|$.*

Proof. Let $\alpha \in \mathbb{C}$ and $z \in E(\lambda)$. Then

$$\|\alpha z\|_\lambda = \|\lambda(\alpha z)\| = \|\alpha\lambda z\| = |\alpha|\|\lambda z\| = |\alpha|\|z\|_\lambda.$$

Secondly, for $y, z \in E(\lambda)$ we have

$$\|y + z\|_\lambda = \|\lambda(y + z)\| = \|\lambda y + \lambda z\| \leq \|\lambda y\| + \|\lambda z\| = \|y\|_\lambda + \|z\|_\lambda,$$

so the triangle inequality holds.

Now, suppose $\|z\|_\lambda = 0$. Then $\|\lambda z\| = 0$ and since $\|\cdot\|$ is a norm we have $\lambda_k z_k = 0$ for each k. Since λ_k are nonzero, we have $z = \theta$. □

2 Main Results

Lemma 2.1. *Let E be a normed sequence space with norm $\|\cdot\|$. Then E is a UBK space if and only if there exists $C > 0$ such that for all $z = (z_k) \in E$, $|z_k| \leq C\|z\|$ for all $k \in \mathbb{N}$.*

Proof. Let E be a UBK space and suppose such a $C > 0$ does not exist. Then, there exists a sequence $(z^{(s)})$ in E with $\|z^{(s)}\| = 1$ such that for an index set (k_s) we have $|z^{(s)}_{k_s}| > s^2$ for $s \in \mathbb{N}$. Now let $y^{(s)} = \frac{1}{s}z^{(s)}$ for $s \in \mathbb{N}$. Then $y^{(s)} \to \theta$ as $s \to \infty$, but

$$\lim_{s\to\infty} |y^{(s)}_{k_s}| = \infty.$$

So, we get to the contradiction, coordinates are not uniformly continuous. The inverse implication is straightforward. □

Theorem 2.2. *If $(E, \|\cdot\|)$ is a Banach sequence space and λ be a sequence of nonzero terms, then $(E(\lambda), \|\cdot\|_\lambda)$ is a Banach space.*

Proof. Let $(E, \|\cdot\|)$ be a Banach sequence space. Then, $(E(\lambda), \|\cdot\|_\lambda)$ is a normed space by the previous theorem. Now, let (x_n) be a Cauchy sequence in $E(\lambda)$. Let $y_n = \lambda x_n$. Then (y_n) is a sequence in E and is Cauchy in E since

$$\|x_n - x_m\|_\lambda = \|\lambda(x_n - x_m)\| = \|\lambda x_n - \lambda x_m\| = \|y_n - y_m\|.$$

Let $y \in E$ such that $\lim y_n = y$ in E. Let $x = \lambda^{-1}y$. Then

$$\|x_n - x\|_\lambda = \|\lambda(x_n - x)\| = \|y_n - y\| \to 0.$$

□

Corollary 2.3. *The multiplier of a BK space is a BK space.*

Corollary 2.4. *Let λ and μ be sequences with nonzero terms, and let E and F be BK spaces. Then any matrix operator $A \in (E(\lambda), F(\mu))$ is continuous.*

Corollary 2.5. *Let λ, γ, μ and δ be sequences with nonzero terms. Then the matrix operators between $\ell_p(\lambda), c_0(\gamma), c(\mu)$ and $\ell_\infty(\delta)$ are continuous.*

Remark 2.6. *The multiplier of a UBK space is not a UBK space, in general. For example, the sequence $(k) \in c((1/k))$ has norm 1 according to the norm given by*

$$\sup_{x=(x_k)} \left\{\frac{1}{k}|x_k|\right\},$$

which is the corresponding norm of $c((1/k))$ according to Theorem 1.2. So the space $c((1/k))$ is not UBK by Lemma 2.1.

For a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonzero terms, define $\lambda^{-1} = (\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots)$. Let D_λ be the diagonal matrix where the diagonal entries are the entries of the sequence λ, i.e.

$$D_\lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $e = (1, 1, \ldots)$. So, D_e is the identity operator. For any two sequences $\mu = (\mu_k)$ and $\lambda = (\lambda_k)$ and a matrix operator $A = (a_{nk})$, we can see that the operator $D_\mu A D_\lambda$ is represented by the matrix $A(\mu, \lambda) = (\alpha_{nk})$ where

$$\alpha_{nk} = \mu_n a_{nk} \lambda_k.$$

Theorem 2.7. *Let E and F be two sequence spaces, λ and μ be two sequences of nonzero terms. Then for a matrix A, $A \in (E(\lambda), F(\mu))$ if and only if $A(\mu, \lambda^{-1}) \in (E, F)$.*

Proof. Suppose $A \in (E(\lambda), F(\mu))$ and let $z = (z_k) \in E$. Then, clearly $\lambda^{-1} z \in E(\lambda)$. Hence, we have $A\lambda^{-1} z \in F(\mu)$, and so $\mu A \lambda^{-1} z \in F$, which is equivalent to saying $D_\mu A D_{\lambda^{-1}} z \in F$.

