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Normal Ideals in Generalized Almost Distributive Lattices

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Abstract : In this paper, we introduced the concept of an annihilator preserving homomorphism from a GADL L into a GADL L' and studied some basic properties of these homomorphisms. We derived a sufficient condition for a homomorphism to be annihilator preserving homomorphism. We introduced the concept of a normal ideal in a GADL L and proved that the set $\mathcal{N}(L)$ of all normal ideals of L forms a Boolean algebra.

Keywords : generalized almost distributive lattice (GADL); annihilator preserving homomorphism; disjunctive GADL; normal ideal.
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1 Introduction

The concept of a Generalized Almost Distributive Lattice (GADL) was introduced by Rao et al. [1] as a generalization of an Almost Distributive Lattice (ADL) [2]. The class of GADLs inherit almost all the properties of a distributive lattice except possibly the commutativity of $\wedge, \vee$, the right distributivity of either of the operations $\vee$ or $\wedge$ over the other. The class of GADLs include the class of

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ADLs properly and retain many important properties of ADLs. In section 3, we introduce the concept of an annihilator preserving homomorphism from a GADL L into a GADL L' and study some basic properties of these homomorphisms. We derive a sufficient condition for a homomorphism to be annihilator preserving homomorphism. In section 4, we define the notion of a dense element in a GADL and the concept of a disjunctive GADL and we prove that, in a GADL L, every left identity element is a dense element and the converse holds when L is disjunctive. We introduce the concept of a normal ideal in a GADL L and prove that the set $\mathcal{N}(L)$ of all normal ideals of L forms a Boolean algebra.

2 Preliminaries

First, we recall certain definitions and properties of GADLs from [1–3] that are required in the paper.

Definition 2.1 ([2])**.** An *Almost Distributive Lattice (ADL)* is an algebra $(L, \vee, \wedge)$ of type (2, 2) satisfying

1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$;

2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

3) $(x \vee y) \wedge y = y$;

4) $(x \vee y) \wedge x = x$;

5) $x \vee (x \wedge y) = x$.

If there is an element $0 \in L$ such that $0 \wedge a = 0$ for all $a \in L$, then $(L, \vee, \wedge, 0)$ is called an ADL with 0.

Definition 2.2 ([2])**.** Let X be a non-empty set. Fix some element $x_0 \in X$. Then, for any $x, y \in X$ define $\vee$ and $\wedge$ on X by,

$$x \vee y = \begin{cases} x, & \text{if } x \neq x_0 \\ y, & \text{if } x = x_0, \end{cases} \qquad x \wedge y = \begin{cases} y, & \text{if } x \neq x_0 \\ x_0, & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL, with x_0 as its zero element. This ADL is called a *discrete ADL.*

Definition 2.3 ([1])**.** An algebra $(L, \vee, \wedge)$ of type $(2, 2)$ is called a *Generalized Almost Distributive Lattice* if it satisfies the following axioms:

$(As\wedge)$ $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;

$(LD\wedge)$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

$(LD\vee)$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

(A_1) $x \wedge (x \vee y) = x$;

(A_2) $(x \vee y) \wedge x = x$;

(A_3) $(x \wedge y) \vee y = y$.

Example 2.4. *Let $L = \{a, b, c\}$. Define two binary operations $\vee$ and $\wedge$ on L as follows:*

$\vee$	a	b	c
a	a	b	a
b	b	b	b
c	c	c	c

$\wedge$	a	b	c
a	a	a	c
b	a	b	c
c	a	a	c

Hence the algebra $(L, \vee, \wedge)$ is a Generalized Almost Distributive Lattice.

For brevity, we will refer to this Generalized Almost Distributive Lattice as GADL. The GADL $(L, \vee, \wedge)$ in Example 2.4 is not an ADL for $(c \vee b) \wedge b \neq b$. Let $(L, \vee, \wedge)$ be a GADL. For any $a, b \in L$ define $a \leq b$ if and only if $a \wedge b = a$ or, equivalently, $a \vee b = b$. Then $\leq$ is a partial ordering on L. In this section, L stands for a GADL unless otherwise mentioned.

