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A New Approximation For Singular Inverse Sturm-Liouville Problem

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Abstract : In this paper, we study that inverse Sturm-Liouville problem having singularity of type $q(x) = \frac{\delta}{x^p} + q_0(x)$ at zero point. We prove that the difference between two potential functions $q(x)$ and $\tilde{q}$ (x) becomes sufficiently small whenever the spectral datas $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ and $\{\tilde{\lambda}_n, \tilde{\alpha}_n\}_{n=0}^{\infty}$ for $q(x)$ and $\tilde{q}$ (x) respectively are chosen sufficiently close to each other.

Keywords : spectrum; singular Sturm-Liouville operator; inverse problem; normalized constants.
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1 Introduction and Preliminaries

An Inverse spectral problem means a way to rebuild the potential from the spectral data. It is known that the spectral characteristics are spectra, spectral functions, scattering data, norming constansts, etc. Since the concept of inverse problem plays very important role in mathematics and physics, various scientists study on that concept. The first result in this subject, giving imputes for the further development of inverse problem theory was proved in [1]. Also inverse problems for regular and singular equations have been showed in the monographs in [2-11]. Later, Marchenko [12] has shown that the spectral data determine the Sturm-Liouville problem. Gelfand-Levitan [13] gave an algorithm for construction of $q(x), h$ and H. Later, Mizutani [14] improved a different algorithm, which is a slight modification of Gelfand-Levitan's model. In this paper, we studied algorithm

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for singular Sturm-Liouville problem. Let's give the main problem and necessary datas, as follows:

Now, considering $\{\lambda_n\}_{n=0}^{\infty}$ as a spectrum of the following singular Sturm-Liouville Problem:

$$Ly = -y^{''} + \left[\frac{\delta}{x^p} + q_0(x)\right] y = \mu y \quad (\mu = \lambda^2,\ 0 \leq x \leq \pi) \tag{1.1}$$

$$y(0) = 0 \tag{1.2}$$

$$y^{'}(\pi) - hy(\pi) = 0 \tag{1.3}$$

where the real potential $q_0(x)$ satisfies the condition

$$\int_0^{\pi} x|q(x)|dx < \infty \tag{1.4}$$

and δ constant, $q(x) \in L_2[0,\ \pi],\ 1 < p < 2$. Let us consider the second Sturm-Liouville Problem

$$\tilde{L}y = -y^{''} + \left[\frac{\delta}{x^p} + \tilde{q}_0(x)\right] y = \tilde{\mu} y \quad (\tilde{\mu} = \tilde{\lambda}^2,\ 0 \leq x \leq \pi) \tag{1.5}$$

with (1.2)-(1.3) hold, where $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ be spectrum of the (1.5) and the real potential $\tilde{q}(x) = \frac{\delta}{x^p} + \tilde{q}_0(x)$ satisfies the condition

$$\int_0^{\pi} x|\tilde{q}(x)|dx < \infty. \tag{1.6}$$

One has the following asymptotic formulas for solutions the problem (1.1-1.3), (see [6, 7]),

$$\Phi(x, \lambda) = \frac{\sin \lambda x}{\lambda} + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^{5-2p}}\right), \tag{1.7}$$

$$\Phi^{'}(x, \lambda) = \cos \lambda x + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^{4-2p}}\right). \tag{1.8}$$

Furthermore note that this type of inverse problems for singular Sturm-Liouville operator is investigated in which is also well known in [5-8] that

$$\lambda_n = n - \frac{1}{2} - \frac{h}{\pi(n - \frac{1}{2})} + \frac{a}{\pi(n - \frac{1}{2})} + O\left(\frac{1}{n^{4-2p}}\right), \tag{1.9}$$

$$\tilde{\lambda}_n = n - \frac{1}{2} - \frac{h}{\pi(n - \frac{1}{2})} + \frac{\tilde{a}}{\pi(n - \frac{1}{2})} + O\left(\frac{1}{n^{4-2p}}\right), \tag{1.10}$$

$$\alpha_n = \|\Phi_n\|^2 = \int_0^{\pi} \Phi_n^2(x)dx = \frac{\pi}{2(n - \frac{1}{2})^2} + O\left(\frac{\tau_n}{n^2}\right), \tag{1.11}$$

$$\tilde{\alpha}_n = \|\tilde{\Phi}_n\|^2 = \int_0^\pi \tilde{\Phi}_n^2(x)dx = \frac{\pi}{2(n-\frac{1}{2})^2} + O\left(\frac{\tilde{\tau}_n}{n^2}\right) \tag{1.12}$$

where for $\tau_n = \int_0^{\frac{2}{n}} t|q(t)|dt + \frac{1}{n}\int_{\frac{1}{2n}}^{\pi} |q(t)|dt + \frac{1}{n},\ \tilde{a} = \int_0^\pi \sin^2(n-\frac{1}{2})t\tilde{q}(t)dt.$

