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Non-Archimedean Fuzzy Versions of Approximate Ring Homomorphisms and Derivations

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Abstract : In this paper we define the non-Archimedean fuzzy normed algebras and consider the stability problem for approximate ring homomorphisms and ring derivations in the non-Archimedean fuzzy normed algebras.

Keywords : fuzzy normed space; non-Archimedean fuzzy normed space; approximate ring homomorphism; approximate ring derivation; stability.
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1 Introduction

The stability of functional equations is an interesting area of research for mathematicians, but it can be also of importance to persons who work outside of the realm of pure mathematics. It seems that the stability problem of functional equations had been first raised by Ulam [1]. Moreover the approximated mappings have been studied extensively in several papers, (see for instance [2–5]).

Fuzzy notion introduced firstly by Zadeh [6] that has been widely involved in different subjects of mathematics. Zadeh's definition of a fuzzy set characterized by a function from a nonempty set X to $[0, 1]$. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in the field of science and engineering. The fuzzy topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. Later in 1984, Katsaras [7] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. With [8] and by modifying the definition of a fuzzy normed space in [9], Mirmostafaee and Sal Moslehian in [10] introduced a notion of a non-Archimedean fuzzy normed space.

Defining non-Archimedean fuzzy normed algebra and the class of approximate solutions of a given functional equation one can ask whether every mapping from this class can be somehow approximated by an exact solution of the considered equation in the non-Archimedean fuzzy normed algebras. To answer this question, we use here the definition of non-Archimedean fuzzy normed algebras to exhibit some reasonable notions of fuzzy approximately ring homomorphisms in the non-Archimedean fuzzy normed algebras and we will prove that under some suitable conditions an approximately ring homomorphism f from an algebra X into the non-Archimedean fuzzy normed algebra Y can be approximated in a fuzzy sense by a homomorphism T from X to Y.

2 Preliminaries

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions.

Definition 2.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a *fuzzy norm on* X if for all $x, y \in X$ and all $t, s \in \mathbb{R}$,

(N1) $N(x, c) = 0$ for $c \leq 0$;

(N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;

(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;

(N5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;

(N6) for $x \neq 0$, $N(x, \cdot)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space.

Definition 2.2. Let (X, N) be a fuzzy normed linear space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

Definition 2.3. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and furthermore the fuzzy normed space is called a fuzzy Banach space.

Definition 2.4. Let X be an algebra and (X, N) be complete fuzzy normed space. The pair (X, N) is said to be a *fuzzy Banach algebra* if for every $x, y \in X$ and $s, t \in \mathbb{R}$ we have: $N(xy, st) \geq \min\{N(x, s), N(y, t)\}$.

Definition 2.5. Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|.| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

(1) $|a| \geq 0$ and equality holds if and only if $a = 0$;

(2) $|ab| = |a||b|$;

(3) $|a + b| \leq \max\{|a|, |b|\}$.

Definition 2.6. Let X be a linear space over a non-Archimedean field $\mathbb{K}$. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a *non-Archmedean fuzzy norm on* X if for all $x, y \in X$ and all $s, t \in \mathbb{K}$,

(N1) $N(x, c) = 0$ for $c \leq 0$;

(N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;

(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(NA4) $N(x + y, \max\{s, t\}) \geq \min\{N(x, s), N(y, t)\}$;

(N5) $\lim_{t\to\infty} N(x, t) = 1$.

The pair (X, N) is called a non-Archimedean fuzzy normed space. Clearly, if (NA4) holds then so is

(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$.

It is easy to see that (NA4) is equivalent to the following condition:

(NA4') $N(x + y, t) \geq \min\{N(x, t), N(y, t)\}$ for all $x, y \in X$ and all $t \in \mathbb{R}$.

The definitions of convergent sequence and cauchy sequence in a non-Archimedean fuzzy normed space are similar with Definitions 2.2 and 2.3. Due to

$$N(x_{n+p} - x_n, t) \geq \min\{N(x_{n+p} - x_{n+p-1}, t), ..., N(x_{n+1} - x_n, t)\}$$

the sequence $\{x_n\}$ is cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ we have $N(x_{n+1} - x_n, t) > 1 - \varepsilon$.

