Thai Journal of Mathematics
Volume 11 (2013) Number 1 : 21–34

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Convergence Theorems of Three-Step Iterations Scheme with Errors for I-Asymptotically Nonexpansive Mappings

Seyit Temir

Department of Mathematics, Art and Science Faculty
Harran University, Sanliurfa 63200, Turkey
e-mail : temirseyit@harran.edu.tr

Abstract : The purpose of this paper is to establish weak and strong convergence theorems of three-step iterations with errors for I-asymptotically nonexpansive mappings in Banach space. The results obtained in this paper extend and improve the recent ones announced by Jeong [1], Takahashi and Tamura [2], Nammanee et al. [3] and many others.

Keywords : I-asymptotically nonexpansive mappings; common fixed point; convergence theorems.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction

Let X be a real normed space and K be a nonempty closed convex subset of X. Let T be a self-mapping of K. Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping T.

A mapping $T : K \longrightarrow K$ is called *nonexpansive* mapping if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$. T is called *asymptotically nonexpansive* mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all $x, y \in K$ and $n \ge 1$.

The class of asymptotically nonexpansive maps which an important generalization of the class nonexpansive maps was introduced by Goebel and Kirk [4]. They proved that every asymptotically nonexpansive self-mapping of a nonempty closed convex bounded subset of a uniformly convex Banach space has a fixed point.

We introduce the following definitions and statements which will be used in our main results (see [5, 6, 7]).

Let $T, I : K \longrightarrow K$. Then T is called *I-nonexpansive* on K if

$$\|Tx - Ty\| \le \|Ix - Iy\|$$

for all $x, y \in K$.

T is called I- *asymptotically nonexpansive* on K if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|I^n x - I^n y\|$$

for all $x, y \in K$ and $n \ge 1$.

Recently, in [5], [6] and [7], the convergence theorems for I-nonexpansive and I-asymptotically quasi-nonexpansive mapping defined for some iterative schemes in Banach spaces were proved. In this paper, we consider the new type of three step iterative process for I-asymptotically nonexpansive mapping.

In 2000, Noor [8] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [9] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [9] that the three-step iterative scheme gives better numerical results than the Mann-type (one-step) and the Ishikawa-type (two-step) approximate iterations. Xu and Noor [10] introduced and studied a three-step iterative for asymptotically nonexpansive mappings and they proved weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space.

Very recently, Suantai [11] introduced the following iterative scheme and used it for the weak and strong convergence of fixed points in an uniformly convex Banach space. The scheme is defined as follows.

$$\begin{cases} x_1 = x \in K \\ z_n = a_n T^n x_n + (1 - a_n) x_n \\ y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n \\ x_{n+1} = \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n) x_n, \forall n \ge 1, \end{cases} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ in $[0, 1]$ satisfy certain conditions. The iterative scheme (1.1) is called the modified Noor iterative scheme for asymptotically nonexpansive mappings. If $\{c_n\} = \{\beta_n\} = 0$, then (1.1) reduces to Noor iterations defined by Xu and Noor [10] as follows:

$$\begin{cases} x_1 = x \in K \\ z_n = a_n T^n x_n + (1 - a_n)x_n \\ y_n = b_n T^n z_n + (1 - b_n)x_n \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \forall n \geq 1. \end{cases} \tag{1.2}$$

If $\{a_n\} = \{c_n\} = \{\beta_n\} = 0$, then (1.1) reduces to modified Ishikawa iterations [12] as follows:

$$\begin{cases} x_1 = x \in K \\ y_n = b_n T^n x_n + (1 - b_n)x_n \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \forall n \geq 1. \end{cases} \tag{1.3}$$

If $\{a_n\} = \{b_n\} = \{c_n\} = \{\beta_n\} = 0$, then (1.1) reduces to Mann iterative process [13] as follows:

$$\begin{cases} x_1 = x \in K \\ x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n)x_n, \forall n \geq 1. \end{cases} \tag{1.4}$$

Many authors starting from Das and Debata [14] and including Takahashi and Tamura [2] and Khan and Takahashi [15] have studied the two mappings case of iterative schemes for different types of mappings. Note that two mappings case has a direct link with the minimization problem. A generalization of Mann and Ishikawa iterative schemes was given by Das and Debata [14] and Takahashi and Tamura [2]. This scheme dealt with two nonexpansive mappings as follows:

$$\begin{cases} x_1 = x \in K \\ y_n = b_n T x_n + (1 - b_n)x_n \\ x_{n+1} = a_n S y_n + (1 - a_n)x_n, \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\{a_n\}$, $\{b_n\}$ in $[0, 1]$ satisfy certain conditions.

Further generalization of the iterative scheme (1.5) for two asymptotically nonexpansive mappings was given Jeong [1] as follows:

$$\begin{cases} x_1 = x \in K \\ z_n = a_n S^n x_n + (1 - a_n)x_n \\ y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n \\ x_{n+1} = \alpha_n S^n y_n + \beta_n S^n z_n + (1 - \alpha_n - \beta_n)x_n, \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ in $[0, 1]$ satisfy certain conditions.

