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Higher Order Approximation by Iterates of Modified Beta Operators[1]

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Abstract : The present paper is a continuation of our work [1] wherein we studied some direct results in ordinary and simultaneous approximation by an iterative combination of modified beta operators [2]. In this paper, we obtain an error estimate in terms of higher order modulus of continuity in simultaneous approximation for these operators.

Keywords : iterative combination; simultaneous approximation; modulus of continuity.
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1 Introduction

Following [3], for the class of functions integrable on $[0,\infty)$, the modified beta operators with the weight function of Baskakov operators are defined as

$$B_n(f;x) = \frac{n-1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)\,dt, \tag{1.1}$$

where

$$p_{n,k}(t) = \binom{n+k-1}{k} t^k (1+t)^{-(n+k)},$$

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$$b_{n,k}(x) = \frac{1}{B(k+1,n)}\, x^k\,(1+x)^{-(n+k+1)},\ \ 0 \le x < \infty.$$

Let us now define a new class of functions e.g. $H[0,\infty)$ as follows:

$$H[0,\infty) =: \Big\{ f : \int_0^\infty \frac{|f(t)|}{(1+t)^n} dt < \infty \text{ for some positive integer } n, f \text{ being a Lebesgue measurable function on } [0,\infty) \Big\}.$$

The motivation to consider the class $H[0,\infty)$ lies in the fact that the operators (1.1) are defined over this class which contains the class of all integrable functions on $[0,\infty)$.

We can easily show that $b_{n,k}(x) = n\, p_{n+1,k}(x)$. Therefore, the operators (1.1) may be rewritten as

$$\begin{aligned} B_n(f;x) &= (n-1)\sum_{k=0}^{\infty} p_{n+1,k}(x) \int_0^\infty p_{n,k}(t) f(t)\, dt \\ &= \int_0^\infty W_n(t,x) f(t)\, dt, \end{aligned}$$

where $W_n(t,x) = (n-1)\sum_{k=0}^{\infty} p_{n+1,k}(x)\, p_{n,k}(t)$ is the kernel of the operators B_n.

It turns out that the order of approximation by these operators is at best $O(n^{-1})$, however smooth the function may be. In order to speed up the rate of convergence by the operators B_n, we [1] considered an iterative combination $L_{n,k} : H[0,\infty) \to C^\infty[0,\infty)$ of the operators $B_n(f;x)$ defined as:

$$L_{n,k}(f(t);x) = \big(I - (I - B_n)^k\big)(f;x) = \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} B_n^r(f(t);x),$$

where $B_n^0 = I$ and $B_n^r = B_n(B_n^{r-1})$ for $r \in \mathbb{N}$ and established an error estimate of the degree of approximation in terms of the ordinary modulus of continuity by $L_{n,k}^{(p)}(.;x)$ for smooth functions [1, Theorem 5]. The aim of this paper is to obtain the corresponding general result in terms of $2k-$th order modulus of continuity by using the properties of a linear method of approximation namely Steklov means.

Throughout this paper, let $0 < a < b < \infty$, $I = [a,b]$, $0 < a_1 < a_2 < b_2 < b_1 < \infty$, $I_i = [a_i, b_i], i = 1,2$, $\|.\|_{C(I)}$ denotes the sup-norm on the interval I and C a constant not necessarily the same in different cases.

2 Preliminaries

In the sequel we shall require the following results:

Lemma 2.1 ([4]). *For the function $u_{n,m}(t), m \in N^0$ (the set of non-negative integers) defined as*

$$u_{n,m}(t) = \sum_{\nu=0}^{\infty} p_{n,\nu}(t) \left(\frac{\nu}{n} - t\right)^m,$$

we have $u_{n,0}(t) = 1$ and $u_{n,1}(t) = 0$. Further, there holds the recurrence relation

$$nu_{n,m+1}(t) = t\left[u'_{n,m}(t) + mu_{n,m-1}(t)\right], \;\; m = 1, 2, 3, ...$$

Consequently,

(i) $u_{n,m}(t)$ is a polynomial in t of degree $[m/2]$, where $[\alpha]$ denotes the integral part of α;

(ii) for every $t \in [0, \infty), u_{n,m}(t) = O\left(n^{-[(m+1)/2]}\right)$.

Lemma 2.2 ([1]). *For the function $p_{n,k}(x)$, there holds*

$$x^r(1+x)^r \frac{d^r}{dx^r}(p_{n,k}(x)) = \sum_{\substack{2i+j \le r,\\ i,j \ge 0}} n^i (k - nx)^j q_{i,j,r}(x)\, p_{n,k}(x),$$

where $q_{i,j,r}(x)$ are certain polynomials in x independent of n and k.