It is known that every convergent sequence in a non-Archimedean fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then the non-Archimedean fuzzy norm is said to be complete and furthermore the non-Archimedean fuzzy normed space is called a non-Archimedean fuzzy Banach space.

Theorem 2.7. *Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and f be a mapping from X to a non-Archimedean fuzzy Banach space Y. Suppose that φ is a function from $X \times X$ to a non-Archimedean fuzzy normed space (Z, N') such that*

$$N(f(x+y) - f(x) - f(y), t) \geq N'(\varphi(x,y), t).$$

for all $x, y \in X$ and $t > 0$. If for some $\alpha > 0$ and some positive integer k with $|k| < \alpha$,

$$N'(\varphi(k^{-1}x, k^{-1}y), t) \geq N'(\varphi(x,y), \alpha t)$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$N(T(x) - f(x), t) \geq M(x, \alpha t),$$

for all $x \in X$ and $t > 0$, where

$$M(x,t) = \min\{N'(\varphi(x,x),t), N'(\varphi(x,2x),t), ..., N'(\varphi(x,(k-1)x),t)\}$$

for all $x \in X$ and all $t > 0$.

Proof. See [11]. □

3 Non-Archimedean Fuzzy Stability of Approximately Ring Homomorphisms

We start our work with definition of non-Archimedean fuzzy normed algebras.

Definition 3.1. Let X be a non-Archimedean fuzzy normed space. X is said to be a *non-Archimedean fuzzy normed algebra* if it is satisfying the condition $N(xy, \min\{t,s\}) \geq \{N(x,t), N(y,s)\}$.

Theorem 3.2. *Let $\mathbb{K}$ be a non-Archimedean field, X an algebra over $\mathbb{K}$ and f be a mapping from X to a non-Archimedean fuzzy Banach algebra Y. Suppose that φ is a function from $X \times X$ to a non-Archimedean fuzzy normed algebra (Z, N') such that*

$$N(f(x+y) - f(x) - f(y), t) \geq N'(\varphi(x,y), t) \tag{3.1}$$

and

$$N(f(xy) - f(x)f(y), t) \geq N'(\varphi(x,y), t),$$

for all $x, y \in X$ and $t > 0$. If for some $\alpha > 0$ and some positive integer k with $|k| < \alpha$,

$$N'(\varphi(k^{-1}x, k^{-1}y), t) \geq N'(\varphi(x,y), \alpha t)$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique ring homomorphism $T : X \to Y$ such that

$$N(T(x) - f(x), t) \geq M(x, \alpha t), \tag{3.2}$$

for all $x \in X$ and $t > 0$, where

$$M(x,t) = \min\{N'(\varphi(x,x),t), N'(\varphi(x,2x),t), ..., N'(\varphi(x,(k-1)x),t)\}$$

for all $x \in X$ and all $t > 0$.

Proof. Theorem 2.7 shows that there exists an additive function $T : X \to Y$ such that the inequality (3.2) satisfied. Now we only need to show that T is a multiplicative map. Our inequality follows that

$$N(f(nx) - T(nx), t) \geq M(nx, \alpha t)$$

for all $x \in X$ and all $t > 0$. Thus

$$N(n^{-1}f(nx) - n^{-1}T(nx), n^{-1}t) \geq M(nx, \alpha t)$$

for all $x \in X$ and all $t > 0$. By the additivity of T it is easy to see that

$$N(n^{-1}f(nx) - T(x), t) \geq M(nx, n\alpha t) \tag{3.3}$$

for all $x \in X$ and all $t > 0$. By taking n tend to infinity in (3.3) we see that

$$T(x) = N - \lim_{n\to\infty} n^{-1}f(nx) \tag{3.4}$$

for all $x \in X$. Using inequality (3.1), we get

$$N(f((nx)y) - f(nx)f(y), t) \geq N'(\varphi(nx, y), t)$$

for all $x, y \in X$ and all $t > 0$. Thus

$$N(n^{-1}f((nx)y) - n^{-1}f(nx)f(y), t) \geq N'(\varphi(nx, y), nt) \tag{3.5}$$

for all $x, y \in X$ and all $t > 0$. By taking n tend to infinity in (3.5) we see that

$$N - \lim_{n\to\infty} n^{-1}f((nx)y) = N - \lim_{n\to\infty} n^{-1}f(nx)f(y). \tag{3.6}$$