The aim of this paper is to introduce and study convergence problem of the three-step iterative sequence with errors for I-asymptotically nonexpansive mappings in an uniformly convex Banach space. The results presented in this paper generalize and extend some recent Jeong [1], Takahashi and Tamura [2], Nammanee et al.[3] and many others.

2 Preliminaries

Let X be a Banach space with dimension $X \geq 2$. The modulus of X is function $\delta_X : (0,2] \to [0,1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = 1, \|y\| = 1, \|x-y\| = \varepsilon\right\}.$$

A Banach space X is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Recall that a Banach space X is said to satisfy Opial's condition [16] if, for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is well known from [16] that all l_r spaces for $1 < r < \infty$ have this property. However, the L_r space do not have unless $r = 2$.

Lemma 2.1 ([17])**.** *Let X be a uniformly convex Banach space, K a nonempty closed convex subset of X and $T : K \longrightarrow K$ be a asymptotically nonexpansive mapping. Then $E - T$(E is identity mapping) is demiclosed at zero.*

Lemma 2.2 ([18])**.** *Let $\{s_n\}$, $\{t_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real sequences satisfying the following conditions: $\forall n \geq 1$, $s_{n+1} \leq (1+\sigma_n)s_n + t_n$, where $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Then $\lim_{n\to\infty} s_n$ exists.*

Lemma 2.3 ([19])**.** *Let $p > 1, r > 0$ be two fixed numbers.Then a Banach space X is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x-y\|)$$

for all $x, y, z \in B_r := \{x \in X : \|x\| \leq r\}$ and $\lambda \in [0,1]$, where $w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

Lemma 2.4 ([3])**.** *Let X be a uniformly convex Banach space and $B_r := \{x \in X : \|x\| \leq r\}, r > 0$. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \nu z + \kappa w\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \nu\|z\|^2 + \kappa\|w\|^2 - (\lambda\mu)g(\|x-y\|)$$

for all $x, y, z, w \in B_r$ and $\lambda, \mu, \nu, \kappa \in [0,1]$ with $\lambda + \mu + \nu + \kappa = 1$.

Lemma 2.5 ([11, Lemma 2.7])**.** *Let X be a Banach space which satisfies Opial's condition and let x_n be a sequence in X. Let $q_1, q_2 \in X$ be such that $\lim_{n\to\infty} \|x_n - q_1\|$ and $\lim_{n\to\infty} \|x_n - q_2\|$ exist. If $\{x_{n_k}\}$ and $\{x_{n_j}\}$ are the subsequences of $\{x_n\}$ which converge weakly to q_1 and q_2, respectively. Then $q_1 = q_2$.*

3 Main Results

Let X be a real uniformly convex Banach space and K be a nonempty closed, bounded and convex subset of X. In this section, we prove theorems of weak and strong of the three-step iterative scheme with errors given in (3.1) to a fixed point for I-asymptotically nonexpansive mappings in a uniformly convex Banach space. Let $T : K \to K$ be a I-asymptotically nonexpansive mapping and $I : K \to K$ be an asymptotically nonexpansive mapping. In order to prove our main results the following iteration scheme is studied:

$$\begin{cases} x_1 = x \in K \\ z_n = a_n I^n x_n + (1 - a_n - \mu_n)x_n + \mu_n u_n \\ y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \nu_n)x_n + \nu_n v_n \\ x_{n+1} = \alpha_n I^n y_n + \beta_n I^n z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n, \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\lambda_n\}$, $\{a_n+\mu_n\}$, $\{b_n+c_n+\nu_n\}$ and $\{\alpha_n+\beta_n+\lambda_n\}$ are appropriate sequences in $[0, 1]$, and $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K.

The iterative scheme (3.1) is called the modified Noor iterative scheme with errors for asymptotically nonexpansive mappings. If $T = I$ and $\mu_n = \nu_n = \lambda_n = 0$, then (3.1) reduces to the (1.1) defined by [11].

In order to prove our main results, the following lemmas are needed.