For the inverse implication, suppose $D_\mu A D_{\lambda^{-1}} \in (E, F)$ and let $z \in E(\lambda)$. Then $\lambda z \in E$ and so $D_\mu A D_{\lambda^{-1}} \lambda z \in F$. This is equivalent to saying $\mu A \lambda^{-1} \lambda z \in F$, and so $\mu A z \in F$ and $Az \in F(\mu)$. □

Corollary 2.8. *Let E and F be two normed sequence spaces, λ and μ be two sequences of nonzero terms. Then for a matrix A, $A \in B(E(\lambda), F(\mu))$ if and only if $A(\mu, \lambda^{-1}) \in B(E, F)$. In this case, we have*

$$\|A\|_{(E(\lambda), F(\mu))} = \|A(\mu, \lambda^{-1})\|_{(E,F)}.$$

Proof. It is enough to show the equality:

$$\begin{aligned} \|A\|_{(E(\lambda), F(\mu))} &= \sup_{\theta \neq z \in E(\lambda)} \frac{\|Az\|_{F(\mu)}}{\|z\|_{E(\lambda)}} = \sup_{\theta \neq \lambda z \in E} \frac{\|\mu A z\|_F}{\|\lambda z\|_E} = \sup_{\theta \neq \lambda z \in E} \frac{\|\mu A \lambda^{-1} \lambda z\|_F}{\|\lambda z\|_E} \\ &= \sup_{\theta \neq \lambda z \in E} \frac{\|A(\mu, \lambda^{-1}) \lambda z\|_F}{\|\lambda z\|_E} = \|A(\mu, \lambda^{-1})\|_{(E,F)}. \end{aligned}$$

□

3 Examples and Applications

Example 3.1. *The matrix*

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in (c, c(n^{-1})),$$

since the Cesàro operator $C = T((n^{-1}), e) \in (c, c)$. The space $c(n^{-1}) = \{(z_n) \in \omega : \lim_n \frac{z_n}{n}$ exists$\}$ is a Banach space by Theorem 2.2.

Example 3.2. *A lower triangular matrix $A = (a_{nk})$ is said to be factorable if there exists sequences (a_n) and (b_n) such that $a_{nk} = a_n b_k$ for all $k, n \in \mathbb{N}$. If $A = (a_n b_k)$ is factorable, then $A = T((a_n), (b_n))$ where T is the matrix in Example 3.1. So, if (a_n) and (b_n) have nonzero terms, we have $T = A((a_n^{-1})(b_n^{-1})) \in (c, c(n^{-1}))$ and $A \in (c(b_n), c(\frac{1}{na_n}))$.*

Theorem 3.3. *Let E be a UBK space and let $A = (a_{nk})$ be a matrix operator with rows in ℓ_1. Then, there exists a Banach space F such that $A \in B(E, F)$.*

Proof. Let $\|\cdot\|$ denote the norm of E and $\|\cdot\|_\infty$ be the norm of ℓ_∞. Let $M_n = \sum_{k=1}^\infty |a_{nk}|$, and let $A_n \in (E, \mathbb{C})$ be the operator corresponding to the n-th row of A, i.e. for $z = (z_k) \in E$ we have $A_n z = \sum_{k=1}^\infty a_{nk} z_k \in \mathbb{C}$. Then, since E is a UBK space, there exists $C > 0$ such that $|z_k| \le C\|z\|$ for all k and all $z \in E$, and so

$$\|A_n\| = \sup_{\theta \neq z \in E} \frac{|A_n z|}{\|z\|} = \sup_{\theta \neq z \in E} \frac{|\sum_{k=1}^\infty a_{nk} z_k|}{\|z\|} \le C M_n < \infty.$$

Then $A \in (E, \omega)$. Now define the sequence $\mu = (\mu_k)$ by

$$\mu_k = \begin{cases} \frac{1}{M_k} & \text{if } M_k \neq 0 \\ 1 & \text{if } M_k = 0 \end{cases}.$$