Theorem 1.1. *Let E be a linear topological space and $E_1, E_2 \subset E$. The transformation operator, $X = X_{L,\tilde{L}}$, mapping E_1 to E_2 can be realized as follows:*

$$X[\Phi(x,\lambda)] = \tilde{\Phi}(x,\lambda) = \Phi(x,\lambda) + \int_0^x K(x,s)\Phi(s,\lambda)ds \tag{1.13}$$

The kernel of operator (1.13) is a solution of the differential equation

$$\frac{\partial^2 K(x,s)}{\partial x^2} - \tilde{q}(x)K(x,s) = \frac{\partial^2 K(x,s)}{\partial s^2} - q(s)K(x,s) \tag{1.14}$$

and also satisfies the following conditions

$$K(x,x) = \frac{1}{2}\int_0^x (\tilde{q}_0(s) - q_0(s))ds,\ K(x,0) = 0. \tag{1.15}$$

Lemma 1.2. *There exists a constant $M > 0$, such that*

$$|\Phi(x,\lambda)| + \frac{|\Phi'(x,\lambda)|}{\lambda} \leq M, \tag{1.16}$$

$$\lambda|\dot{\Phi}(x,\lambda)| + \dot{\Phi}'(x,\lambda) \leq M, \tag{1.17}$$

hold for every $\lambda \geq 1$ and $0 \leq x \leq \pi$, ($\dot{\Phi} = \frac{d\Phi}{d\lambda}$).

Proof. The right side of inequality is limited, by virtue of $0 \leq x \leq \pi$ and $-1 \leq \cos x \leq 1$, $-1 \leq \sin x \leq 1$.

$$|\Phi(x,\lambda)| + \frac{|\Phi'(x,\lambda)|}{\lambda} \leq \left|\frac{\sin\lambda x}{\lambda} + O\left(\frac{|e^{Im\lambda x|}}{\lambda^{5-2p}}\right)\right| + \left|\frac{\cos\lambda x}{\lambda} + O\left(\frac{e^{|Im\lambda x|}}{\lambda^{5-2p}}\right)\right| \tag{1.18}$$

Therefore there exists a constant $M > 0$ such that

$$|\Phi(x,\lambda)| + \frac{|\Phi'(x,\lambda)|}{\lambda} \leq M.$$

We can obtain inequality (1.16), in a similar way. □

Theorem 1.3. *Letting*

$$F(x,s) = \sum_{n=0}^{\infty}\left[\frac{\Phi(x,\tilde{\lambda}_n)\Phi(s,\tilde{\lambda}_n)}{\tilde{\alpha}_n} - \frac{\Phi(x,\lambda_n)\Phi(s,\lambda_n)}{\alpha_n}\right] \tag{1.19}$$

then, we obtain

$$K(x,s) + \int_0^x K(x,t)F(t,s)dt + F(x,s) = 0 \quad for\ 0 \leq s \leq x \leq \pi. \tag{1.20}$$

Mizutani [14] showed the uniqueness of the potential function for Sturm Liouville problem according to normalizing constants and eigenvalues. The purpose of our study is to give the sctructure concerning the difference $q(x)-\tilde{q}(x)$ for the differential operators having the singularity type $\frac{\delta}{x^p}+q_0(x)$, by using Mizutani method. Now, let's give the main theorem and its proof.

2 Main Results

Theorem 2.1. *Let us consider the equality*

$$A \equiv \sum_{n=0}^{\infty}[|\tilde{\alpha}_n-\alpha_n|+|\tilde{\lambda}_n^2-\lambda_n^2|]. \tag{2.1}$$

If it is sufficiently small, then we get

$$\max_{0\le x\le\pi}|\tilde{q}_0(x)-q_0(x)|\le C_1A^{''} \tag{2.2}$$

where $C>0$ is a constant depending only on $q(x)$ and h.