Lemma 3.1. *$\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{\nu_n\}$ are in sequences in $[0, 1]$ such that $\limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$ and $\{k_n\}$ and $\{\ell_n\}$ are sequences of real numbers with $\ell_n, k_n \geq 1$ for all $n \geq 1$ and $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \ell_n = 1$, then there exists a positive integer N_1 and $\gamma \in (0, 1)$ such that $a_n c_n \ell_n^2 k_n < \gamma$ for all $n \geq N_1$.*

Proof. By $\limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$, there exists a positive integer N_0 and $\delta \in (0, 1)$ such that

$$a_n c_n \leq c_n \leq b_n + c_n + \nu_n < \delta, \quad \forall n \geq N_0.$$

Let $\delta' \in (0, 1)$ with $\delta' > \delta$. From $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \ell_n = 1$, then there exists a positive integer $N_1 \geq N_0$ such that

$$\ell_n^2 k_n - 1 < \frac{1}{\delta'} - 1, \quad \forall n \geq N_1,$$

from which we have $\ell_n^2 k_n < \frac{1}{\delta'}$, $\forall n \geq N_1$. Put $\gamma = \frac{\delta}{\delta'}$. Then we have $a_n c_n \ell_n^2 k_n < \gamma$ for all $n \geq N_1$. □

Lemma 3.2. *Let X be a real uniformly convex Banach space and K be a nonempty closed, bounded and convex subset of X. Let $T : K \to K$ be a I-asymptotically nonexpansive mapping with $\{k_n\}$ a sequence of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $I : K \to K$ be an asymptotically nonexpansive mapping with $\{\ell_n\}$ a sequence of real numbers such that $\ell_n \geq 1$ and $\sum_{n=1}^{\infty}(\ell_n - 1) < \infty$. Suppose further that the set $F(T) \cap F(I)$ (i.e., $F(T) := \{x \in K : x = Tx\}, F(I) :=$*

$\{x \in K : x = Ix\}$) is nonempty. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$, such that $\{a_n+\mu_n\}$, $\{b_n+c_n+\nu_n\}$ and $\{\alpha_n+\beta_n+\lambda_n\}$ in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\nu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$, and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences in K defined by (3.1), then we have the following conclusions:

(1) *If q is a common fixed point of T and I, then $\lim_{n\to\infty}\|x_n - q\|$ exists.*

(2) *If $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$, then $\lim_{n\to\infty}\|T^n z_n - x_n\| = 0$.*

(3) *$0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then $\lim_{n\to\infty}\|I^n y_n - x_n\| = 0$.*

Proof. Let $q \in F(T)\cap F(I)$. Since $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K, there exists $\vartheta > 0$ such that $\max\{\sup_{n\geq 1}\|u_n - q\|^2, \sup_{n\geq 1}\|v_n - q\|^2, \sup_{n\geq 1}\|w_n - q\|^2\} \leq \vartheta$. Then

$$\begin{aligned}
\|z_n - q\|^2 &= \|a_n I^n x_n + (1-a_n-\mu_n)x_n + \mu_n u_n - q\|^2 \\
&\leq \|a_n(I^n x_n - q) + (1-a_n-\mu_n)(x_n - q) + \mu_n(u_n - q)\|^2 \\
&\leq a_n\|I^n x_n - q\|^2 + (1-a_n-\mu_n)\|x_n - q\|^2 + \mu_n\|u_n - q\|^2 \\
&\quad - w_2(a_n)g(\|I^n x_n - x_n\|) \\
&\leq a_n\ell_n^2\|x_n - q\|^2 + (1-a_n-\mu_n)\|x_n - q\|^2 + \mu_n\|u_n - q\|^2 \\
&\leq (1 + a_n\ell_n^2 - a_n - \mu_n)\|x_n - q\|^2 + \mu_n\vartheta \\
&= \left(1 + a_n\left(\ell_n^2 - 1\right) - \mu_n\right)\|x_n - q\|^2 + \mu_n\vartheta,
\end{aligned}$$

$$\begin{aligned}
\|y_n - q\|^2 &= \|(b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n-\nu_n)x_n + \nu_n v_n) - q\|^2 \\
&\leq b_n\|T^n z_n - q\|^2 + c_n\|T^n x_n - q\|^2 + (1-b_n-c_n-\nu_n)\|x_n - q\|^2 \\
&\quad + \nu_n\|v_n - q\|^2 - b_n(1-b_n-c_n)g_2(\|T^n z_n - x_n\|) \\
&\leq b_n k_n^2\|I^n z_n - q\|^2 + c_n k_n^2\|I^n x_n - q\|^2 + (1-b_n-c_n-\nu_n)\|x_n - q\|^2 \\
&\quad + \nu_n\|v_n - q\|^2 - b_n(1-b_n-c_n)g_2(\|T^n z_n - x_n\|) \\
&\leq b_n k_n^2\ell_n^2\|z_n - q\|^2 + c_n k_n^2\ell_n^2\|x_n - q\|^2 + (1-b_n-c_n-\nu_n)\|x_n - q\|^2 \\
&\quad + \nu_n\vartheta - b_n(1-b_n-c_n)g_2(\|T^n z_n - x_n\|) \\
&\leq b_n k_n^2\ell_n^2\left(1 + a_n\left(\ell_n^2 - 1\right) - \mu_n\right)\|x_n - q\|^2 \\
&\quad + \left(c_n k_n^2\ell_n^2 + (1-b_n-c_n-\nu_n)\right)\|x_n - q\|^2 \\
&\quad + b_n k_n^2\ell_n^2\mu_n\vartheta + \nu_n\vartheta - b_n(1-b_n-c_n-\nu_n)g(\|T^n z_n - x_n\|),
\end{aligned}$$