Applying (3.4) and (3.6) we have

$$\begin{aligned} T(xy) &= N - \lim_{n\to\infty} n^{-1}f(n(xy)) = N - \lim_{n\to\infty} n^{-1}f((nx)y) \\ &= N - \lim_{n\to\infty} n^{-1}f(nx)f(y) = T(x)f(y) \end{aligned} \tag{3.7}$$

for all $x, y \in X$. From this equation by the additivity of T we have

$$T(x)f(ny) = T(x(ny)) = T((nx)y) = T(nx)f(y) = nT(x)f(y)$$

for all $x, y \in X$. Therefore,

$$T(x)n^{-1}f(ny) = T(x)f(y)$$

for all $x, y \in X$. Again by taking n tend to infinity and by (3.4) we see that

$$T(x)T(y) = T(x)f(y)$$

for all $x, y \in X$. Combining this formula with equation (3.7) we have that T is a ring homomorphism.

To prove the uniqueness property of T, assume that T' is another ring homomorphism satisfying (3.2). Since both T and T' are additive we deduce that

$$\begin{aligned} N(T(x) - T'(x), t) &= N(T(nx) - T'(nx), nt) \\ &\geq \min\{N(T(nx) - f(nx), nt/2), N(f(nx) - T'(nx), nt/2)\} \\ &\geq M(nx, \alpha nt/2) \end{aligned}$$

for all $x \in X$ and all $t > 0$. Taking n tend to infinity we find that

$$N(T(x) - T'(x), t) = 1$$

for all $x \in X$ and all $t > 0$. Hence $T(x) = T'(x)$ for all $x \in X$. □

Theorem 3.3. *Let $\mathbb{K}$ be a non-Archimedean field, X an algebra over $\mathbb{K}$ and f be a mapping from X to a non-Archimedean fuzzy Banach algebra Y. Suppose that φ is a function from $X \times X$ to a non-Archimedean fuzzy normed algebra (Z, N') such that*

$$N(f(x+y) - f(x) - f(y), t) \geq N'(\varphi(x, y), t)$$

and

$$N(f(x^2) - f(x)^2, t) \geq N'(\varphi(x, x), t),$$

for all $x, y \in X$ and $t > 0$. If for some $\alpha > 0$ and some positive integer k with $|k| < \alpha$,

$$N'(\varphi(k^{-1}x, k^{-1}y), t) \geq N'(\varphi(x, y), \alpha t)$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique additive Jordan mapping $T : X \to Y$ such that

$$N(T(x) - f(x), t) \geq M(x, \alpha t), \tag{3.8}$$

for all $x \in X$ and $t > 0$, where

$$M(x, t) = \min\{N'(\varphi(x, x), t), N'(\varphi(x, 2x), t), ..., N'(\varphi(x, (k-1)x), t)\}$$

for all $x \in X$ and all $t > 0$.

Proof. Theorem 2.7 shows that there exists an additive function $T : X \to Y$ such that (3.8) is satisfied. Now we only need to show that T is a Jordan map. If $a = 0$, since $T(0) = 0$ it is obvious. In the other case,

$$N(n^{-2}f(n^2a^2) - n^{-2}T(n^2a^2), n^{-2}t) \geq M(n^2a^2, \alpha t),$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. By the additivity of T it is easy to see that

$$N(n^{-2}f(n^2a^2) - T(a^2), t) \geq M(n^2a^2, \alpha tn^2), \tag{3.9}$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. Letting n tend to infinity in (3.9) we see that

$$T(a^2) = N - \lim_{n\to\infty} n^{-2}f(n^2a^2). \tag{3.10}$$

Also with a similar argument represented above shows that:

$$N(n^{-1}f(na) - T(a), t) \geq M(na, \alpha tn),$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. Hence we have

$$T(a) = N - \lim_{n\to\infty} n^{-1}f(na). \tag{3.11}$$

On the other hand,

$$N(f(n^2a^2) - f(na)^2, t) \geq N'(\varphi(na, na), t)),$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. We observe that

$$N(n^{-2}f(n^2a^2) - n^{-2}f(na)^2, t) \geq M(\varphi(na, na), n^2t),$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. So again by taking n tend to infinity we have

$$N - \lim_{n\to\infty} n^{-2}f(n^2a^2) = N - \lim_{n\to\infty} n^{-2}f(na)^2. \tag{3.12}$$