Lemma 2.5 ([1])**.** *Let L be a GADL with 0. For any $a, b \in L$, the followings hold:*

(1) $a \vee a = a$;

(2) $a \wedge a = a$;

(3) $a \vee (a \wedge b) = a$;

(4) $a \vee (b \wedge a) = a$;

(5) $a \wedge b = b \Rightarrow a \vee b = a$;

(6) $a \vee b = b \Leftrightarrow a \wedge b = a$;

(7) $a \vee (a \vee b) = a \vee b$;

(8) $b \wedge (a \wedge b) = a \wedge b$;

(9) $a \wedge (b \wedge a) = b \wedge a$;

(10) $a \leq c, b \leq c$ *if and only if* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$;

(11) $a \wedge b \wedge c = b \wedge a \wedge c$;

(12) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$;

(13) $a \vee 0 = a = 0 \vee a$ *and* $a \wedge 0 = 0$;

(14) For any $m \in L, m$ is maximal with respect partial ordering '$\leq$' if and only if $m \vee x = m$ for all $x \in L$.

Definition 2.6 ([1])**.** Let L be a GADL. An element $e \in L$ is said to be *left identity element* in L if $e \wedge x = x$ for all $x \in L$.

Note that every left identity element is maximal element but converse need not to be true. In Example 2.4, we observe that c is maximal but not a left identity element.

Definition 2.7 ([3])**.** A non-empty subset I of L is said to be an *ideal* of L if (i) $a, b \in I$ implies $a \vee b \in I$ and (ii) $a \in I, x \in L$ implies $a \wedge x \in I$.

Theorem 2.8 ([3])**.** *Let $(L, \vee, \wedge, 0)$ be a GADL with 0 and I an ideal of L then for any $a, b \in L$, the followings hold:*

(i) $a \wedge b \in I \Leftrightarrow b \wedge a \in I$;

(ii) The set $S = \{a \wedge x \mid x \in L\}$ is the smallest ideal of L containing a. We denote by $S = (a]$;

(iii) $(a] \cap (b] = (a \wedge b]$ *and* $a \in (b] \Leftrightarrow b \wedge a = a$.

3 Annihilator Preserving Homomorphisms

In this section, we introduce the concept of an annihilator preserving homomorphism from a GADL L into a GADL L' and study some basic properties of these homomorphisms. We derive a sufficient condition for a homomorphism to be annihilator preserving homomorphism. In the following we give the definition of a homomorphism between two GADLs with zero and the Kernel of a homomorphism in a natural way.

Definition 3.1. Let L and L' be two GADLs with zeros. Then a mapping $f : L \to L'$ is called a homomorphism if it satisfies the following:

(1) $f(a \vee b) = f(a) \vee f(b)$;

(2) $f(a \wedge b) = f(a) \wedge f(b)$;

(3) $f(0) = 0'$ (where $0'$ is the zero element of L');

and the set

$$Kerf = \{x \in L \mid f(x) = 0'\}$$

is called the *Kernel* of the homomorphism f.

We first prove the following lemma which is useful in the forth coming results.

Lemma 3.2. *Let L and L' be two GADLs with 0 and $0'$, respectively, and $f : L \to L'$ a homomorphism. Then we have the followings:*

(1) For any ideal J of L', $f^{-1}(J)$ is an ideal of L containing $Kerf$.

(2) If f is onto, then for any ideal I of L, $f(I)$ is an ideal of L'.

Proof. (1) Let J be an ideal of L'. Since $f(0) = 0' \in J$, we obtain $0 \in f^{-1}(J)$. Let $a, b \in f^{-1}(J)$ where $a, b \in L$. Then $f(a), f(b) \in J$. Since J is an ideal in L', we obtain that $f(a \vee b) = f(a) \vee f(b) \in J$. Therefore $a \vee b \in f^{-1}(J)$. Again, let $x \in f^{-1}(J)$ and $r \in L$. Then $f(x) \in J$. Now $f(x \wedge r) = f(x) \wedge f(r) \in J$. Hence $x \wedge r \in f^{-1}(J)$. Therefore $f^{-1}(J)$ is an ideal of L. Since $0' \in J$, we obtain $Kerf = f^{-1}(\{0'\}) \subseteq f^{-1}(J)$.