Then, for the matrix $A(\mu, e)$ we have

$$\begin{aligned} \sup_{\theta \neq z \in E} \frac{\|A(\mu, e)z\|_\infty}{\|z\|} &= \sup_{\theta \neq z \in E} \sup_n \frac{|\mu_n \sum_{k=1}^\infty a_{nk} z_k|}{\|z\|} \\ &\le C \sup_n \mu_n \sum_{k=1}^\infty |a_{nk}| \\ &\le C \sup_n \mu_n M_n \le C. \end{aligned}$$

Hence $A(\mu, e) \in B(E, \ell_\infty)$, and by Corollary 2.8 we have $A \in B(E(e^{-1}), \ell_\infty(\mu)) = B(E, \ell_\infty(\mu))$ and $\ell_\infty(\mu)$ is a Banach space by Theorem 2.2. □

Corollary 3.4. *If $A \in (E, \omega)$ where E is one of the sequence spaces c_0, c or ℓ_∞. Then there exists a Banach sequence space F such that $A \in B(E, F)$.*

Proof. c_0, c and ℓ_∞ are UBK spaces. If A is in (c_0, ω), in (c, ω) or in (ℓ_∞, ω), then rows of A are in ℓ_1 and so the theorem can be applied. □

Let us list the following conditions:

$$\lim_{n\to\infty} \mu_n a_{nk} \text{ exists for each } k \tag{3.1}$$

$$\lim_{n\to\infty} \mu_n a_{nk} = 0 \text{ for each } k \tag{3.2}$$

$$\lim_{n\to\infty} \mu_n \sum_{k=1}^{\infty} \frac{a_{nk}}{\lambda_k} \text{ exists} \tag{3.3}$$

$$\sup_n |\mu_n| \sum_{k=1}^{\infty} \left|\frac{a_{nk}}{\lambda_k}\right| < \infty \tag{3.4}$$

$$\lim_{n\to\infty} \mu_n \sum_{k=1}^{\infty} \frac{a_{nk}}{\lambda_k} = 0 \tag{3.5}$$

$$\sum_{k=1}^{\infty} \left|\frac{\mu_n a_{nk}}{\lambda_k}\right| \text{ converges uniformly in } n \tag{3.6}$$

$$\lim_{n\to\infty} |\mu_n| \sum_{k=1}^{\infty} \left|\frac{a_{nk}}{\lambda_k}\right| = 0 \tag{3.7}$$

$$\sup_{n,k} \left|\frac{\mu_n a_{nk}}{\lambda_k}\right| < \infty \tag{3.8}$$

$$\sup_k \sum_{n=1}^{\infty} \left|\frac{\mu_n a_{nk}}{\lambda_k}\right|^p < \infty \tag{3.9}$$

$$\sup_{N\in f(\mathbb{N})} \sum_{k=1}^{\infty} \left|\frac{1}{\lambda_k} \sum_{n\in N} \mu_n a_{nk}\right| < \infty \tag{3.10}$$

where $f(\mathbb{N})$ in (3.10) denotes the collection of all finite subsets of $\mathbb{N}$.

Theorem 2.7 has many applications. As an example we give the following theorem, which characterizes the matrix operators between multipliers of the classical sequence spaces. We give this theorem without proof, since the results are direct applications of Theorem 2.7 to the well known characterizations of the matrix mappings between the classical sequence spaces (see e.g. [4–6]).

Theorem 3.5. *Let λ and μ be two sequences of nonzero terms, and $A = (a_{nk})$. Then*

$$A \in (\ell_\infty(\lambda), \ell_\infty(\mu)) \text{ if and only if } (3.4) \text{ holds,} \tag{3.11}$$

$$A \in (c(\lambda), \ell_\infty(\mu)) \text{ if and only if } (3.4) \text{ holds,} \tag{3.12}$$

$$A \in (c_0(\lambda), \ell_\infty(\mu)) \text{ if and only if } (3.4) \text{ holds,} \tag{3.13}$$

$$A \in (\ell_\infty(\lambda), c(\mu)) \text{ if and only if } (3.1) \text{ and } (3.6) \text{ hold,} \tag{3.14}$$

$$A \in (c(\lambda), c(\mu)) \text{ if and only if } (3.1), (3.3) \text{ and } (3.4) \text{ hold,} \tag{3.15}$$

$$A \in (c_0(\lambda), c(\mu)) \text{ if and only if } (3.1) \text{ and } (3.4) \text{ hold,} \tag{3.16}$$

$$A \in (\ell_\infty(\lambda), c_0(\mu)) \text{ if and only if } (3.7) \text{ holds,} \tag{3.17}$$

