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Multihomomorphisms from Groups into Groups of Real Numbers

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Abstract : By a *multihomomorphism* from a group G into a group G' we mean a multifunction f from G into G' such that

$$f(xy) = f(x)f(y) \ \big(= \{st \mid s \in f(x) \text{ and } t \in f(y)\}\big)$$

for all $x, y \in G$. We denote by $\mathrm{MHom}(G, G')$ the set of all multihomomorphisms from G into G'. It is shown that if $f \in \mathrm{MHom}(G, G')$ where G' is a subgroup of $(\mathbb{R}, +)$, then either f is a homomorphism or there is an infinite cardinal number η such that $|f(x)| = \eta$ for all $x \in G$. If $f \in \mathrm{MHom}(G, G')$ where G' is a subgroup of $(\mathbb{R}^*, \cdot)$, then (i) f is a homomorphism, (ii) $|f(x)| = 2$ for all $x \in G$ or (iii) there is an infinite cardinal number η such that $|f(x)| = \eta$ for all $x \in G$.

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1 Introduction

The cardinality of a set X will be denoted by $|X|$. A *multifunction* from a nonempty set X into a nonempty set Y is a function $f : X \to P^*(Y)$ where $P(Y)$ is the power set of Y and $P^*(Y) = P(Y)\backslash\{\emptyset\}$.

Upper and lower semicontinuity of multifunctions between two topological spaces were studied by Whyburn [4] and Smithson [2]. Feichtinger [1] also gave a characterization of lower semi-continuous multifunctions. The authors in [3] were motivated by these works to study multifunctions in an algebraic sense. "Multihomomorphisms" between cyclic groups were characterized in [3]. The definition of a multihomomorphism between groups was given naturally as follows :

A multifunction f from a group G into a group G' is called a *multihomomorphism* if

$$f(xy) = f(x)f(y) \ \big(= \{st \mid s \in f(x) \text{ and } t \in f(y)\}\big) \text{ for all } x, y \in G$$

and let $\mathrm{MHom}(G, G')$ denote the set of all multihomomorphisms from G into G'. Then every homomorphism from G into G' belongs to $\mathrm{MHom}(G, G')$ and for $f \in \mathrm{MHom}(G, G')$, f is a homomorphism if and only if $|f(x)| = 1$ for all $x \in G$.

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. Then $(\mathbb{R}, +)$ and $(\mathbb{R}^*, \cdot)$ are abelian groups where $+$ and $\cdot$ are the usual addition and multiplication, respectively. Observe that there are infinitely many subgroups of $(\mathbb{R}, +)$ and of $(\mathbb{R}^*, \cdot)$. Let $\mathbb{R}^+$ denote the set of positive real numbers.

Our purpose is to give remarkable necessary conditions of $f \in \mathrm{MHom}(G, G')$ where G' is a subgroup of $(\mathbb{R}, +)$ and $(\mathbb{R}^*, \cdot)$. It will be shown that if $f \in \mathrm{MHom}(G, G')$ where G' is a subgroup of $(\mathbb{R}, +)$, then either

(i) f is a homomorphism or

(ii) there is an infinite cardinal number η such that $|f(x)| = \eta$ for all $x \in G$.

Also, if $f \in \mathrm{MHom}(G, G')$ where G' is a subgroup of $(\mathbb{R}^*, \cdot)$, then one of the following statements holds.

(i) f is a homomorphism.

(ii) For every $x \in G$, $f(x) = \{-x', x'\}$ for some $x' \in G'$.

(iii) There is an infinite cardinal number η such that $|f(x)| = \eta$ for all $x \in G$.

2 Main Results

To obtain our main results, the following two lemmas are needed.