(2) Since f is a homomorphism and $0 \in I$, we obtain that $0' = f(0) \in f(I)$. Let $f(a), f(b) \in f(I)$, where $a, b \in I$. Since I is an ideal, $a \vee b \in I$ and hence $f(a \vee b) \in f(I)$. Therefore $f(a) \vee f(b) \in f(I)$. Again, let $f(a) \in f(I)$ and $r \in L'$, where $a \in I$. Since f is onto, there exists $s \in L$ such that $f(s) = r$. Now $f(a) \wedge r = f(a) \wedge f(s) = f(a \wedge s) \in f(I)$. Therefore $f(I)$ is an ideal in L'. □

Definition 3.3. For any non-empty subset A of a GADL L with 0, define

$$A^* = \{x \in L \mid x \wedge a = 0, \text{ for all } a \in A\}.$$

This A^* is an ideal of L and is called the *annihilator ideal* of A. For any $a \in L$, we write $[a]^*$ for $\{a\}^*$ and is called *annulet* of L.

It can be easily observed that, for any subset A of L, $A \cap A^* = \{0\}$. In the following we prove some properties of annihilator ideals.

Lemma 3.4. *For any ideals I, J of a GADL L with 0, we have the following:*

(1) $I^* = \bigcap_{a \in I} [a]^*$;

(2) If $I \subseteq J$ *then* $J^* \subseteq I^*$;

(3) $I \subseteq I^{**}$;

(4) $I^{***} = I^*$;

(5) $I \cap J = (0] \Leftrightarrow I \subseteq J^*$.

Proof. (1) Clearly $I^* \subseteq \bigcap_{a \in I} [a]^*$. Let $x \in [a]^*$ for all $a \in I$. Let $b \in I$. Then $x \in [b]^*$ and hence $x \wedge b = 0$. Therefore $x \in I^*$.

(2) Let $I \subseteq J$. Let $a \in J^*$ and $b \in I$. Then $b \in J$ and $a \wedge b = 0$. Hence $a \in I^*$. Therefore $J^* \subseteq I^*$.

(3) Let $x \in I$ and $a \in I^*$. Then $x \wedge a \in I$ and $x \wedge a \in I^*$. So that $x \wedge a \in I \cap I^* = \{0\}$. Hence $x \wedge a = 0$ for all $a \in I^*$. We get $x \in I^{**}$. Therefore $I \subseteq I^{**}$.

(4) Since $I \subseteq I^{**}$, we have, by (2), $I^{***} \subseteq I^*$. Now, let $x \in I^*$ and $a \in I^{**}$. Then $x \wedge a \in I^* \cap I^{**} = \{0\}$. Hence $x \wedge a = 0$ for all $a \in I^{**}$. Therefore $x \in I^{**}$, we get $I^* \subseteq I^{***}$. Thus $I^{***} = I^*$.

(5) Suppose $I \cap J = (0]$. Let $x \in I$ and $a \in J$. Then $x \wedge a \in I$ and $x \wedge a \in J$ and hence $x \wedge a \in I \cap J = (0]$. Therefore $x \wedge a = 0$, we get $x \in J^*$. Thus $I \subseteq J^*$. Conversely, assume that $I \subseteq J^*$. Let $x \in I \cap J$. Then $x \in I$ implies that $x \in J^*$ and hence $x = x \wedge x = 0$. Therefore $I \cap J = (0]$. □

The following lemma can be verified easily.

Lemma 3.5. *Let L be a GADL with 0. For any $x, y \in L$, we have the following:*

(1) $[x \wedge y]^* = [y \wedge x]^*$;

(2) $x \leq y \Rightarrow [y]^* \subseteq [x]^*$;

(3) $[x \wedge y]^{**} = [x]^{**} \cap [y]^{**}$;

(4) $[x]^{***} = [x]^*$;

(5) $[x \vee y]^* = [x]^* \cap [y]^* = [y]^* \cap [x]^* = [y \vee x]^*$.

Now we prove the following.

Lemma 3.6. *Let L and L' be two GADLs with 0, $0'$, respectively. If $f : L \to L'$ is a homomorphism, then for any non-empty subset A of L, we have*

$$f(A^*) \subseteq \{f(A)\}^*.$$

Proof. Let $x \in f(A^*)$ and $y \in f(A)$. Then there exists $a \in A^*$ and $b \in A$ such that $x = f(a)$ and $y = f(b)$. Now $x \wedge y = f(a) \wedge f(b) = f(a \wedge b) = f(0)$ ($\because a \in A^*$ and $b \in A$)$= 0'$. That is $x \wedge y = 0'$ for all $y \in f(A)$. Hence $x \in \{f(A)\}^*$. Therefore $f(A^*) \subseteq \{f(A)\}^*$. □

If L is a GADL with 0, then for any $A \subseteq L$, $\{f(A)\}^* = f(A^*)$ is not true in general. Consider the following example.

