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Dimension Formulae for Tensor-Product Spline Spaces with Homogeneous Boundary Conditions over Regular T-meshes[1]

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Abstract : A regular T-mesh is basically a rectangular grid that allows T-junctions over a rectangular domain. In this paper, we mainly study the dimension of bivariate tensor-product spline space $S^{\alpha,\beta}_{m,n}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ with homogeneous boundary conditions over a regular T-mesh $\mathcal{T}$. By using B-net method, we construct a minimal determining set for $S^{\alpha,\beta}_{m,n}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ by removing some unwanted domain points from the minimal determining set for $S^{\alpha,\beta}_{m,n}(\mathcal{T})$ given by Deng et al.. The new results are useful in the fields of computer aided geometric design, such as surface approximation, model design, and so on.

Keywords : bivariate tensor-product spline; regular T-mesh; dimension of spline space; homogeneous boundary condition; minimal determining set.
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1 Introduction

It is well known that the traditional tensor-product B-spline surfaces ([1, 2]) have been widely applied in the fields of computer aided geometric design (CAGD).

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However, it is well known that the control points of tensor-product B-spline surfaces must lie topologically in a rectangular grid, so it is not easy to modify these surfaces locally. This trouble is caused by the large number of superfluous control points arisen from the inherent mathematical properties of the traditional tensor-product B-splines. It is necessary for engineers to use a new kind of spline other than the traditional tensor-product B-splines to overcome the above-mentioned limitation.

Recently, many works have been done on this topic [3–10]. In particular, for example, Deng et al. introduced the so-called splines over T-meshes ([3]). Obviously, T-mesh is a generalization of a traditional rectangular grid. The splines over T-meshes, as polynomial splines different to the so-called T-splines ([11–14]) which are rational, have been already used in CAGD ([5, 9]). The new splines have many advantages over the traditional tensor-product B-splines and the so-called T-splines. In fact, the local refinement becomes very simple and easy if we apply this new kind of splines in CAGD. In order to found the general theory frame for the spline spaces over T-meshes, Deng et al. studied the dimensions for these linear spaces. Their results (see Theorem 2.1 in Section 2) only involve the topological quantities of the corresponding T-mesh ([3]).

In geometric modeling, the surfaces constrained by some boundary conditions are also important. Hence, it is necessary to study the new spline spaces. In this paper, we will pay attention to the spline spaces with homogeneous boundary conditions over regular T-meshes. The new spline space has not been well studied before. Only a recent paper ([4]) gives some simple results. In [4], the authors only consider a simple case, i.e. $S^{\alpha,\beta}_{m,n}(\mathcal{T};\alpha,\alpha,\beta,\beta)$. In this paper, based on the results in [3], we will construct the minimal determining sets for the general spline spaces $S^{\alpha,\beta}_{m,n}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ with homogeneous boundary conditions over regular T-meshes, so their dimensions are obtained. Our results are new and more general than [4] and can be applied in surface interpolation and fitting, geometric modeling, and so on.

The remainder of this paper is organized as follows: in Section 2, we give some preliminaries for spline spaces over T-meshes; in Section 3, we first present the definition of bivariate tensor-product spline spaces with homogeneous boundary conditions over regular T-meshes and the definition of minimal determining set, and then by using B-net method, we derive the dimension formulae for these spline spaces based on the cardinalities of the minimal determining sets, we also give an example to show our method; finally, we conclude this paper with some remarks in the last section.

2 Preliminaries

A T-mesh, i.e. a union of a set of rectangles, is essentially a rectangular grid that allows T-junctions arising from the so-called T-splines ([12, 13]). In other words, a T-mesh is formed by some horizontal grid line segments and some vertical grid line segments. If the domain occupied by the T-mesh is a rectangular domain, then we call it a regular T-mesh, see Figure 1.

Given a T-mesh, the subrectangles are called cells, the grid line segments are called edges and the grid points are called vertices. The edges that lie on the boundary of the T-mesh are called boundary edges; others are called interior edges. Furthermore, if a vertex lies on the boundary of the T-mesh, then we call it a boundary vertex; otherwise, we call it an interior vertex. For example, see Figure 1, $B_i (i = 1, 2, \ldots, 17)$ are boundary vertices, $V_i (i = 1, 2, 3)$ are interior vertices; V_1B_3 and V_2V_3 are interior edges, while B_5B_6 is a boundary edge.