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\alpha_n I^n y_n + \beta_n I^n z_n + (1-\alpha_n-\beta_n-\lambda_n)x_n + \lambda_n w_n - q\|^2 \\
&\le \alpha_n \|I^n y_n - q\|^2 + \beta_n \|I^n z_n - q\|^2 + (1-\alpha_n-\beta_n-\lambda_n)\,\|x_n-q\|^2 \\
&\quad + \lambda_n \|w_n - q\|^2 - (1-\alpha_n-\beta_n-\lambda_n)\,(\alpha_n g(\|I^n y_n - x_n\|)) \\
&\le \alpha_n \ell_n^2 \|y_n - q\|^2 + \beta_n \ell_n^2 \|z_n - q\|^2 + (1-\alpha_n-\beta_n-\lambda_n)\,\|x_n-q\|^2 \\
&\quad + \lambda_n \|w_n - q\|^2 - \alpha_n\,(1-\alpha_n-\beta_n-\lambda_n)\,g(\|I^n y_n - x_n\|) \\
&\le \alpha_n \ell_n^2 \Big[b_n k_n^2 \ell_n^2 \|x_n - q\|^2 + c_n k_n^2 \ell_n^2 \|x_n - q\|^2 + (1-b_n-c_n-\nu_n)\|x_n-q\|^2 \\
&\quad + \alpha_n \ell_n^2 \left(b_n k_n^2 \ell_n^2 \mu_n \vartheta + \nu_n \vartheta \right) - b_n(1-b_n-c_n-\nu_n) g(\|T^n z_n - x_n\|) \Big] \\
&\quad + \beta_n \ell_n^2 \|z_n - q\|^2 + (1-\alpha_n-\beta_n-\lambda_n)\,\|x_n-q\|^2 + \lambda_n \vartheta \\
&\quad - \alpha_n\,(1-\alpha_n-\beta_n-\lambda_n)\,g(\|I^n y_n - x_n\|) \\
&\le \alpha_n \ell_n^2 \Big[b_n k_n^2 \ell_n^2 (1 + a_n \ell_n^2 - a_n - \mu_n) \|x_n - q\|^2 + c_n k_n^2 \ell_n^2 \|x_n - q\|^2 \\
&\quad + (1-b_n-c_n-\nu_n)\|x_n-q\|^2 - b_n(1-b_n-c_n-\nu_n) g_2(\|T^n z_n - x_n\|) \Big] \\
&\quad + \beta_n \ell_n^2 (1 + a_n \ell_n^2 - a_n - \mu_n) \|x_n - q\|^2 + (1-\alpha_n-\beta_n-\lambda_n)\,\|x_n-q\|^2 \\
&\quad + \lambda_n \vartheta + \alpha_n \ell_n^2 \left(b_n k_n^2 \ell_n^2 \mu_n \vartheta + \nu_n \vartheta \right) + \beta_n \mu_n \ell_n^2 \vartheta \\
&\quad - \alpha_n\,(1-\alpha_n-\beta_n-\lambda_n)\,g(\|I^n y_n - x_n\|) \\
&\le \Big[1 + c_n \alpha_n \ell_n^2 [k_n^2 \ell_n^2 - 1] + \alpha_n [\ell_n^2 - 1] + \beta_n [\ell_n^2 - 1] + \beta_n a_n \ell_n^2 [\ell_n^2 - 1] \\
&\quad + \alpha_n a_n b_n k_n^2 \ell_n^4 [\ell_n^2 - 1] + \alpha_n b_n \ell_n^2 [k_n^2 \ell_n^2 - 1] \Big] \|x_n - q\|^2 \\
&\quad + (1-b_n-c_n-\nu_n) g(\|T^n z_n - x_n\|) \\
&\quad - \alpha_n (1-\alpha_n-\beta_n-\lambda_n) g(\|I^n y_n - x_n\|) \\
&\le \Big(1 + c_n \alpha_n \ell_n^2 [(k_n^2-1)(\ell_n^2-1) + (k_n^2-1) + (\ell_n^2-1)] + \alpha_n [\ell_n^2 - 1] \\
&\quad + \beta_n [\ell_n^2 - 1] + \beta_n a_n \ell_n^2 [\ell_n^2 - 1] + \alpha_n a_n b_n k_n^2 \ell_n^4 [\ell_n^2 - 1] \\
&\quad + \alpha_n b_n \ell_n^2 [(k_n^2-1)(\ell_n^2-1) + (k_n^2-1) + (\ell_n^2-1)] \Big) \|x_n - q\|^2 \\
&\quad + \lambda_n \vartheta + \alpha_n \ell_n^2 (b_n k_n^2 \ell_n^2 \mu_n \vartheta + \nu_n \vartheta) + \beta_n \mu_n \ell_n^2 \vartheta \\
&\quad - \alpha_n \ell_n^2 b_n (1-b_n-c_n-\nu_n) g(\|T^n z_n - x_n\|) \\
&\quad - \alpha_n (1-\alpha_n-\beta_n-\lambda_n) g(\|I^n y_n - x_n\|).
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &\le \|x_n - q\|^2 + \{ (k_n^2-1)\{ c_n \alpha_n \ell_n^2 + \alpha_n b_n \ell_n^2 \} \\
&\quad + (\ell_n^2-1)\{ c_n \alpha_n \ell_n^2 + \alpha_n + \beta_n + \beta_n a_n \ell_n^2 + \alpha_n a_n b_n k_n^2 \ell_n^4 + \alpha_n b_n \ell_n^2 \} \\
&\quad + (k_n^2-1)(\ell_n^2-1)\{ c_n \alpha_n \ell_n^2 + \alpha_n b_n \ell_n^2 \} \} \|x_n - q\|^2 \\
&\quad + \lambda_n \vartheta + \alpha_n \ell_n^2 (b_n k_n^2 \ell_n^2 \mu_n \vartheta + \nu_n \vartheta) + \beta_n \mu_n \ell_n^2 \vartheta \\
&\quad - \alpha_n \ell_n^2 b_n (1-b_n-c_n-\nu_n) g(\|T^n z_n - x_n\|) \\
&\quad - \alpha_n (1-\alpha_n-\beta_n-\lambda_n) g(\|I^n y_n - x_n\|). \qquad (3.2)
\end{aligned}
$$