Example 3.7. *Let $L = \{0, a, b, c\}$ be a discrete ADL. Define a mapping $f : L \longrightarrow L$ by $f(x) = 0$ for all $x \in L$. Then clearly f is a homomorphism on L. Now take $A = \{a, b\}$. Then clearly $A^* = \{0\}$ and $f(A) = \{0\}$. Hence $f(A^*) = \{0\}$ and $\{f(A)\}^* = L$. Therefore $\{f(A)\}^* \neq f(A^*)$.*

This motivates us to introduce the concept of annihilator preserving homomorphism in the following.

Definition 3.8. Let L and L' be two GADLs with 0 and $0'$, respectively. Then a homomorphism $f : L \to L'$ is called *annihilator preserving* if

$$f(A^*) = \{f(A)\}^*$$

for any set A such that $(0] \subset A \subset L$.

Example 3.9. *Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ADLs. Write $L = A \times B = \{(0, 0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)\}$. Then $(L, \vee, \wedge, \bar{0})$ is an GADL with $\bar{0} = (0, 0)$, under point-wise operations. Let $L' = \{0', a', b', c'\}$ be another GADL in which the operations $\vee', \wedge'$ are defined as follows:*

$\vee'$	$0'$	a'	b'	c'
$0'$	$0'$	a'	b'	c'
a'	a'	a'	c'	c'
b'	b'	c'	b'	c'
c'	c'	c'	c'	c'

$\wedge'$	$0'$	a'	b'	c'
$0'$	$0'$	$0'$	$0'$	$0'$
a'	$0'$	a'	$0'$	a'
b'	$0'$	$0'$	b'	b'
c'	$0'$	a'	b'	c'

Now, define the mapping $f : L \longrightarrow L'$ as follows:

$$f((0,0)) = 0';\ f((a,0)) = a'$$

$$f((0,b_1)) = f((0,b_2)) = b';\ f((a,b_1)) = f((a,b_2)) = c'.$$

It can be easily verified that f is a homomorphism from L onto L'.

(i) For $A = \{(0,0)\}$, clearly we get $f(A^*) = L' = \{f(A)\}^*$.

(ii) For $A = \{(a,0)\}$, we get $A^* = \{(0,0),(0,b_1),(0,b_2)\}$ and $f(A) = \{a'\}$. Hence $f(A^*) = \{0', b'\} = \{f(A)\}^*$.

(iii) For $A = \{(0,b_i)\}, i = 1,2$, we get $A^* = \{(0,0),(a,0)\}$ and $f(A) = \{b'\}$. Hence $f(A^*) = \{0', a'\} = \{f(A)\}^*$.

Similarly, for $A = \{(0,0),(a,0)\}$ and $A = \{(0,0),(0,b_i)\}$, we get $f(A^*) = \{f(A)\}^*$. In the remaining cases, $f(A^*) = \{0'\} = \{f(A)\}^*$. Therefore f is an annihilator preserving homomorphism.

If f is a homomorphism of a GADL L with 0 into another GADL L' with $0'$ such that $Kerf = \{0\}$ and f is onto, then f need not be an isomorphism. It may be seen in the following example.

Example 3.10. *Let $L = \{0,a,b\}$ and $L' = \{0', c\}$ be two discrete ADLs. Define a mapping $f : L \longrightarrow L'$ by $f(0) = 0'$ and $f(a) = f(b) = c$. Then clearly f is a homomorphism from L into L' and also f is onto. Also $Kerf = \{0\}$. But f is not one-one. Hence f is not an isomorphism.*

However, we have the following.