Lemma 2.1 *Let G and G' be groups. Then for every $f \in MHom(G, G')$, $|f(x)| = |f(e)|$ for all $x \in G$ where e is the identity of G.*

Proof. Let $f \in \mathrm{MHom}(G, G')$ and $x \in G$. Since $|f(x^{-1})| \geq 1$ and G' is cancellative, we have

$$|f(e)| = |f(xx^{-1})| = |f(x)f(x^{-1})| \geq |f(x)| = |f(xe)| = |f(x)f(e)| \geq |f(e)|,$$

so $|f(x)| = |f(e)|$. □

Lemma 2.2 *Let G be a group with identity e and G' a subgroup of $(\mathbb{R}^*, \cdot)$. If $f \in MHom(G, G')$ is such that $|f(e)| = 2$, then*

(i) $f(e) = \{-1, 1\}$ and

(ii) for every $x \in G$, $f(x) = \{-x', x'\}$ for some $x' \in G'$.

Proof. (i) Let $f(e) = \{a, b\}$ with $a < b$. Then

$$\{a, b\} = f(e) = f(e)f(e) = \{a^2, ab, b^2\}. \tag{2.1}$$

Since $a < b$, it follows that $a^2 < ab < b^2$ if $a > 0$ and $a^2 > ab > b^2$ if $b < 0$. From this fact and (2.1), we deduce that $a < 0$ and $b > 0$. Hence $ab = a$ and $a^2 = b^2 = b$ which imply that $b = 1$ and $a = -1$.

(ii) Let $x \in G$. Then $|f(x)| = 2$ by Lemma 2.1. Let $f(x) = \{y, z\}$ with $y < z$ in G'. Then

$$\{y, z\} = f(x) = f(x)f(e) = \{y, z\}\{-1, 1\} = \{-y, -z, y, z\}. \tag{2.2}$$

Since $0 \neq y < z$, we have $y \neq -y > -z$, so by (2.2), $-y = z$. Hence $-z = y < z$. Consequently, $f(x) = \{-z, z\}$. □

Theorem 2.3 *Let G be a group and G' a subgroup of $(\mathbb{R}, +)$. If $f \in MHom$ (G, G'), then either*

(i) f is a homomorphism or

(ii) there is an infinite cardinal number η such that $|f(x)| = \eta$ for all $x \in G$.

Proof. Let $f \in \mathrm{MHom}(G, G')$ and assume that f is not a homomorphism from G into G'. Then $|f(e)| > 1$ by Lemma 2.1 where e is the identity of G. Suppose that $f(e)$ is a finite subset of $\mathbb{R}$, say $f(e) = \{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$ and $n \geq 2$. Then $a_1 + a_1 < a_2 + a_1 < \ldots < a_n + a_1 < a_n + a_n$ and hence

$$\begin{aligned} n = |\{a_1, a_2, \ldots, a_n\}| = |f(e)| &= |f(e) + f(e)| \\ &= |\{a_1, a_2, \ldots, a_n\} + \{a_1, a_2, \ldots, a_n\}| \\ &\geq |\{a_1 + a_1, a_2 + a_1, \ldots, a_n + a_1, a_n + a_n\}| \\ &= n + 1, \end{aligned}$$

a contradiction. Hence $|f(e)| = \eta$ for some infinite cardinal number η, and by Lemma 2.1, $|f(x)| = \eta$ for all $x \in G$. □

Theorem 2.4 *Let G be a group and G' a subgroup of $(\mathbb{R}^*, \cdot)$. If $f \in MHom(G, G')$, then f satisfies one of the following conditions.*

(i) f is a homomorphism.

(ii) For every $x \in G$, $f(x) = \{-x', x'\}$ for some $x' \in G'$.

(iii) There is an infinite cardinal number η such that $|f(x)| = \eta$ for all $x \in G$.

Proof. Let $f \in \mathrm{MHom}(G, G')$ and assume that f is not a homomorphism from G into G'. It follows from Lemma 2.1, $|f(e)| > 1$. If $|f(e)| = 2$, then by Lemma 2.2, f satisfies (ii). Assume that $|f(e)| > 2$. To show that f satisfies (iii), suppose not. By lemma 2.1, $f(e)$ is a finite subset of $\mathbb{R}^*$, say $f(e) = \{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$ and $n > 2$. Since $f(e)f(e) = f(e)$, we have $\{a_1, a_2, \ldots, a_n\}\{a_1, a_2, \ldots, a_n\} = \{a_1, a_2, \ldots, a_n\}$. It follows that

$$\big\{a_i a_j \mid i, j \in \{1, 2, \ldots, n\}\big\} = \{a_1, a_2, \ldots, a_n\} \tag{2.3}$$

and

$$|\big\{a_i a_j \mid i, j \in \{1, 2, \ldots, n\}\big\}| = n. \tag{2.4}$$

Case 1: $0 < a_1 < \cdots < a_n$. Then $a_1^2 < a_1a_2 < \cdots < a_1a_n < a_n^2$, so $|\{a_1^2, a_1a_2, \ldots, a_1a_n, a_n^2\}| = n+1$ which is contrary to (2.4).