Since $\{k_n\}$, $\{\ell_n\}$ and K are bounded, there exists a constant $\Psi > 0$ such that

$$(c_n\alpha_n\ell_n^2 + \alpha_n b_n\ell_n^2)\|x_n - q\|^2 < \Psi,$$

$$(c_n\alpha_n\ell_n^2 + \alpha_n + \beta_n + \beta_n a_n\ell_n^2 + \alpha_n a_n b_n k_n^2\ell_n^4 + \alpha_n b_n\ell_n^2)\|x_n - q\|^2 < \Psi,$$

for all $n \geq 1$. By (3.2), we have

$$\begin{aligned}\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 &+ \Psi\{(k_n^2 - 1) + (\ell_n^2 - 1) + (k_n^2 - 1)(\ell_n^2 - 1)\}\\ &+ \lambda_n + A\nu_n + A((k_n^2 - 1)(\ell_n^2 - 1) + B)\mu_n\\ &- \alpha_n\ell_n^2 b_n(1 - b_n - c_n - \nu_n)g(\|T^n z_n - x_n\|)\\ &- \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|I^n y_n - x_n\|), \qquad (3.3)\end{aligned}$$

where $A = \sup\{\ell_n^2\vartheta : n \geq 1\}$ and $B = \sup\{(\ell_n^2 + k_n^2)\vartheta : n \geq 1\}$.

It follows that

$$\begin{aligned}&\alpha_n\ell_n^2 b_n(1 - b_n - c_n - \nu_n)g(\|T^n z_n - x_n\|)\\ &\quad\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \Psi\{(k_n^2 - 1) + (\ell_n^2 - 1) + (k_n^2 - 1)(\ell_n^2 - 1)\}\\ &\qquad+ \lambda_n + A\nu_n + A((k_n^2 - 1)(\ell_n^2 - 1) + B)\mu_n \qquad (3.4)\end{aligned}$$

and

$$\begin{aligned}&\alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|I^n y_n - x_n\|)\\ &\quad\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \Psi\{(k_n^2 - 1) + (\ell_n^2 - 1) + (k_n^2 - 1)(\ell_n^2 - 1)\}\\ &\qquad+ \lambda_n + A\nu_n + A((k_n^2 - 1)(\ell_n^2 - 1) + B)\mu_n. \qquad (3.5)\end{aligned}$$

(1) If q is a common fixed point of T and I, the assumption $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, $\sum_{n=1}^{\infty}(\ell_n^2 - 1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, $\sum_{n=1}^{\infty}(\ell_n^2 - 1) < \infty$, then it follows from (3.3) and Lemma 2.2 that $\lim_{n\to\infty}\|x_n - q\|$ exists.

(2) If $0 < \liminf_{n\to\infty}\alpha_n$ and $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$, then there exists a positive integer n_0 and $\eta, \delta, \acute{\delta} \in (0, 1)$ such that $0 < \delta < b_n, 0 < \eta < \alpha_n$ and $b_n + c_n + \nu_n < \acute{\delta} < 1$ for all $n \geq n_0$.