Applying (3.10), (3.11) and (3.12) we have

$$\begin{aligned} T(a^2) &= N - \lim_{n\to\infty} n^{-2}f(n^2a^2) \\ &= N - \lim_{n\to\infty} n^{-2}f(na)^2 \\ &= N - \lim_{n\to\infty} (n^{-1}f(na))^2 = T(a)^2. \end{aligned}$$

To prove the uniqueness property of T, assume that T' is another additive Jordan mapping satisfying

$$N(f(x) - T(x), t) \geq M(x, \alpha t).$$

Since both T and T' are additive we deduce that

$$N(T(a) - T'(a), t) \geq \min\{N(T(a) - n^{-1}f(na), t/2), N(n^{-1}f(na) - T'(a), t/2)\}$$

for all $a \in X$ and all $t > 0$. Letting n tend to infinity we find that $T(a) = T'(a)$ for all $a \in X$. □

Now we consider the stability of ring derivations in the non-Archimeden fuzzy normed algebras.

Theorem 3.4. *Let $\mathbb{K}$ be a non-Archimedean field, X an algebra over $\mathbb{K}$ and f be a mapping from X to a non-Archimedean fuzzy Banach algebra Y. Suppose that φ is a function from $X \times X$ to a non-Archimedean fuzzy normed algebra (Z, N') such that*

$$N(f(x+y) - f(x) - f(y), t) \geq N'(\varphi(x, y), t) \tag{3.13}$$

and

$$N(f(xy) - xf(y) - f(x)y, t) \geq N'(\varphi(x, y), t),$$

for all $x, y \in X$ and $t > 0$. If for some $\alpha > 0$ and some positive integer k with $|k| < \alpha$,

$$N'(\varphi(k^{-1}x, k^{-1}y), t) \geq N'(\varphi(x, y), \alpha t)$$

for all $x, y \in X$ and all $t > 0$. Then there exists a unique ring derivation $T : X \to Y$ such that

$$N(T(x) - f(x), t) \geq M(x, \alpha t), \tag{3.14}$$

for all $x \in X$ and $t > 0$, where

$$M(x, t) = \min\{N'(\varphi(x, x), t), N'(\varphi(x, 2x), t), ..., N'(\varphi(x, (k-1)x), t)\}$$

for all $x \in X$ and all $t > 0$.

Proof. Theorem 2.7 shows that there exists an additive mapping $T : X \to Y$ that is satisfied in (3.14). Now we only need to show that T is a ring derivation. Our inequality implies that

$$N(f(na) - T(na), t) \geq M(na, \alpha t),$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. By the additivity of T it is easy to see that

$$N(n^{-1}f(na) - T(a), t) \geq M(na, \alpha tn), \tag{3.15}$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. Letting n tend to infinity in (3.15) we see that

$$T(a) = N - \lim_{n\to\infty} n^{-1}f(na). \tag{3.16}$$

Using inequality (3.13), we get

$$N(f((na)b) - (na)f(b) - f(na)b, t) \geq N'(\varphi(na, b), t),$$

for all $a \in X$, $t > 0$ and $n \in \mathbb{N}$. Hence we have

$$N - \lim_{n\to\infty}(n^{-1}f((na)b) - n^{-1}naf(b) - n^{-1}f(na)b) = 1. \tag{3.17}$$

Now applying (3.16) in (3.17), we get

$$T(ab) = af(b) + T(a)b \tag{3.18}$$

for all $a, b \in X$.

Let $a, b \in X$ and $n \in \mathbb{N}$ be fixed. Then using (3.18) and the additivity of T, we have

$$\begin{aligned} af(nb) + nT(a)b &= af(nb) + T(a)nb = T(a(nb)) = T((na)b) \\ &= naf(b) + T(na)b = naf(b) + nT(a)b. \end{aligned}$$

Therefore,

$$af(b) = an^{-1}f(nb)$$

for all $a, b \in X$. By taking n tend to infinity and (3.16), we see that

$$af(b) = aT(b)$$

for all $a, b \in X$. Combining this formula with equation (3.18) we have

$$T(ab) = aT(b) + T(a)b$$

for all $a, b \in X$.

The proof of the uniqueness property of T is similar with the proof of the uniqueness in Theorem 3.2. □

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