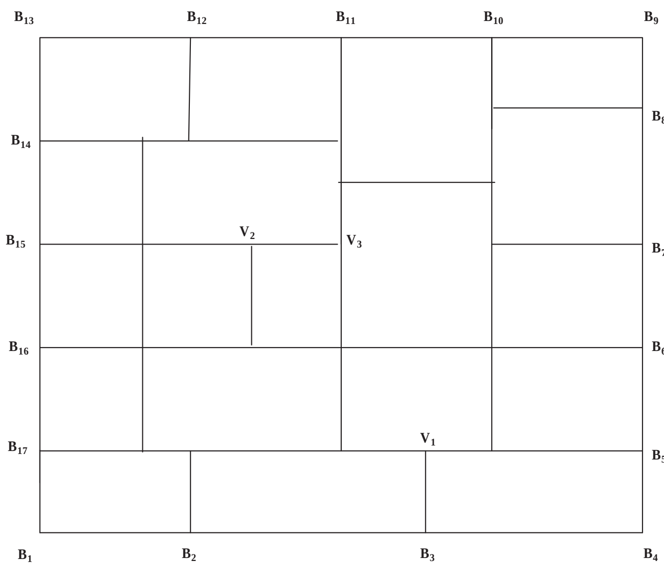

Figure 1. A regular T-mesh.

For a rectangular domain D, let $\mathcal{T}$ be a T-mesh over it, and D_i $(i = 1, 2, \ldots, N)$ be the cells. For nonnegative integers m, n and α, β, the tensor-product spline space over $\mathcal{T}$ is defined as follows ([3]):

$$S^{\alpha,\beta}_{m,n}(\mathcal{T}) = \{s(x, y) \in C^{\alpha,\beta}(D) \mid s(x, y)|_{D_i} \in P_{m,n}, i = 1, 2, \ldots, N\},$$

where $s(x, y)|_{D_i}$ denotes the restriction of $s(x, y)$ over D_i, $P_{m,n}$ is the space of all the polynomials with bi-degree (m, n) with respect to x and y respectively, and $C^{\alpha,\beta}(D)$ is the space of all the bivariate functions which are continuous in D with order α along x direction and with order β along y direction. In [3], by using the B-net method, a dimension formula for $S^{\alpha,\beta}_{m,n}(\mathcal{T})$ with $m \geq 2\alpha + 1$ and $n \geq 2\beta + 1$ is given:

Theorem 2.1 ([3]).

$$dim S^{\alpha,\beta}_{m,n}(\mathcal{T}) = N(m+1)(n+1)-E_h(m+1)(\beta+1)-E_v(\alpha+1)(n+1)+V(\alpha+1)(\beta+1),$$

where N is the number of cells in $\mathcal{T}$, E_h and E_v the number of interior horizontal edges and interior vertical edges respectively, and V the number of interior vertices.

3 Main Results

Let $D = [a, b] \times [c, d]$, for integers α_1, α_2 and β_1, β_2 with $-1 \leq \alpha_1, \alpha_2 \leq \alpha$, and $-1 \leq \beta_1, \beta_2 \leq \beta$, we give the definition of bivariate tensor-product spline space over $\mathcal{T}$ with homogeneous boundary conditions as following:

$$\begin{aligned} & S^{\alpha,\beta}_{m,n}(\mathcal{T}; \alpha_1, \alpha_2, \beta_1, \beta_2) \\ & \qquad := \{s(x, y) \mid s(x, y) \in S^{\alpha,\beta}_{m,n}(\mathcal{T}) \cap S_a(\alpha_1) \cap S_b(\alpha_2) \cap S_c(\beta_1) \cap S_d(\beta_2)\}, \end{aligned}$$

where

$$\begin{aligned} S_a(\alpha_1) &= \left\{ s(x, y) \mid \frac{\partial^{r_1} s(x, y)}{\partial x^{r_1}}|_{x=a} = 0,\ r_1 = 0, 1, \ldots, \alpha_1 \right\}, \\ S_b(\alpha_2) &= \left\{ s(x, y) \mid \frac{\partial^{r_2} s(x, y)}{\partial x^{r_2}}|_{x=b} = 0,\ r_2 = 0, 1, \ldots, \alpha_2 \right\}, \\ S_c(\beta_1) &= \left\{ s(x, y) \mid \frac{\partial^{r_3} s(x, y)}{\partial y^{r_3}}|_{y=c} = 0,\ r_3 = 0, 1, \ldots, \beta_1 \right\}, \\ S_d(\beta_2) &= \left\{ s(x, y) \mid \frac{\partial^{r_4} s(x, y)}{\partial y^{r_4}}|_{y=d} = 0,\ r_4 = 0, 1, \ldots, \beta_2 \right\}, \end{aligned}$$

and if one parameter is -1, we mean that there are no homogeneous conditions over the corresponding boundary.