This implies by (3.4) that

$$\begin{aligned}&\eta\delta(1 - \acute{\delta})g(\|T^n z_n - x_n\|)\\ &\quad\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \Psi\{(k_n^2 - 1) + (\ell_n^2 - 1) + (k_n^2 - 1)(\ell_n^2 - 1)\}\\ &\qquad+ \lambda_n + A\nu_n + A((k_n^2 - 1)(\ell_n^2 - 1) + B)\mu_n \qquad (3.6)\end{aligned}$$

for all $n \geq n_0$. It follows from (3.6) that

$$\sum_{n=n_0}^{m} g(\|T^n z_n - x_n\|) \leq \frac{1}{\eta\delta(1-\delta)} \sum_{n=n_0}^{m} (\|x_n - q\|^2 - \|x_{n+1} - q\|^2) + \Psi \sum_{n=n_0}^{m} \{(k_n^2 - 1) + (\ell_n^2 - 1) + (k_n^2 - 1)(\ell_n^2 - 1)\} + \sum_{n=n_0}^{m} (\lambda_n + A\nu_n + A((k_n^2 - 1)(\ell_n^2 - 1) + B)\mu_n). \quad (3.7)$$

Let $m \to \infty$ in (3.7). Since $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $\sum_{n=1}^{\infty}(\ell_n^2 - 1) < \infty$,then we get $\sum_{n=n_0}^{m} g(\|T^n z_n - x_n\|) < \infty$. Therefore $\lim_{n\to\infty} g(\|T^n z_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T^n z_n - x_n\| = 0$.

(3) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then by using a similar method together with inequality (3.5), it is easy seen that $\lim_{n\to\infty} \|I^n y_n - x_n\| = 0$.

□

Lemma 3.3. *Let X be a real uniformly convex Banach space and K be a nonempty closed, bounded and convex subset of X. Let $T : K \to K$ be a I-asymptotically nonexpansive mapping with $\{k_n\}$ a sequence of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $I : K \to K$ be an asymptotically nonexpansive mapping with $\{\ell_n\}$ a sequence of real numbers such that $\ell_n \geq 1$ and $\sum_{n=1}^{\infty}(\ell_n - 1) < \infty$. Suppose further that the set $F(T)\cap F(I)$ (i.e., $F(T) := \{x \in K : x = Tx\}, F(I) := \{x \in K : x = Ix\}$) is nonempty. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$, such that $\{a_n + \mu_n\}$, $\{b_n + c_n + \nu_n\}$ and $\{\alpha_n + \beta_n + \lambda_n\}$ in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences in K defined by (3.1), then we have the following conclusions:*

(1) *If $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$, and*

(2) *$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,*

then $\lim_{n\to\infty} \|I^n x_n - x_n\| = 0 = \lim_{n\to\infty} \|T^n x_n - x_n\|$.

Proof. By Lemma 3.2 (2) and (3), we have

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0, \quad (3.8)$$

and

$$\lim_{n\to\infty} \|I^n y_n - x_n\| = 0. \quad (3.9)$$

From $z_n = a_n I^n x_n + (1 - a_n - \mu_n)x_n + \mu_n u_n$ and $y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \nu_n)x_n + \nu_n v_n$, we have

$$\begin{aligned}\|x_n - z_n\| &= \|a_n I^n x_n + (1 - a_n - \mu_n)x_n + \mu_n u_n - x_n\| \\ &\le a_n\|I^n x_n - x_n\| + \mu_n\|u_n - x_n\|, \end{aligned} \tag{3.10}$$

$$\begin{aligned}\|x_n - y_n\| &= \|b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \nu_n)x_n + \nu_n v_n - x_n\| \\ &\le b_n\|T^n z_n - x_n\| + c_n\|T^n x_n - x_n\| + \nu_n\|v_n - x_n\|. \end{aligned} \tag{3.11}$$

From (3.10), we have

$$\begin{aligned}\|T^n x_n - x_n\| &\le \|T^n x_n - T^n z_n\| + \|T^n z_n - x_n\| \\ &\le k_n\|I^n x_n - I^n z_n\| + \|T^n z_n - x_n\| \\ &\le k_n \ell_n\|x_n - z_n\| + \|T^n z_n - x_n\| \\ &\le k_n \ell_n a_n\|I^n x_n - x_n\| + k_n \ell_n \mu_n\|u_n - x_n\| + \|T^n z_n - x_n\|. \end{aligned} \tag{3.12}$$

Thus, from (3.11), we have

$$\begin{aligned}\|I^n x_n - x_n\| &\le \|I^n x_n - I^n y_n\| + \|I^n y_n - x_n\| \\ &\le \ell_n\|x_n - y_n\| + \|I^n y_n - x_n\| \\ &\le \ell_n b_n\|T^n z_n - x_n\| + \ell_n^2 k_n c_n a_n\|I^n x_n - x_n\| + \ell_n c_n\|T^n z_n - x_n\| \\ &\quad + \|I^n y_n - x_n\| + \ell_n^2 k_n c_n \mu_n\|u_n - x_n\| + \ell_n \nu_n\|v_n - x_n\| \\ &\le \ell_n(b_n + c_n)\|T^n z_n - x_n\| + \ell_n^2 k_n c_n a_n\|I^n x_n - x_n\| + \|I^n y_n - x_n\| \\ &\quad + \ell_n^2 k_n c_n \mu_n\|u_n - x_n\| + \ell_n \nu_n\|v_n - x_n\|. \end{aligned} \tag{3.13}$$

