Thai Journal of Mathematics
Volume 10 (2012) Number 3 : 635–641
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Common Fixed Point as a Contractive Fixed Point[1]

T. Phaneendra[†,2] and M. Chandra Shekhar[‡]

[†]Applied Analysis Division, School of Advanced Sciences
VIT University, Vellore - 632 014, Tamil Nadu State, India
e-mail : drtp.indra@gmail.com

[‡]Department of Mathematics, Vijay Rural Engineering College
Nizamabad - 503 003, Andhra Pradesh State, India
e-mail : maisa.chandrashekhar@gmail.com

Abstract : In this paper, we prove a common fixed point theorem for a wider class of generalized contraction type mappings relative to a self-map and show that the common fixed point will be a contractive fixed point of the reference map under certain condition on contraction constant. Our result is a generalization of common fixed point theorems of first author, and of Akkouchi.

Keywords : self-map; associated Sequence; common fixed point; attracting and contractive fixed points.
2010 Mathematics Subject Classification : 54H25.

1 Introduction

In this paper X represents a metric space with metric d. If x is a point of X and S a self-map on X, we write Sx for the image of x under S, $S(X)$ for the range of S, and TS for the composition of self-maps T and S.

A point $p \in X$ is a *fixed point* for a self-map S on X if $Sp = p$. We denote by $Fix(S)$, the set of all fixed points of S, and by $Fix(S,T)$ the set of all common fixed points of S and T.

[1]This paper was presented in the Sixteenth Congress of AP Society of Mathematical Sciences held at NGRI, Hyderabad, Andhra Pradesh State-INDIA, in December, 2007
[2]Corresponding author.

Given a reference map S on X and $0 < \alpha < 1$, $B(S,\alpha)$ denotes the class of self-maps T on X satisfying the contractive type condition:

$$d(Sx, TSy) \leq \alpha \max\{d(x,Sx), d(x,Sy), d(Sy,TSy), \frac{1}{2}[d(x,TSy) + d(Sx,Sy)]\} \text{ for all } x, y \in X \tag{1.1}$$

where $0 < \alpha < 1$. Fisher [1] proved the following result:

Theorem 1.1. *Given a self-map S on X and $T \in B(S,\alpha)$, where $0 < \alpha < 1$, suppose that either S or T is continuous on X. If X is complete, then there is a unique point p in X such that $p \in Fix(S,T)$.*

Later in 2004, the author [2] obtained the conclusion of Theorem 1.1 by replacing the completeness of the space X without an appeal to the continuity condition, as given below:

Theorem 1.2. *Given a self-map S on X and $T \in B(S,\alpha)$, where $0 < \alpha < 1$, suppose that the associated sequence $\langle x_n \rangle_{n=1}^{\infty}$ at some $x_0 \in X$ with the choice*

$$x_n = \begin{cases} Sx_{n-1} & (n \text{ is odd}) \\ Tx_{n-1} & (n \text{ is even}) \end{cases} \tag{1.2}$$

has a subsequence converging to a point z in X. Then

(i) the sequence (1.2) *will also converge to z.*

(ii) z will be a unique point such that $Fix(S) = Fix(S,T) = \{z\}$.

In this paper, we prove that a common fixed point can also be obtained by generalizing the condition (1.1), which will be a contractive fixed point for the reference map S, when $0 < \alpha < \frac{1}{2}$.