Obviously, $S^{\alpha,\beta}_{m,n}(\mathcal{T}; \alpha_1, \alpha_2, \beta_1, \beta_2)$ is a linear subspace of $S^{\alpha,\beta}_{m,n}(\mathcal{T})$. In this section, we will give the dimension formula for $S^{\alpha,\beta}_{m,n}(\mathcal{T}; \alpha_1, \alpha_2, \beta_1, \beta_2)$ with $m \geq 2\alpha + 1$ and $n \geq 2\beta + 1$.

Given a spline $s(x, y) \in S^{\alpha,\beta}_{m,n}(\mathcal{T}; \alpha_1, \alpha_2, \beta_1, \beta_2) \subset S^{\alpha,\beta}_{m,n}(\mathcal{T}) \subset S^{0,0}_{m,n}(\mathcal{T})$, for a representative cell $D_k = [x^k_0, x^k_1] \times [y^k_0, y^k_1]$, let s_k denotes the restriction of $s(x, y)$ on D_k. We can express s_k in the Bernstein-Bézier form:

$$s_k = \sum_{i=0}^{m} \sum_{j=0}^{n} b^k_{i,j} B^m_i \left(\frac{x - x^k_0}{x^k_1 - x^k_0} \right) B^n_j \left(\frac{y - y^k_0}{y^k_1 - y^k_0} \right),$$

where

$$B^m_i(\cdot) = \frac{m!}{i!(m - i)!} (\cdot)^i (1 - \cdot)^{m-i},$$

$$B^n_j(\cdot) = \frac{n!}{j!(n - j)!} (\cdot)^j (1 - \cdot)^{n-j},$$

are the univariate Bernstein polynomials with degree m and n respectively, and $\{b^k_{i,j}\}$ $(i = 0, 1, \ldots, m; j = 0, 1, \ldots, n)$ are the Bézier ordinates. Each of the Bézier ordinate $b^k_{i,j}$ is associated with a domain point

$$P^k_{i,j} = \left(\frac{(m - i)x^k_0 + ix^k_1}{m}, \frac{(n - j)y^k_0 + jy^k_1}{n} \right).$$

Let $P(m,n,\mathcal{T})$ denotes the set of domain points. In fact, if given a spline, then there is a one-to-one correspondence between $\{b_{i,j}^k\}(i=0,1,\ldots,m; j=0,1,\ldots,n)$ and $P(m,n,\mathcal{T})$.

For any $p=P_{i,j}^k \in P(m,n,\mathcal{T})$, let λ_p be the linear functional on $S_{m,n}^{\alpha,\beta}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ defined by ([3, 15])

$$\lambda_p s(x,y)=b_{i,j}^k \quad (i=0,1,\ldots,m; j=0,1,\ldots,n; k=1,2,\ldots,N),$$

where $b_{i,j}^k$ is the Bézier ordinate of the spline $s(x,y)$ associated with the domain point $p=P_{i,j}^k$.

A set of domain points $Q\subset P(m,n,\mathcal{T})$ is called a determining set for $S_{m,n}^{\alpha,\beta}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ if and only if for any spline $s(x,y)\in S_{m,n}^{\alpha,\beta}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ and any $p\in Q$, $\lambda_p s(x,y)=0$ implying $s(x,y)\equiv 0$. If each of proper subsets of Q is not a determining set, then Q is called a minimal determining set. If the Bézier ordinates of a spline $s(x,y)$ associated with the domain points in Q are fixed, then the Bézier ordinates associated with the other domain points not in Q will be determined by the smoothness, and hence $s(x,y)$ is obtained. Moreover, Q is crucial for us to determine the dimension of $S_{m,n}^{\alpha,\beta}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ because $dim S_{m,n}^{\alpha,\beta}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ equals the cardinality of Q.

For a T-mesh $\mathcal{T}$ over $D=[a,b]\times[c,d]$, let H_1, H_2 be the numbers of the interior boundary vertices of $y=c$ and $y=d$ respectively, and V_1, V_2 the numbers of the interior boundary vertices of $x=a$ and $x=b$ respectively. See Fig.1, $H_1=2$, $H_2=3$, and $V_1=4$, $V_2=4$.