Theorem 3.11. *Let L and L' be two GADLs with zeroes 0 and $0'$ respectively and $f : L \longrightarrow L'$ a homomorphism. If $Kerf = \{0\}$ and f is onto, then f is annihilator preserving.*

Proof. Assume that f is onto and $Ker\ f = \{0\}$. Let A be a non-empty subset of L. We have always $f(A^*) \subseteq \{f(A)\}^*$. Let $x \in \{f(A)\}^* \subseteq L'$. Since f is onto, there exists $y \in L$ such that $f(y) = x$. By knowing that $f(y) \in \{f(A)\}^*$. Then $f(y) \wedge m = 0$ for all $m \in f(A)$. Let $a \in A$. Then $f(y) \wedge f(a) = 0'$. That is $f(y \wedge a) = 0'$ which means $y \wedge a \in Ker\ f = \{0\}$. Then $y \wedge a = 0$. Hence $y \in A^*$. Therefore $\{f(A)\}^* \subseteq f(A^*)$. Thus $\{f(A)\}^* = f(A^*)$. Therefore f is annihilator preserving. □

Theorem 3.12. *Let L and L' be two GADLs with 0 and $0'$, respectively and $f : L \to L'$ an epimorphism. If $Kerf = \{0\}$, then*

$$A^* = B^* \ \Leftrightarrow\ \{f(A)\}^* = \{f(B)\}^*$$

for any two non-empty subsets A, B of L.

Proof. Since f is an epimorphism and $Ker\ f = \{0\}$, by Theorem 3.11, f is annihilator preserving. Let A, B be two non-empty subsets of L. Assume that $A^* = B^*$. Then clearly $f(A^*) = f(B^*)$. Hence $\{f(A)\}^* = \{f(B)\}^*$. Conversely, assume that $\{f(A)\}^* = \{f(B)\}^*$. Let $t \in A^*$ and $b \in B$. Then $t \wedge a = 0$ for all $a \in A$. Now,

$$\begin{aligned} t \in A^* &\Rightarrow f(t) \in f(A^*) \\ &\Rightarrow f(t) \in \{f(A)\}^* \text{ (by Theorem 3.11)} \\ &\Rightarrow f(t) \in \{f(B)\}^* \text{ (since } f(A^*) = f(B^*)) \\ &\Rightarrow f(t) \wedge f(b) = 0' \text{ (since } f(b) \in f(B)) \\ &\Rightarrow f(t \wedge b) = 0' \\ &\Rightarrow t \wedge b \in Kerf = \{0\} \\ &\Rightarrow t \wedge b = 0 \\ &\Rightarrow t \in B^*. \end{aligned}$$

Hence $A^* \subseteq B^*$. Similarly, we can obtain that $B^* \subseteq A^*$. Therefore $A^* = B^*$. □

4 Normal Ideals

In this section we introduce the concept of a dense element in a GADL, disjunctive GADL and normal ideal in a GADL and we prove that the set of all normal ideals of a GADL forms a Boolean algebra. First we begin with the following.

Definition 4.1. An element a of L is called *dense element* if $[a]^* = \{0\}$.

Theorem 4.2. *In a GADL, every left identity element is a dense element.*

Proof. Let m be a maximal element in L and $x \in [m]^*$. Then $x \wedge m = 0$. So that $0 = x \wedge m = x$. Hence $x = 0$. Therefore $[m]^* = \{0\}$. Thus m is a dense element. □

In the following example we show that a dense element needs not to be a left identity element.

Example 4.3. *Let $L = \{0, a, b, c\}$ and define $\vee$ and $\wedge$ on L as follows:*

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	b
c	c	c	c	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	c
c	0	b	b	c

Then clearly $(L, \vee, \wedge, 0)$ is a GADL with 0. Clearly a, b, c are dense elements but b and c are not left identity elements.

Now we define the notion of a disjunctive GADL.

Definition 4.4. A GADL L with 0, is called disjunctive iff for all $a, b \in L$, $[a]^* = [b]^*$ implies $a = b$.

Example 4.5. *Let $L = \{0, a, b, c\}$ be a set. Define $\vee$ and $\wedge$ on L as follows:*

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

Then clearly $(L, \vee, \wedge, 0)$ is a GADL with 0. Now, $[a]^* = [0]$, $[b]^* = \{0, c\}$ and $[c]^* = \{0, b\}$. We can see that $x \neq y$ and $[x]^* \neq [y]^*$ for all $x, y \in L$. Hence L is disjunctive.

In a GADL L with 0, we know that a left identity element is always a dense element (Theorem 4.2). Now we prove the converse in disjunctive GADL.