By Lemma 3.1, there exists a positive integer N_1 and $\gamma \in (0,1)$ such that $a_n c_n \ell_n^2 k_n < \gamma$ for all $n \ge N_1$. This together with (3.13) implies that for $n \ge N_1$

$$\begin{aligned}(1-\gamma)\|I^n x_n - x_n\| &< (1 - \ell_n^2 k_n c_n a_n)\|I^n x_n - x_n\| \\ &\le \ell_n(b_n + c_n)\|T^n z_n - x_n\| + \|I^n y_n - x_n\| \\ &\quad + \ell_n^2 k_n c_n \mu_n\|u_n - x_n\| + \ell_n \nu_n\|v_n - x_n\|. \end{aligned} \tag{3.14}$$

Taking limit of both sides (3.14), it follows from (3.8) and (3.9) that

$$\lim_{n\to\infty} \|I^n x_n - x_n\| = 0 \tag{3.15}$$

and taking limit of both sides (3.12), it follows from (3.8) and (3.15) that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.16}$$

□

Next, in order to prove the main result we need the following statement. Since

$$\begin{aligned}\|I^n z_n - x_n\| &\leq \|I^n z_n - I^n x_n\| + \|I^n x_n - x_n\| \\ &\leq \ell_n \|x_n - z_n\| + \|I^n x_n - x_n\| \\ &= \ell_n a_n \|I^n x_n - x_n\| + \|I^n x_n - x_n\| + \ell_n \mu_n \|u_n - x_n\|, \end{aligned} \tag{3.17}$$

taking limit of both sides (3.17), it follows from (3.15) that

$$\lim_{n\to\infty} \|I^n z_n - x_n\| = 0. \tag{3.18}$$

Theorem 3.4. *Let X be a real uniformly convex Banach space and K be a non-empty closed, bounded and convex subset of X. Let T, I be completely continuous asymptotically nonexpansive mappings with $\{k_n\} \subset [1,\infty)$, $\{\ell_n\} \subset [1,\infty)$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(\ell_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$, such that $\{a_n + \mu_n\}$, $\{b_n + c_n + \nu_n\}$ and $\{\alpha_n + \beta_n + \lambda_n\}$ in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K and*

1. *If* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$, *and*
2. $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,

Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences in K defined by (3.1). If $F(T) \cap F(I) \neq \emptyset$, then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to a common fixed point of T and I.

Proof. By (3.8), (3.9), (3.15), (3.16) and (3.18) in Lemma 3.3, we have

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0, \lim_{n\to\infty} \|I^n y_n - x_n\| = 0, \lim_{n\to\infty} \|I^n x_n - x_n\| = 0,$$

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0, \lim_{n\to\infty} \|I^n z_n - x_n\| = 0.$$

Since $x_{n+1} - x_n = \alpha_n(I^n y_n - x_n) + \beta_n(I^n z_n - x_n) + \lambda_n(w_n - x_n)$, we have

$$\begin{aligned}\|x_{n+1} - I^n x_{n+1}\| &\leq \|x_{n+1} - x_n\| + \|I^n x_n - I^n x_{n+1}\| + \|x_n - I^n x_n\| \\ &\leq (1 + \ell_n)\|x_{n+1} - x_n\| + \|x_n - I^n x_n\| \\ &\leq (1 + \ell_n)\alpha_n \|I^n y_n - x_n\| + (1 + \ell_n)\beta_n \|I^n z_n - x_n\| \\ &\quad + (1 + \ell_n)\lambda_n \|w_n - x_n\| + \|x_n - I^n x_n\|\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - T^n x_{n+1}\| &\leq \|x_{n+1} - x_n\| + \|T^n x_n - T^n x_{n+1}\| + \|x_n - T^n x_n\| \\ &\leq (1 + k_n \ell_n)\|x_{n+1} - x_n\| + \|x_n - T^n x_n\| \\ &\leq (1 + k_n \ell_n)\alpha_n \|I^n y_n - x_n\| + (1 + k_n \ell_n)\beta_n \|I^n z_n - x_n\| \\ &\quad + (1 + k_n \ell_n)\lambda_n \|w_n - x_n\| + \|x_n - I^n x_n\|.\end{aligned}$$

It follows from (3.9), (3.15), (3.16) and (3.18) in Lemma 3.3 that we obtain

$$\lim_{n\to\infty} \|x_{n+1} - I^n x_{n+1}\| = 0 \text{ and } \lim_{n\to\infty} \|x_{n+1} - T^n x_{n+1}\| = 0.$$