In the following, we construct a minimal determining set for $S_{m,n}^{\alpha,\beta}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ by removing some unwanted domain points from the special minimal determining set for $S_{m,n}^{\alpha,\beta}(\mathcal{T})$ given by Deng et al. in [3]. Deng's minimal determining set includes five parts of domain points. Considering the definition of $s(x,y)\in S_{m,n}^{\alpha,\beta}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$, we find an important fact that the homogeneous boundary conditions can only vanish the Bézier ordinates for the domain points in the second part, which are close to the four boundary edges. Take the boundary edge $y=c$ for example, the restrictions of the spline $s(x,y)$ on each of the H_1+1 cells near $y=c$ can be expressed in form of $(y-c)^{\beta_1+1}q(x,y)$, where $q(x,y)$ is a polynomial called smoothing cofactor ([6, 16]) with bi-degree $(m,n-\beta_1-1)$. So the Bézier ordinates of the domain points on the β_1+1 rows near $y=c$ in these H_1+1 cells are vanished by

$$\frac{\partial^{r_3}s(x,y)}{\partial y^{r_3}}|_{y=c}=0,\ r_3=0,1,\ldots,\beta_1.$$

Similarly, the Bézier ordinates of the domain points on the β_2+1 rows, the α_1+1 columns and the α_2+1 columns near $y=d$, $x=a$ and $x=b$ in the corresponding H_2+1 cells, V_1+1 cells and V_2+1 cells are vanished. So, if we remove these unwanted domain points from this part, then we get a minimal determining set for $S_{m,n}^{\alpha,\beta}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$. Counting these special points carefully near the four boundary edges and at the four corners of the T-mesh, we get the key total number of these unwanted domain points is:

$$\begin{aligned} d = {} & (H_1+1)(m-\alpha)(\beta_1+1) && \text{(near the bottom boundary)} \\ & +(H_2+1)(m-\alpha)(\beta_2+1) && \text{(near the top boundary)} \\ & +(V_1+1)(\alpha_1+1)(n-\beta) && \text{(near the left boundary)} \\ & +(V_2+1)(\alpha_2+1)(n-\beta) && \text{(near the right boundary)} \\ & +(\beta_1+1)(\alpha-\alpha_2) && \text{(at the bottom-right corner)} \\ & +(\alpha_2+1)(\beta-\beta_2) && \text{(at the top-right corner)} \\ & +(\beta_2+1)(\alpha-\alpha_1) && \text{(at the top-left corner)} \\ & +(\alpha_1+1)(\beta-\beta_1). && \text{(at the bottom-left corner)} \end{aligned} \tag{3.1}$$

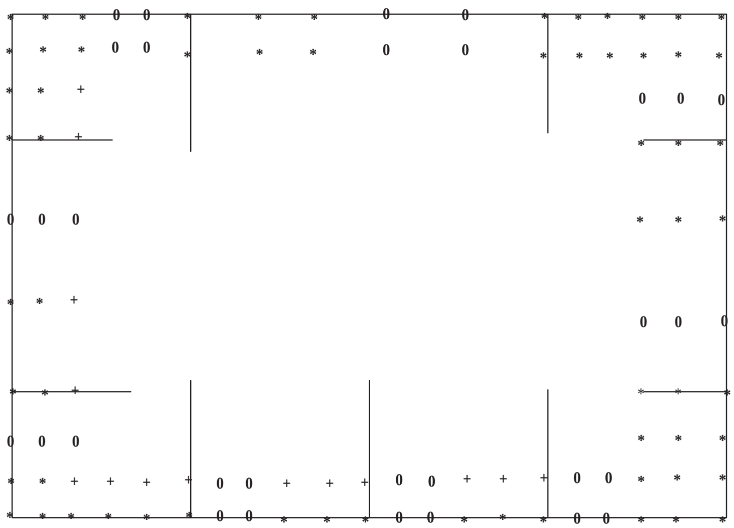

Figure 2. Boundary domain points for $S^{2,1}_{5,3}(\mathcal{T};1,2,0,1)$.
$m=5$, $n=3$; $\alpha=2$, $\beta=1$;
$H_1=3$, $\beta_1=0$; $H_2=2$, $\beta_2=1$;
$V_1=2$, $\alpha_1=1$; $V_2=2$, $\alpha_2=2$.