Theorem 4.6. *If L is a disjunctive GADL, then every dense element of L is a left identity element.*

Proof. Assume that L is disjunctive. Let m be a dense element of L. That is $[m]^* = \{0\}$. For any $x \in L$, $[m \wedge x]^{**} = [m]^{**} \cap [x]^{**} = L \cap [x]^{**} = [x]^{**}$ and hence $[m \wedge x]^{***} = [x]^{***}$. So that $[m \wedge x]^* = [x]^*$. Since L is disjunctive, we get that $m \wedge x = x$. Therefore m is a left identity element of L. $\square$

Now we define the concept of a normal ideal in a GADL.

Definition 4.7. Let L be a GADL with 0. An ideal I of L is called a *normal ideal* if $I = I^{**}$, or equivalentlty, $I = S^* = \{y \in L \mid y \wedge s = 0, \text{ for all } s \in S\}$ for some non-empty subset S of L. We denote the set of all normal ideals of L by $\mathcal{N}(L)$.

Example 4.8. *Let $L = \{0, a, b, c\}$ and define $\vee$ and $\wedge$ on L as follows:*

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

Then clearly $(L, \vee, \wedge, 0)$ is a GADL with 0. Consider the set $I = \{0, b\} \subseteq L$. Then clearly I is an ideal in L. Now $I^* = \{0, c\}$ and also $I^{**} = \{0, b\} = I$. Thus I is normal ideal in L. Similarly, the ideal $J = \{0, c\}$ of L, is another normal ideal in L.

Now, we prove the following

Theorem 4.9. *Let L and L' be two GADLs with zeroes* 0 *and* $0'$, *respectively and $f : L \to L'$ a homomorphism. Then we have the following:*

(1) If f is annihilator preserving and onto, then $f(I)$ is a normal ideal of L' for every normal ideal I of L.

(2) If in addition $Kerf = \{0\}$, then $f^{-1}(J)$ is a normal ideal of L, for every normal ideal J of L'.

Proof. (1) Let I be a normal ideal of L. Then by Lemma 3.2(2), $f(I)$ is an ideal of L'. Since f is annihilator preserving, $\{f(I)\}^{**} = f(I^{**}) = f(I)$. Therefore $f(I)$ is a normal ideal in L'.

(2) Let J be a normal ideal of L'. Then by Lemma 3.2(1), $f^{-1}(J)$ is an ideal of L. Let $x \in \{f^{-1}(J)\}^{**}$ and $y \in J^*$. Then $y = f(t)$ for some $t \in L$. Let $s \in f^{-1}(J)$. Then $f(s) \in J$ and hence $y \wedge f(s) = 0$. Therefore $t \wedge s = 0$. Hence $t \in (f^{-1}(J))^*$. Therefore $x \wedge t = 0$ and hence $f(x) \wedge f(t) = 0'$. Thus $f(x) \in J^{**} = J$. Hence $x \in f^{-1}(J)$. Therefore $f^{-1}(J)$ is a normal ideal in L. □

Corollary 4.10. *Let L and L' be two GADLs with zeros 0 and $0'$ respectively and $f : L \to L'$ is annihilator preserving homomorphism. Assume that $Kerf = \{0\}$ and f is onto. Then J is normal ideal of L' if and only if $f^{-1}(J)$ is a normal ideal of L.*

Theorem 4.11. *Let L be a GADL with 0. Then the set $\mathcal{N}(L)$ of all normal ideals of L forms a Boolean algebra.*