Case 2: $a_1 < \cdots < a_n < 0$. Then $a_ia_j > 0$ for all $i, j \in \{1, 2, \ldots, n\}$. Hence every element of $\{a_1, a_2, \ldots, a_n\}$ is negative but every element of $\{a_ia_j \mid i, j \in \{1, 2, \ldots, n\}\}$ is positive which are contrary to (2.3).

Case 3: $a_1 < \cdots < a_{n-1} < 0 < a_n$. Then $\{a_1, a_2, \ldots, a_n\}$ contains only one positive real number. But $a_1^2 > a_1a_2 > \cdots > a_1a_{n-1} > 0$ and $n-1 > 1$, so $\{a_ia_j \mid i, j \in \{1, 2, \ldots, n\}\}$ contains at least $n-1$ positive real numbers. These contradict (1) since $n > 2$.

Case 4: $a_1 < \cdots < a_k < 0 < a_{k+1} < \cdots < a_n$ and $k+1 < n$. Then $\{a_1, a_2, \ldots, a_n\}$ contains exactly $n-k$ positive real numbers. Since $0 < a_{k+1}^2 < a_{k+1}a_{k+2} < \cdots < a_{k+1}a_n < a_n^2$, we have that $\{a_ia_j \mid i, j \in \{1, 2, \ldots, n\}\}$ contains at least $n-k+1$ positive real numbers. These contradict (2.3).
Hence the theorem is proved. □

It is clearly seen that if H is a subsemigroup of G' containing e', then the multifunction from G into G' defined by

$$f(x) = H \text{ for all } x \in G$$

is a multihomomorphism from G into G'. Such a multihomomorphism is called a *constant multihomomorphism.* Since for every $a \in \mathbb{R}^+$, $[a, \infty) \cup \{0\}$ is a subsemigroup of $(\mathbb{R}, +)$ containing 0, it follows that there are uncountably many constant multihomomorphisms from G into $(\mathbb{R}, +)$. The next example shows that there are also uncountably many nonconstant multihomomorphisms in $\mathrm{MHom}\,((\mathbb{R}, +), (\mathbb{R}, +))$.

Example 2.5 For each $a \in \mathbb{R}$, define

$$f_a(x) = [ax, \infty) \text{ for all } x \in \mathbb{R}.$$

If $a, x, y \in \mathbb{R}$, then

$$f_a(x+y) = [a(x+y), \infty) = [ax, \infty) + [ay, \infty) = f_a(x) + f_a(y).$$

Hence $f_a \in \mathrm{MHom}\,((\mathbb{R}, +), (\mathbb{R}, +))$ for all $a \in \mathbb{R}$ and f_a is a constant multihomomorphism if and only if $a = 0$. If $a, b \in \mathbb{R}$ are such that $a \neq b$, then $f_a(1) = [a, \infty) \neq [b, \infty) = f_b(1)$. Therefore $\{f_a \mid a \in \mathbb{R}\backslash\{0\}\}$ is an uncountably infinite subset of $\mathrm{MHom}\,((\mathbb{R}, +), (\mathbb{R}, +))$. Observe that $|f_a(x)| = \aleph_1$ for all $x \in \mathbb{R}$.