$$A \in (c(\lambda), c_0(\mu)) \text{ if and only if } (3.2), (3.4) \text{ and } (3.5) \text{ hold,} \tag{3.18}$$

$$A \in (c_0(\lambda), c_0(\mu)) \text{ if and only if } (3.2) \text{ and } (3.4) \text{ hold,} \tag{3.19}$$

$$A \in (\ell_1(\lambda), \ell_\infty(\mu)) \text{ if and only if } (3.8) \text{ holds,} \quad (3.20)$$

$$A \in (\ell_1(\lambda), \ell_p(\mu)) \text{ for } (1 \leq p < \infty) \text{ if and only if } (3.9) \text{ holds,} \quad (3.21)$$

$$A \in (\ell_\infty(\lambda), \ell_1(\mu)) \text{ if and only if } (3.10) \text{ holds.} \quad (3.22)$$

Moreover, all the matrix mappings in (3.11)-(3.22) *are continuous.*

Theorem 3.6. *Let* $A = (a_{nk}) \in (c, \omega)$ *be a matrix satisfying the conditions:*

(i) $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = \infty$,

(ii) $a_k := \lim_{n\to\infty} \frac{a_{nk}}{\sum_{k=1}^{\infty}|a_{nk}|}$ *exists for each* $k \in \mathbb{N}$ *and*

(iii) $a_0 := \lim_{n\to\infty} \frac{\sum_{k=1}^{\infty} a_{nk}}{\sum_{k=1}^{\infty}|a_{nk}|}$ *exists*

with $(a_0, a_1, \ldots) \neq \theta$. *Then there exists a convergent sequence* $z = (z_n)$ *such that* $\lim_n |(Az)_n| = \infty$.

Proof. Since $A \in (c, \omega)$, the rows of A are in ℓ_1. Define $\mu = (\mu_n)$ by

$$\mu_n = \begin{cases} \sum_{k=1}^{\infty} |a_{nk}| & \text{if } \sum_{k=1}^{\infty} |a_{nk}| \neq 0 \\ 1 & \text{if } \sum_{k=1}^{\infty} |a_{nk}| = 0 \end{cases}.$$

So the sequence (μ_n) has nonzero terms and $\mu_n \to \infty$ by condition (i). Then, using conditions (ii) and (iii) we get $A \in (c, c(\mu^{-1}))$ by (3.15), and $A \notin (c, c_0(\mu^{-1}))$ by (3.18). So, there exists $z = (z_n) \in c$ such that $y = (y_n) = Az \in c(\mu^{-1})\backslash c_0(\mu^{-1})$. Hence the sequence $(y_n/\mu_n) \in c\backslash c_0$, and so there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\lim_n \frac{y_n}{\mu_n} = \alpha$. Then, since $\lim_n \mu_n = \infty$, we have $\lim_n |y_n| = \infty$. □

Remark 3.7. *The characterization of a matrix* $A \in (c, \omega)$ *that guarantees at least one convergent sequence is sent to a sequence that goes to infinity, is an important open problem. Theorem* 3.6 *gives some sufficient conditions for such matrices.*

Finally we give a theorem which is a c_0 version of Theorem 3.6.

Theorem 3.8. *Let* $A = (a_{nk}) \in (c_0, \omega)$ *be a matrix satisfying the conditions:*

(i) $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = \infty$ *and*

(ii) $a_k := \lim_{n\to\infty} \frac{a_{nk}}{\sum_{k=1}^{\infty}|a_{nk}|}$ *exists for each* $k \in \mathbb{N}$

with $(a_1, a_2, \ldots) \neq \theta$. *Then there exists a sequence* $z = (z_n) \in c_0$ *such that* $\lim_n |(Az)_n| = \infty$.

Proof. Since $A \in (c_0, \omega)$, the rows of A are in ℓ_1. Let $\mu = (\mu_n)$ be defined as in the proof of Theorem 3.6. Using condition (ii) we get $A \in (c_0, c(\mu^{-1}))$ by (3.16), and $A \notin (c_0, c_0(\mu^{-1}))$ by (3.19). So, there exists $z = (z_n) \in c_0$ such that $y = (y_n) = Az \in c(\mu^{-1})\backslash c_0(\mu^{-1})$. Hence the sequence $(y_n/\mu_n) \in c\backslash c_0$, and so there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\lim_n \frac{y_n}{\mu_n} = \alpha$. Then, since $\lim_n \mu_n = \infty$, we have $\lim_n |y_n| = \infty$. □

Acknowledgement : The author thanks the referees for their valuable comments and suggestions.

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(Received 23 June 2011)
(Accepted 13 February 2012)