Proof. For $I, J \in \mathcal{N}(L)$, define $I \wedge J = I \cap J$ and $I \underline{\vee}\, J = (I^* \cap J^*)^*$. Let $I, J \in \mathcal{N}(L)$. Then $I^{**} = I$ and $J^{**} = J$. Hence $(I \cap J)^{**} = I^{**} \cap J^{**} = I \cap J$. Thus $I \cap J \in \mathcal{N}(L)$. We have also $I \underline{\vee}\, J \in \mathcal{N}(L)$. It can be easily observed that $(\mathcal{N}(L), \wedge, \underline{\vee})$ is a lattice. Since $[0]^* = L$ and $L^* = (0]$, we get that $(0], L \in \mathcal{N}(L)$ are the least and the greatest elements of $\mathcal{N}(L)$, respectively. Therefore, $(\mathcal{N}(L), \wedge, \underline{\vee})$ is a bounded lattice. Let $I \in \mathcal{N}(L)$. Then clearly $I^* \in \mathcal{N}(L)$ and $I \wedge I^* = I \cap I^* = (0], I \underline{\vee}\, I^* = (I^* \cap I^{**})^* = (I^* \cap I)^* = \{0\}^* = [0]^* = L$. Thus I^* is the complement of I for any $I \in \mathcal{N}(L)$. Therefore $(\mathcal{N}(L), \wedge, \underline{\vee}, *, (0], L)$ is a complemented lattice. Let $I, J, K \in \mathcal{N}(L)$. We prove that $I \underline{\vee}\, (J \wedge K) = (I \underline{\vee}\, J) \wedge (I \underline{\vee}\, K)$. We first prove that $(I \underline{\vee}\, J) \wedge K \subseteq I \underline{\vee}\, (J \wedge K)$. We have $I \cap K \cap [I^* \cap (J \cap K)^*] = (0]$, so that $K \cap I^* \cap (J \cap K)^* \subseteq I^*$. Similarly $J \cap K \cap [I^* \cap (J \cap K)^*] = (0]$ implies that $K \cap I^* \cap (J \cap K)^* \subseteq J^*$. Hence $K \cap I^* \cap (J \cap K)^* \subseteq I^* \cap J^*$. Thus, by Lemma 3.4, we get that $[K \cap I^* \cap (J \cap K)^*] \cap (I^* \cap J^*)^* = (0]$. That is $I^* \cap (J \cap K)^* \cap [K \cap (I^* \cap J^*)^*] = (0]$. Thus $K \cap (I^* \cap J^*)^* \subseteq [I^* \cap (J \cap K)^*]^*$. Hence $(I \underline{\vee}\, J) \wedge K \subseteq I \underline{\vee}\, (J \wedge K)$.

We prove the distributivity. $(I \underline{\vee}\, J) \cap (I \underline{\vee}\, K) \subseteq I \underline{\vee} [J \cap (I \underline{\vee}\, K)] = I \underline{\vee}\, [(I \underline{\vee}\, K) \cap J] \subseteq I \underline{\vee}\, [I \underline{\vee}\, (K \cap J)] = I \underline{\vee}\, (J \cap K)$. Clearly, $I \underline{\vee}\, (J \cap K) \subseteq (I \underline{\vee}\, J) \cap (I \underline{\vee}\, K)$. Thus $(\mathcal{N}(L), \wedge, \underline{\vee}, *, (0], L)$ is a Boolean algebra. □

It can be easily observed that every annulet, for any $x \in L$, $[x]^*$ is a normal ideal in L. We denote the set of all annulets of L by $\mathcal{N}_0(L)$. That is, $\mathcal{N}_0(L) = \{[x]^* | x \in L\}$. Annulets have many important properties. We give some of them in the following lemma which can be easily verified.

Lemma 4.12. *Let L be a GADL with 0 and $x, y \in L$. Then we have:*

(1) $x \leq y \Rightarrow [y]^* \subseteq [x]^*$;

(2) $[x \wedge y]^* = [y \wedge x]^*$;

(3) $[x \vee y]^* = [y \vee x]^*$;

(4) $[x \vee y]^* = [x]^* \cap [y]^*$.

Since each annulet is a normal ideal, we can have the following:

$$[x]^* \underline{\vee} [y]^* = [[x]^{**} \cap [y]^{**}]^* = [(x \wedge y)^{**}]^* = [x \wedge y]^*$$
$$[x]^* \wedge [y]^* = [x]^* \cap [y]^* = [x \vee y]^*.$$

Then we prove in the following theorem that the set $\mathcal{N}_0(L)$ of all annulets of a GADL L forms a distributive lattice.

Theorem 4.13. *Let L be a GADL with 0. Then $(\mathcal{N}_0(L), \cap, \underline{\vee})$ is a sublattice of the Boolean algebra $\langle \mathcal{N}(L), \cap, \underline{\vee}, ^*, (0], L \rangle$ of normal ideals of L and hence it is a distributive lattice. $\mathcal{N}_0(L)$ has the same greatest element $L = [0]^*$ as $\mathcal{N}(L)$. $\mathcal{N}_0(L)$ has the smallest element if and only if L possesses a dense element.*

Proof. Let $[x]^*, [y]^* \in \mathcal{N}_0(L)$, where $x, y \in L$. Then
1. $[x]^* \wedge [y]^* = [x]^* \cap [y]^* = [x \vee y]^* \in \mathcal{N}_0(L)$ and
2. $[x]^* \underline{\vee}\, [y]^* = [x \wedge y]^* \in \mathcal{N}_0(L)$.