Next, for $a \in \mathbb{R}$, define

$$g_a(x) = x + a\mathbb{Z}_0^+ \text{ for all } x \in \mathbb{R}$$

where $\mathbb{Z}^+$ is the set of positve integers and $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$. Hence

$$g_a(x) = \{x, x+a, x+2a, \ldots\} \text{ for all } x \in \mathbb{R}.$$

Since for any $a \in \mathbb{R}$, $a\mathbb{Z}_0^+$ is a subsemigroup of $(\mathbb{R}, +)$ containing 0, we have $a\mathbb{Z}_0^+ + a\mathbb{Z}_0^+ = a\mathbb{Z}_0^+$ for every $a \in \mathbb{R}$. It follows that $g_a \in \mathrm{MHom}((\mathbb{R}, +), (\mathbb{R}, +))$ which is nonconstant for every $a \in \mathbb{R}$. Clearly, $a\mathbb{Z}_0^+ \neq b\mathbb{Z}_0^+$ for all distinct $a, b \in \mathbb{R}$. If $a, b \in \mathbb{R}$ are such that $a \neq b$, then $g_a(0) = a\mathbb{Z}_0^+ \neq b\mathbb{Z}_0^+ = g_b(0)$. Hence $\{g_a \mid a \in \mathbb{R}\}$ is an uncountable subset of $\mathrm{MHom}((\mathbb{R}, +), (\mathbb{R}, +))$. Notice that $|g_a(x)| = \aleph_0$ for all $a \in \mathbb{R}\backslash\{0\}$ and $x \in \mathbb{R}$.

If $a \in \mathbb{R}$ and $a \geq 1$, then $[a, \infty) \cup \{1\}$ is a subsemigroup of $(\mathbb{R}^*, \cdot)$ containing 1. Therefore there are uncountably many constant multihomomorphisms from any group G into $(\mathbb{R}^*, \cdot)$. We show in the last example that $\mathrm{MHom}((\mathbb{R}^*, \cdot), (\mathbb{R}^*, \cdot))$ contains an uncountably many nonconstant f which satisfy (ii) of Theorem 2.4 and an uncountably many nonconstant f which satisfy (iii) of Theorem 2.4.

Example 2.6 For each $a \in \mathbb{R}$, define

$$\begin{aligned} &h_a(x) = \{-|x|^a, |x|^a\},\ k_a(x) = [|x|^a, \infty) \ \text{ and} \\ &l_a(x) = |x|^a\mathbb{Z}^+ \text{ for all } \ x \in \mathbb{R}^*. \end{aligned}$$

If $a \in \mathbb{R}$ and $x, y \in \mathbb{R}^*$, then

$$\begin{aligned} h_a(xy) = \{-|xy|^a, |xy|^a\} &= \{-|x|^a|y|^a, |x|^a|y|^a\} \\ &= \{-|x|^a, |x|^a\}\{-|y|^a, |y|^a\} = h_a(x)h_a(y), \\ k_a(xy) = [|xy|^a, \infty) &= [|x|^a|y|^a, \infty) \\ &= [|x|^a, \infty)[|y|^a, \infty) = k_a(x)k_a(y) \quad \text{and} \\ l_a(xy) = |xy|^a\mathbb{Z}^+ &= |x|^a|y|^a\mathbb{Z}^+ \\ &= (|x|^a\mathbb{Z}^+)(|y|^a\mathbb{Z}^+) = l_a(x)l_a(y). \end{aligned}$$

Hence $h_a, k_a, l_a \in \mathrm{MHom}((\mathbb{R}^*, \cdot), (\mathbb{R}^*, \cdot))$ for all $a \in \mathbb{R}$. Also, h_a is nonconstant if and only if $a \neq 0$, and this is also true for k_a and l_a.

Moreover, if $a, b \in \mathbb{R}$ are such that $a \neq b$, then $2^a \neq 2^b$, so $h_a(2) \neq h_b(2)$, $k_a(2) \neq k_b(2)$ and $l_a(2) \neq l_b(2)$. Hence $\{h_a \mid a \in \mathbb{R}\}$, $\{k_a \mid a \in \mathbb{R}\}$ and $\{l_a \mid a \in \mathbb{R}\}$ are uncountable subsets of $\mathrm{MHom}((\mathbb{R}^*, \cdot), (\mathbb{R}^*, \cdot))$, and $|h_a(x)| = 2$, $|k_a(x)| = \aleph_1$ and $|l_a(x)| = \aleph_0$ for all $a \in \mathbb{R}$ and $x \in \mathbb{R}^*$.

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