Proof. Let us solve the integral equation (1.19). Firstly let us start with $F(s,t)$, then construct the iterated kernels $F^{(n)}(s,t;x)$, (n=1,2,...). We get

$$F^{(1)}(s,t;x)=F(s,t),\quad F^{(n+1)}(s,t;x)=\int_0^x F(s,u)F^(n)(u,t;x)\;du,\;\; n\ge 1. \tag{2.3}$$

Now let us take

$$S(s,t;x)=\sum_{n=1}^{\infty}(-1)^n\;F^{(n)}(s,t;x),$$

and assuming that

$$\int_0^\pi\int_0^\pi\;|F(s,t)|^2\;ds\,dt<1. \tag{2.4}$$

It is easy to see that

$$K(x,s)=S(x,s;x)\;\; for\;\, 0\le s\le x\le\pi. \tag{2.5}$$

Then it follows from (1.15) that

$$\frac{1}{2}(q_0(x)-\tilde{q}_0(x))=-\frac{dK(x,x)}{dx}. \tag{2.6}$$

Let's give the following Lemma to complete the proof of the Theorem 2.1. □

Lemma 2.2. *Let us consider $F(x,s)$ defined by (1.19). Then we get*

$$\frac{1}{2}(q_0(x)-\tilde{q}_0(x))=\frac{dF(x,x)}{dx}-K^2(x,x)+2\int_0^x F_x(x,u)K(x,u)du \tag{2.7}$$

where $F(x,s)$ has a continuous derivative and the condition (2.4) is satisfied.

Proof. Using formula (2.3)-(2.6), we obtain the following equation

$$\frac{1}{2}(q_0(x) - \tilde{q}_0(x)) = -\frac{d}{dx}\left(\sum_{n=1}^{\infty}(-1)^n \frac{d}{dx}F^{(n)}(x,x;x)\right)$$

$$\frac{1}{2}(q_0(x) - \tilde{q}_0(x)) = \frac{dF(x,x)}{dx} + \sum_{n=1}^{\infty}(-1)^n \frac{d}{dx}F^{(n+1)}(x,x;x). \tag{2.8}$$

Now, we estimate

$$\begin{aligned}\frac{d}{dx}F^{(n+1)}(x,x;x) &= \left\{F_s^{(n+1)} + F_t^{(n+1)} + F_x^{(n+1)}\right\}_{s=x,t=x} \\ &= 2\int_0^x F_x(x,u)F^{(n)}(x,u;x)du \\ &\quad + \sum_{k=1}^{n} F^{(k)}(x,x;x)F^{(n+1-k)}(x,x;x). \end{aligned}\tag{2.9}$$

Using equation (2.9) in (2.8), the following equation

$$\begin{aligned}\frac{1}{2}(q_0(x) - \tilde{q}_0(x)) &= \frac{dF(x,x)}{dx} + 2\int_0^x F_x(x,u)K(x,u)du \\ &\quad + \sum_{n=1}^{\infty}(-1)^n \sum_{n=1}^{\infty} F^{(k)}(x,x;x)F^{(n+1-k)}(x,x;x) \\ &= \frac{dF(x,x)}{dx} + 2\int_0^x F_x(x,u)K(x,u)du - K^2(x,x)\end{aligned}$$

is obtained.

Now, we can prove the main theorem. Let us take A_0 as follows:

$$A_0 = \inf_n \alpha_n. \tag{2.10}$$

A_0 is positive from the asymptotic formula (1.9)-(1.12). We suppose that

$$A = \sum_{n=0}^{\infty}[|\tilde{\alpha}_n - \alpha_n| + |\tilde{\lambda}_n^2 - \lambda_n^2|] \leq A_0. \tag{2.11}$$

Then we get

$$\alpha_n \geq 2A_0 \quad \text{and} \quad \tilde{\alpha}_n \geq A_0. \tag{2.12}$$

Differentiating formally the right side of (1.19) with respect to x, we obtain the following equation

$$F_x(x,s) = \sum_{n=1}^{\infty}\left[\frac{\Phi'(x,\tilde{\lambda}_n)\Phi(s,\tilde{\lambda}_n)}{\tilde{\alpha}_n} - \frac{\Phi'(x,\lambda_n)\Phi(s,\lambda_n)}{\alpha_n}\right].$$

Adding and subtracting $\frac{\Phi^{'}(x,\lambda)_n\Phi(s,\lambda_n)}{\tilde{\alpha}_n}$ to the right side of the last equation, we obtain

$$F_x(x,s)=\sum_{n=1}^{\infty}\left[\left(\frac{\alpha_n-\tilde{\alpha}_n}{\alpha_n\tilde{\alpha}_n}\right)\Phi^{'}(x,\tilde{\lambda_n})\Phi^{'}(s,\tilde{\lambda}_n)+\frac{1}{\tilde{\alpha}_n}\int_{\lambda_n}^{\tilde{\lambda}_n}(\Phi^{'}(x,\lambda)\Phi(s,\lambda))^{\cdot}d\lambda\right].$$