See Figure 2 for example. The domain points symbolized with "$*$" and "$+$" constitute the second part of Deng's minimal determining set for $S^{2,1}_{5,3}(\mathcal{T})$, and the number of domain points in this part is

$$d_2 := (H_1+H_2+2)(m-\alpha)(\beta+1)+(V_1+V_2+2)(\alpha+1)(n-\beta) = 78,$$

where the numbers of "$*$" and "$+$" are 64 and 14 respectively. The Bézier ordinates of the domain points symbolized with "0" can be determined by the C^2 and C^1 smoothing conditions along x and y direction respectively, so they are not in Deng's minimal determining set. The Bézier ordinates of the domain points symbolized with "$*$" are vanished by the additional homogeneous boundary conditions, hence they are the unwanted domain points and should be deleted from this part. By the above formula (3.1) of d, it is easy to get the number of " $*$ " is $d = 64$. Finally, only the domain points symbolized with " $+$ " are reserved, where the number of " $+$ " is $d_2 - d = 14$. We give the following theorem.

Theorem 3.1.

$$\begin{aligned} dimS^{\alpha,\beta}_{m,n}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2) &= dimS^{\alpha,\beta}_{m,n}(\mathcal{T}) - d \\ &= dimS^{\alpha,\beta}_{m,n}(\mathcal{T}) - (H_1+1)(m-\alpha)(\beta_1+1) \\ &\quad - (H_2+1)(m-\alpha)(\beta_2+1) - (V_1+1)(\alpha_1+1)(n-\beta) \\ &\quad - (V_2+1)(\alpha_2+1)(n-\beta) - (\beta_1+1)(\alpha-\alpha_2) \\ &\quad - (\alpha_2+1)(\beta-\beta_2) - (\beta_2+1)(\alpha-\alpha_1) \\ &\quad - (\alpha_1+1)(\beta-\beta_1) \end{aligned}$$

where H_1, H_2 are the numbers of the interior boundary vertices of $y = c$ and $y = d$ respectively, and V_1, V_2 the numbers of the interior boundary vertices of $x = a$ and $x = b$ respectively.

4 Remarks

In this section, we give some remarks.

- This paper is devoted to the dimension for spline space $S^{\alpha,\beta}_{m,n}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ with homogeneous boundary conditions over regular T-mesh with the constraints $m \geq 2\alpha+1$ and $n \geq 2\beta+1$.
- The dimension of $S^{\alpha,\beta}_{m,n}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ equals the cardinality of its minimal determining set Q. It is obtained by deleting some special points from the minimal determining set in [3]. See Section 3 for our methods. For further reading, also see [3].
- If $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = -1$, by (3.1), we have $d = 0$, it means that there are no homogeneous boundary conditions over $\mathcal{T}$, so we have

$$dimS^{\alpha,\beta}_{m,n}(\mathcal{T};-1,-1,-1,-1) = dimS^{\alpha,\beta}_{m,n}(\mathcal{T}).$$

- In [4], a dimension formula $dimS^{0,0}_{1,1}(\mathcal{T};0,0,0,0) = V^+$ is given. Noting $dimS^{0,0}_{1,1}(\mathcal{T}) = V^+ + V^b$ and $d = V^b = H_1 + H_2 + V_1 + V_2 + 4$, we remark that our Theorem 3.1 can also offer us the same results, where V^+ is the number of crossing vertices, and V^b the number of boundary vertices. Besides, the strategy used in [4] is linear space embedding with the operator of mixed partial derivative. We remark that it is more complicated than our method.
- In [4], $dimS^{1,1}_{2,2}(\mathcal{T};1,1,1,1)$ over a hierarchical T-mesh $\mathcal{T}$ is given. However, the results are valid only for hierarchical T-meshes. For $dimS^{1,1}_{2,2}(\mathcal{T};0,0,0,0)$ over a hierarchical T-mesh or a general regular T-mesh, readers should refer Theorem 3.1.

- Generally, $dimS^{\alpha,\beta}_{m,n}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ without the constraints $m \geq 2\alpha+1$ and $n \geq 2\beta+1$ is very difficult to obtained. In future, we will continue to study this problem by consulting some relative references. For instance, in [8], Li et al. have improved Deng's results with some relaxed constraint depending on the order of the smoothness, the degree of the spline and the structure of the T-mesh as well.
- Besides, the nonnegative and local supported basis splines for $S^{\alpha,\beta}_{m,n}(\mathcal{T};\alpha_1,\alpha_2,\beta_1,\beta_2)$ are also needed in CAGD, so this is also our future work.

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