Hence $\mathcal{N}_0(L)$ is a sublattice of $\mathcal{N}(L)$. Since $\mathcal{N}(L)$ is distributive, we have that $\mathcal{N}_0(L)$ is also distributive. Clearly, $[0]^*$ is the greatest element of $\mathcal{N}(L)$. Now for any $[x]^* \in \mathcal{N}_0(L)$, we get $[x]^* \cap [0]^* = [x \vee 0]^* = [x]^*$ and $[x]^* \underline{\vee}\, [0]^* = [x \wedge 0]^* = [0]^*$. It shows that $[0]^*$ is the greatest element in $\mathcal{N}_0(L)$. Now, it remains to prove the final condition of the theorem. Assume $\mathcal{N}_0(L)$ has the smallest element, say $[d]^*$ where $d \in L$. Suppose $x \in [d]^*$. Then $x \wedge d = 0$. Since $[d]^*$ is the least element, we get $[x]^* = [x]^* \underline{\vee}\, [d]^* = [x \wedge d]^* = [0]^* = L$. Hence $x = 0$. Thus $[d]^* = (0]$. Therefore d is a dense element in L.

Conversely, suppose that L possesses a dense element, say d. So $[d]^* = (0]$. Clearly, $[d]^* \in \mathcal{N}_0(L)$. Now for any $x \in L$, consider $[x]^* \cap [d]^* = [x]^* \cap (0] = [x]^* \cap (0] = (0]$. Also $[x]^* \underline{\vee}\, [d]^* = \{[x]^{**} \cap [d]^{**}\}^* = \{[x]^{**} \cap [0]^*\}^* = \{[x]^{**} \cap L\}^* = [x]^{***} = [x]^*$. Hence $[d]^*$ is the smallest element in $\mathcal{N}_0(L)$. □

In general, the mapping $x \mapsto [x]^*$ of L into $\mathcal{N}_0(L)$ is a dual onto homomorphism. In fact, we have the following result.

Theorem 4.14. *A disjunctive GADL L is dually isomorphic to $\mathcal{N}_0(L)$.*

Proof. Let L be a disjunctive GADL. Define a mapping $\Phi : L \longrightarrow \mathcal{N}_0(L)$ by $\Phi(x) = [x]^*$, for all $x \in L$. Clearly, Φ is well-defined. Let $x, y \in L$ such that $\Phi(x) = \Phi(y)$. Then $[x]^* = [y]^*$. Since L is disjunctive, we obtain that $x = y$. Therefore Φ is one to one. Let $I \in \mathcal{N}_0(L)$. Then $I = [x]^*$, for some $x \in L$. Hence $\Phi(x) = [x]^* = I$. Therefore Φ is onto.

Let$[x]^*, [y]^* \in \mathcal{N}_0(L)$, where $x, y \in L$. Then $\Phi(x \wedge y) = [x \wedge y]^* = [x]^* \underline{\vee}\, [y]^* = \Phi(x) \underline{\vee}\, \Phi(y)$ and $\Phi(x \vee y) = [x \vee y]^* = [x]^* \cap [y]^* = \Phi(x) \cap \Phi(y)$. Hence Φ is a dual isomorphism. □

Conclusion and Future Work

In this paper we have introduced the concept of an annihilator preserving homomorphism and studied some basic properties of these homomorphisms. We derived a sufficient condition for a homomorphism to be annihilator preserving homomorphism. We introduced the concept of a normal ideal in a GADL L and proved that the set $\mathcal{N}(L)$ of all normal ideals of L forms a Boolean algebra. In [3], we have proved that the set of all ideals of a GADL with 0 forms a complete lattice under set inclusion but we were unable to characterize the nature of the supremum of the ideals in this lattice. Also, the ideal generated by any nonempty subset S of a GADL, except the case when S contains only one element, was not characterized. Investigations in this direction are going on in order to give a topological characterization and sheaf representation of a GADL.

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