By virtue of (2.11), (2.12) and Lemma 1.2, it is seen that $F(x,s)$ has a continuous derivative and

$$|F_x(x,s)|\leq C^{'}\sum_{n=0}^{\infty}\left[|\tilde{\alpha}_n-\alpha_n|+|\tilde{\lambda}_n^2-\lambda_n^2|\right]\equiv C^{'}A. \tag{2.13}$$

We can also write

$$\left|\frac{d}{dx}F(x,x)\right|\leq 2C^{'}A. \tag{2.14}$$

Using the same method, we obtain

$$F(x,s)=\sum_{n=0}^{\infty}\left[\left(\frac{\alpha_n-\tilde{\alpha}_n}{\alpha_n\tilde{\alpha}_n}\right)\Phi^2(x,\tilde{\lambda}_n)+\frac{1}{\tilde{\alpha}_n}\int_{\lambda_n}^{\tilde{\lambda}_n}(2\dot{\Phi}(x,\lambda)\Phi(s,\lambda))d\lambda\right].$$

By means of formulas (1.7), (1.8) for $0\leq x\leq\pi$ and $-1\leq\cos x\leq 1$, $-1\leq\sin x\leq\ 1$,

$$\begin{aligned}&|F(x,x)|\\&=\left|\sum_{n=0}^{\infty}\left(\frac{\alpha_n-\tilde{\alpha}_n}{\alpha_n\tilde{\alpha}_n}\right)\left(\frac{\sin\lambda x}{\lambda}+O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^{5-2p}}\right)\right)^2\right|\\&\quad+\frac{1}{\tilde{\alpha}_n}\int_{\lambda_n}^{\tilde{\lambda}_n}2\left[\frac{x\cos\lambda x-\sin\lambda x}{\lambda^2}+O\left(\frac{xe^{|Im\lambda|x}}{|\lambda|^{5-2p}}\right)\right]\left(\frac{\sin\lambda s}{\lambda}+O\left(\frac{e^{|Im\lambda|s}}{|\lambda|^{5-2p}}\right)\right)d\lambda.\end{aligned}$$

It follows from the last equation, we have

$$\begin{aligned}|F(x,x)|&\leq\sum_{n=0}^{\infty}\left|\left(\frac{\alpha_n-\tilde{\alpha}_n}{\alpha_n\tilde{\alpha}_n}\right)c_1+\frac{1}{\tilde{\alpha}_n}\int_{\lambda_n}^{\tilde{\lambda}_n}2\left(\frac{\pi}{\lambda^2}-\frac{1}{\lambda^3}\right)c_2d\lambda\right|\\&=C^{''}\sum_{n=0}^{\infty}\left[|(\tilde{\alpha}_n-\alpha_n)|+|\tilde{\lambda}_n^2-\lambda_n^2|\right].\end{aligned} \tag{2.15}$$

From (2.15), we obtain

$$|F(x,x)|\leq C^{''}A, \tag{2.16}$$

where $C^{'}$ and $C^{''}$ are constants depending only on $q(x)$ and h. If $\pi C^{''}A$ is sufficiently small, e.g. $\pi C^{''}A<\frac{1}{2}$ using formula (2.5), we can write the equation

$$|K(x,s)|=\sum_{n=0}^{\infty}(-1)^nF^{(n)}(x,s;x). \tag{2.17}$$

Because of formula (2.3), we construct the iterated kernels$F^{(n)}$ as follows:

$$|F^{(1)}(x,x)| = |F(x,x)| \leq C^{''}A$$
$$|F^{(2)}(x,x)| = \left|\int_0^x FF^{(1)}du\right| = \left|\int_0^x FFdu\right| \leq \left|\int_0^x (C^{''}A)^2 du\right| = (C^{''}A)^2\pi$$
$$\vdots$$
$$|F^{(n)}| \leq \frac{1}{\pi}(\pi C^{''}A)^n. \tag{2.18}$$

Using (2.18) in (2.17), we have

$$|K(x,s)| \leq \left|\sum_{n=1}^{\infty}\frac{1}{\pi}(\pi C^{''}A)^n\right| \leq 2C^{''}A. \tag{2.19}$$

By virtue of Lemma 2.2 and using (2.13)-(2.19), consequently, we obtain

$$|\tilde{q}_0(x) - q_0(x)| \leq C_1\sum_{n=0}^{\infty}\left[|\tilde{\alpha}_n - \alpha_n| + |\tilde{\lambda}_n^2 - \lambda_n^2|\right] \tag{2.20}$$

for $A \leq \min\{A_0, (2\pi C^{''})^{-1}\}$. This completes the proof. □

3 Conclusion

The more norming constants and spectrums which are taken as spectral data of singular inverse problem is close to each other, the more the potential difference of these two problems is small, sufficiently.

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