Let $f \in C(I)$ and $I_1 \subset (a, b)$. Then, for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of mth order corresponding to f is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left(f(t) + (-1)^{m-1} \Delta^m_{\sum_{i=1}^m t_i} f(t)\right) \prod_{i=1}^m dt_i, \;, t \in I_1,$$

where Δ^m_h is the forward difference operator with step length h.

Lemma 2.3 ([2]). *For the function $f_{\eta,m}$, we have*

(a) $f_{\eta,m}$ has derivatives up to order m over I_1;

(b) $\|f^{(r)}_{\eta,m}\|_{C(I_1)} \le C_r\ \omega_r(f, \eta, I), r = 1, 2, ..., m$;

(c) $\|f - f_{\eta,m}\|_{C(I_1)} \le C_{m+1}\ \omega_m(f, \eta, I)$;

(d) $\|f_{\eta,m}\|_{C(I_1)} \le C_{m+2}\ \eta^{-m}\|f\|_{C(I)}$;

(e) $\|f^{(r)}_{\eta,m}\|_{C(I_1)} \le C_{m+3}\ \|f\|_{C(I)}$,

where $C'_i s$ are certain constants that depend on i but are independent of f and η.

Lemma 2.4 ([1]). *For the function $T_{n,m}(x), m \in N^0$ defined by*

$$T_{n,m}(x) = \int_0^\infty W_n(t,x)(t-x)^m \, dt,$$

we have

$$T_{n,0}(x) = 1, \; T_{n,1}(x) = \frac{1+3x}{n-2}, \;\; n > 2$$

and for $m \geq 1$

$$(n-m-2)T_{n,m+1}(x) = x(1+x)[T'_{n,m}(x) + 2mT_{n,m-1}(x)] + \{(m+1)(1+2x)+x\}T_{n,m}(x),$$

where $n > m+2$. Consequently,

(i) $T_{n,m}(x)$ is a polynomial in x of degree m;

(ii) for every $x \in [0,\infty)$, $T_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

For every $m \in N^0$, the m-th order moment $T^{\{p\}}_{n,m}$ for the operator B^p_n is defined by

$$T^{\{p\}}_{n,m}(x) = B^p_n\left((t-x)^m; x\right).$$

We adopt the convention $T^{\{1\}}_{n,m}(x) = T_{n,m}(x)$. From Lemma 2.4, it follows that $T^{\{p\}}_{n,m}(x)$ is a polynomial in x of degree m.

Lemma 2.5 ([1]). *For every $x \in [0,\infty)$, we have*

$$T^{\{p\}}_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right).$$

Theorem 2.6 ([1]). *Let $f \in H[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$ and $f(t) = O(t^\alpha)$ as $t \longrightarrow \infty$ for some $\alpha > 0$. If $f^{(2k+p)}(x)$ exists at a point $x \in (0,\infty)$ then we have*

$$\lim_{n\to\infty} n^k \left\{ L^{(p)}_{n,k}(f;x) - f^{(p)}(x) \right\} = \sum_{\nu=p}^{2k+p} Q(\nu,k,p,x) f^{(\nu)}(x), \tag{2.1}$$

where $Q(\nu,k,p,x)$ are certain polynomials in x. Further,if $f^{(2k+p)}$ is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then (2.1) holds uniformly in $[a,b]$.

3 Main Result

We establish the following direct theorem:

Theorem 3.1. *Let $f \in H[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$ and $f(t)=O(t^{\alpha})$ as $t \longrightarrow \infty$ for some $\alpha>0$. If $f^{(p)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, for some $\eta>0$ then*

$$\left\|L_{n,k}^{(p)}(f(t);x)-f^{(p)}(x)\right\|_{C(I_2)} \le C\left\{n^{-k}\|f\|_{C(I_1)}+\omega_{2k}\big(f^{(p)},n^{-1/2},I_1\big)\right\},$$

where C is independent of f and n.

Proof. We can write

$$\begin{aligned}
&\left\|L_{n,k}^{(p)}(f(t);x)-f^{(p)}(x)\right\|_{C(I_2)}\\
&\qquad\le \left\|L_{n,k}^{(p)}(f-f_{\eta,2k};x)\right\|_{C(I_2)}+\left\|L_{n,k}^{(p)}(f_{\eta,2k};x)-f_{\eta,2k}^{(p)}(x)\right\|_{C(I_2)}\\
&\qquad\quad+\left\|f^{(p)}(x)-f_{\eta,2k}^{(p)}(x)\right\|_{C(I_2)}\\
&\qquad= S_1+S_2+S_3, \text{ say.}
\end{aligned}$$