Thus

$$\begin{aligned} \|x_{n+1} - Ix_{n+1}\| &\le \|x_{n+1} - I^{n+1}x_{n+1}\| + \|Ix_{n+1} - I^{n+1}x_{n+1}\| \\ &\le \|x_{n+1} - I^{n+1}x_{n+1}\| + \ell_n \|x_{n+1} - I^n x_{n+1}\| \\ &\to 0 \quad as \quad n \to \infty \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + \|Tx_{n+1} - T^{n+1}x_{n+1}\| \\ &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1\ell_1 \|x_{n+1} - T^n x_{n+1}\| \\ &\to 0 \quad as \quad n \to \infty \end{aligned}$$

which imply that

$$\lim_{n\to\infty} \|x_n - Ix_n\| = 0 \tag{3.19}$$

and

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.20}$$

Since I, T are completely continuous and $\{x_n\} \subseteq K$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore, from (3.19), (3.20), $\{x_{n_k}\}$ converges. Let $\lim_{k\to\infty} x_{n_k} = q$. By the continuity of T, I and (3.19), (3.20) we have $Tq = q$ and $Iq = q$, so q is a common fixed point T and I. By Lemma 3.2 (1), $\lim_{n\to\infty} \|x_n - q\|$ exists. But $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$. Thus $\lim_{n\to\infty} \|x_n - q\| = 0$. Since

$$\|y_n - x_n\| \le b_n \|T^n z_n - x_n\| + c_n \|T^n x_n - x_n\| + \nu_n \|v_n - x_n\| \to 0 \quad as \quad n \to \infty$$

and

$$\|z_n - x_n\| \le a_n \|I^n x_n - x_n\| + \mu_n \|u_n - x_n\| \to 0 \quad as \quad n \to \infty,$$

it follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. □

Finally, we prove the weak convergence of the iterative scheme (3.1) for I-asymptotically nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.5. *Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed, bounded and convex subset of X. Let $T : K \to K$ be a I-asymptotically nonexpansive mapping with $\{k_n\}$ a sequence of real numbers such that $k_n \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $I : K \to K$ be an asymptotically nonexpansive mapping with $\{\ell_n\}$ a sequence of real numbers such that $\ell_n \ge 1$ and $\sum_{n=1}^{\infty}(\ell_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$*

and $\{\lambda_n\}$ *be sequences of real numbers in* $[0,1]$*, such that* $\{a_n+\mu_n\}$*,* $\{b_n+c_n+\nu_n\}$ *and* $\{\alpha_n+\beta_n+\lambda_n\}$ *in* $[0,1]$ *for all* $n \geq 1$*, and* $\sum_{n=1}^{\infty}\mu_n < \infty$*,* $\sum_{n=1}^{\infty}\nu_n < \infty$*,* $\sum_{n=1}^{\infty}\lambda_n < \infty$*, and* $\{u_n\}$*,* $\{v_n\}$ *and* $\{w_n\}$ *are bounded sequences in* K *and*

(1) *If* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n+c_n+\nu_n) < 1$*, and*

(2) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n+\beta_n+\lambda_n) < 1$*,*

Let $\{x_n\}$*,* $\{y_n\}$*,* $\{z_n\}$ *be the sequences in* K *defined by* (3.1)*. If* $F(T)\cap F(I) \neq \emptyset$*, then* $\{x_n\}$*,* $\{y_n\}$*,* $\{z_n\}$ *converge weakly to a common fixed point of* T *and* I*.*

Proof. It follows from Theorem 3.4 that $\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|x_n - Ix_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to q_1$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.1, we have $q_1 \in F(T)\cap F(I)$. We assume that q_1 and q_2 are weak limits of the subsequences $\{x_{n_k}\}$, $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By (3.19), (3.20) and $E-T$ and $E-I$ are demiclosed by Lemma 2.1, $Tq_1 = q_1$, $Iq_1 = q_1$ and in the same way, $Tq_2 = q_2$, $Iq_2 = q_2$. Therefore, we have $q_1, q_2 \in F(T)\cap F(I)$. By Lemma 3.2 (1), $\lim_{n\to\infty}\|x_n - q_1\|$ and $\lim_{n\to\infty}\|x_n - q_2\|$ exist. It follows from Lemma 2.5 that $q_1 = q_2$. Therefore $\{x_n\}$ converges weakly to a common fixed point of T and I. Moreover, $\lim_{n\to\infty}\|y_n - x_n\| = 0 = \lim_{n\to\infty}\|z_n - x_n\|$ as proved in Theorem 3.4 and $x_n \to q_1$ weakly as $n \to \infty$, therefore $y_n \to q_1$ weakly as $n \to \infty$ and $z_n \to q_1$ weakly as $n \to \infty$. This completes the proof. □

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(Received 9 November 2010)
(Accepted 19 March 2012)