Since $f_{\eta,2k}^{(p)}(x)=\big(f^{(p)}\big)_{\eta,2k}(x)$, by property (c) of the Steklov mean, we get

$$S_3 \le C\ \omega_{2k}(f^{(p)},\eta,I_1).$$

Next, applying Theorem 2.6 and the interpolation property [5] it follows that

$$\begin{aligned}
S_2 &\le C\,n^{-k}\sum_{m=p}^{2k+p}\left\|f_{\eta,2k}^{(m)}\right\|_{C(I_2)}\\
&\le C\,n^{-k}\left(\left\|f_{\eta,2k}\right\|_{C(I_2)}+\left\|\left(f_{\eta,2k}^{(p)}\right)^{(2k)}\right\|_{C(I_2)}\right).
\end{aligned}$$

Hence, by properties (b) and (d) of Steklov mean, we have

$$S_2 \le Cn^{-k}\left(\|f\|_{C(I_1)}+\eta^{-2k}\omega_{2k}\big(f^{(p)},\eta,I_1\big)\right).$$

Let a^* and b^* be such that $0<a_1<a^*<a_2<b_2<b^*<b_1<\infty$.

To estimate S_1, let $F \equiv f-f_{\eta,2k}$ then by our hypothesis we can write

$$\begin{aligned}
F(t)=\sum_{m=0}^{p}\frac{F^{(m)}(x)}{m!}(t-x)^m+\frac{F^{(p)}(\xi)-F^{(p)}(x)}{p!}(t-x)^p\psi(t)&\\
+h(t,x)\,(1-\psi(t))\,,&\qquad(3.1)
\end{aligned}$$

where ξ lies between t and x, and ψ is the characteristic function of the interval $[a^*,b^*]$. For $t\in[a^*,b^*]$ and $x\in[a_2,b_2]$, we get

$$F(t)=\sum_{m=0}^{p}\frac{F^{(m)}(x)}{m!}(t-x)^m+\frac{F^{(p)}(\xi)-F^{(p)}(x)}{p!}(t-x)^p,$$

and for $t \in [0,\infty) \setminus [a^*,b^*], x \in [a_2,b_2]$ we define

$$h(t,x) = F(t) - \sum_{m=0}^{p} \frac{F^{(m)}(x)}{m!}(t-x)^m.$$

Now, operating on both sides of (3.1) by $L_{n,k}^{(p)}$, we get three terms J_1, J_2, and J_3, corresponding to the three terms in the right hand side of (3.1). By using Lemma 2.4, we get

$$\begin{aligned} J_1 &= \sum_{m=0}^{p} \frac{F^{(m)}(x)}{m!} \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} D^p \left(B_n^i((t-x)^m; x)\right),\ D \equiv \frac{d}{dx} \\ &= \frac{F^{(p)}(x)}{p!} \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \int_0^\infty W^{(p)}(n,x,u) B_n^{i-1}(t^p; u)\, du \\ &= \frac{F^{(p)}(x)}{p!} \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} D^p \left(B_n^i(t^p; x)\right) \\ &\to F^{(p)}(x),\ \text{as } n \to \infty, \text{ uniformly in } I_2. \end{aligned}$$

Therefore, $\|J_1\|_{C(I_2)} \le C \left\| f^{(p)} - f_{\eta,2k}^{(p)} \right\|_{C(I_2)}$. Next,

$$\begin{aligned} \|J_2\|_{C(I_2)} \le \frac{2}{p!} \left\| f^{(p)} - f_{\eta,2k}^{(p)} \right\|_{C[a^*,b^*]} \sum_{i=1}^{k} \binom{k}{i} (n-1) \sum_{\nu=0}^{\infty} \left| p_{n+1,\nu}^{(p)}(x) \right| \\ \times \int_0^\infty p_{n,\nu}(v) B_n^{i-1}(|t-x|^p; v)\, dv. \end{aligned}$$

Now, using Lemma 2.2, Cauchy Schwarz inequality three times, Lemma 2.1 and Lemma 2.5 (in that order), it follows that

$$\begin{aligned} &(n-1) \sum_{\nu=0}^{\infty} \left| p_{n+1,\nu}^{(p)}(x) \right| \int_0^\infty p_{n,\nu}(v) B_n^{i-1}(|t-x|^p; v)\, dv \\ &\quad \le C \sum_{\substack{2i+j \le p \\ i,j \ge 0}} (n-1)(n+1)^i |q_{i,j,p}(x)| (x(1+x))^{-p} \sum_{\nu=0}^{\infty} p_{n+1,\nu}(x) |\nu - (n+1)x|^j \\ &\quad\quad \times \int_0^\infty p_{n,\nu}(v) B_n^{i-1}(|t-x|^p; v)\, dv \end{aligned}$$

$$\leq C \sum_{\substack{2i+j\leqslant p\\ i,j\geqslant 0}} (n+1)^i \left(\sum_{\nu=0}^{\infty} p_{n+1,\nu}(x)(\nu-(n+1)x)^{2j}\right)^{1/2}$$

$$\times \left((n-1)\sum_{\nu=0}^{\infty} p_{n+1,\nu}(x) \int_0^{\infty} p_{n,\nu}(v) B_n^{i-1}((t-x)^{2p};v)\,dv\right)^{1/2}$$

$$= C \sum_{\substack{2i+j\leq p\\ i,j\geq 0}} (n+1)^i O\left(n^{j/2}\right) O\left(n^{-p/2}\right) = O(1), \text{ as } n\to\infty, \text{ uniformly in } I_2.$$

Therefore,

$$\|J_2\|_{C(I_2)} \leq C \left\|f^{(p)} - f^{(p)}_{\eta,2k}\right\|_{C[a^*,b^*]}.$$

Since, $t \in [0,\infty)\setminus[a^*,b^*]$ and $x \in I_2$, we can choose a $\delta>0$ in such a way that $|t-x|\geq\delta$. If β is any integer $\geq \max\{\alpha,p\}$, then we can find a constant $M>0$ such that $|h(t,x)| \leq M|t-x|^{\beta}$ whenever $|t-x|\geq\delta$. Again, applying Lemma 2.2, Cauchy Schwarz inequality three times , Lemma 2.1 and Lemma 2.5 (in that order), we get

$$|J_3| \leq C \sum_{i=1}^{k} \binom{k}{i} \sum_{\substack{2i+j\leq p\\ i,j\geq 0}} (n+1)^i \left(\sum_{\nu=0}^{\infty} p_{n+1,\nu}(x)(\nu-(n+1)x)^{2j}\right)^{1/2}$$

$$\times \left((n-1)\sum_{\nu=0}^{\infty} p_{n+1,\nu}(x) \int_0^{\infty} p_{n,\nu}(v) B_n^{i-1}((1-\psi(t))(t-x)^{2\beta};v)\,dv\right)^{1/2}$$

$$\leq C \sum_{i=1}^{k} \binom{k}{i} \sum_{\substack{2i+j\leq p\\ i,j\geq 0}} (n+1)^i \left(\sum_{\nu=0}^{\infty} p_{n+1,\nu}(x)(\nu-(n+1)x)^{2j}\right)^{1/2}$$

$$\times \left((n-1)\sum_{\nu=0}^{\infty} p_{n+1,\nu}(x) \int_0^{\infty} p_{n,\nu}(v) B_n^{i-1}\left(\frac{(t-x)^{2m}}{\delta^{2m-2\beta}};v\right)dv\right)^{1/2}$$

$$\leq C \sum_{\substack{2i+j\leq p\\ i,j\geq 0}} (n+1)^i O\left(n^{j/2}\right) O\left(n^{-m/2}\right), \text{for any integer } m>\beta.$$

Hence, it follows that $J_3 = o(1)$, as $n\to\infty$, uniformly in I_2. Combining the estimates of $J_1 - J_3$, in view of property (c) of Steklov mean we obtain

$$S_1 \leq C \left\|f^{(p)} - f^{(p)}_{\eta,2k}\right\|_{C[a^*,b^*]} \leq C\,\omega_{2k}\left(f^{(p)},\eta,I_1\right).$$

Therefore, with $\eta = n^{-1/2}$, the theorem follows. □

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References

[1] P.N. Agrawal, K.K. Singh, Approximation by iterates of modified beta operators, Proc. of the First Int. Conf. on Math. and Stat., American University of Sharjah, Sharjah, U.A.E., March 18-21, Vol. 2010 Art. ID 100169, 6 pages.

[2] P. Maheshwari, V. Gupta, On rate of approximation by modified beta operators, Int. J. Math. Math. Sci. Vol. 2009, Art. ID 205649, 8 pages.

[3] P. Maheshwari, Saturation theorem for the combinations of modified beta operators in L_p-spaces, Appl. Math. E-Notes 7 (2007) 32–37 (electronic).

[4] R.P. Sinha, P.N. Agrawal, V. Gupta, On simultaneous approximation by modified Baskakov operators, Bull. Soc. Math. Belg. Sér. B 43 (2) (1991) 217–231.

[5] S. Goldberg, A. Meir, Minimum moduli of ordinary differential operators, Proc. London Math. Soc. 23 (1971) 1–15.

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