2 Notation

Let $\beta \geq 0$. Given $0 < \alpha < 1$, and self-map S on X, we shall denote by $B_\beta(S,\alpha)$ the class of all mappings $T : X \to X$ satisfying the condition:

$$\begin{aligned} &[1 + \beta d(x,Sy)]\, d(Sx,TSy) \\ &\quad \leq \alpha \max\left\{d(x,Sx), d(x,Sy), d(Sy,TSy), \frac{1}{2}[d(x,TSy) + d(Sx,Sy)]\right\} \\ &\qquad + \beta[d(x,Sx)d(Sy,TSy) + d(x,TSy)d(Sx,Sy)] \text{ for all} x, y \in X \end{aligned} \tag{2.1}$$

Remark 2.1. *Writing $\beta = 0$ in* (2.1), *we get* (1.1). *Thus $B_0(S,\alpha) = B(S,\alpha)$. We note that if $B^*(S,\alpha) = \bigcup_{\beta \geq 0} B_\beta(S,\alpha)$, then $B^*(S,\alpha)$ includes $B(S,\alpha)$.*

Let S be a self-map on X and $x \in X$. The S-*orbit* or simply *orbit* at x is the sequence $O_S(x)$ of iterates: $Sx, S^2x, \ldots$. If $p \in Fix(S)$ has a neighbourhood N in X such that the S-of orbit at each x in N converges to p, then p is called an *attracting fixed point* of S. In case $N = X$, p is called a *contractive fixed point* [3].

Example 2.2. *Consider $S : R \to R$ defined by $Sx = x^3$ for all x. Then $0, 1,$ and -1 are the fixed points of S, and $O_S(x) = \langle x^{3n} \rangle_{n=1}^{\infty}$. Note that for $|x| < 1$, the orbit converges to 0, while for $|x| > 1$ the orbit diverges to $\pm\infty$ respectively. Thus 0 is an attracting but not a contractive fixed point.*

Example 2.3. *Let $S : R \to R$ given by $Sx = \frac{x}{2}$ for all x.Then 0 is the only fixed point of S, which is also a contractive one, to which the orbit $O_S(x) = \left\langle \frac{x}{2^n} \right\rangle_{n=1}^{\infty}$ at each real x converges.*

3 Main Result

Our main result is

Theorem 3.1. *Given $0 < \alpha < 1$, $\beta \geq 0$, and self-map S on X, let $T \in B_\beta(S, \alpha)$. Suppose that the associated sequence $\langle x_n \rangle_{n=1}^{\infty}$ at some $x_0 \in X$ with the choice (1.2) has a subsequence converging to a point z of X. Then the following assertions will be true:*

(i) $\lim_{n\to\infty} x_n = z$

(ii) $z \in Fix(S) \subset Fix(T)$

(iii) $Fix(S) = Fix(T) = Fix(S,T) = \{z\}$ *whenever* $T(X) \subset S(X)$

(iv) *z is a unique common fixed point of S and T*

(v) *S and TS are continuous at z.*

Further if $0 < \alpha < \frac{1}{2}$, the unique common fixed point z will be a contractive fixed point for the reference map S.

Proof. Let x_0 be arbitrary point of X. We write $r_n = d(x_n, x_{n+1}), n = 1, 2, 3, ...$ First we establish that

$$r_n \leq \alpha \max\{r_n, r_{n-1}\} \quad \text{for all } n. \tag{3.1}$$

Suppose that n is even. Writing $x = x_n$ and $y = x_{n-2}$ in (2.1), we get

$$\begin{aligned}
&[1 + \beta d(x_n, x_{n-1})]\, d(x_{n+1}, x_n) \\
&\qquad \leq \alpha \max\left\{ d(x_n, x_{n+1}), d(x_n, x_{n-1}), 0, \frac{1}{2} d(x_{n+1}, x_{n-1}) \right\} \\
&\qquad\quad + \beta[d(x_n, x_{n+1}) d(x_{n-1}, x_n) + 0]
\end{aligned}$$

or

$$(1+\beta r_{n-1})\, r_n \le \alpha \max\left\{r_n, r_{n-1}, \frac{1}{2}d(x_{n+1}, x_{n-1})\right\} + \beta r_n r_{n-1}$$

from which (3.1) follows, since $\frac{1}{2}(a+b) \le \max\{a,b\}$. While if n is odd, we take $x = y = x_{n-1}$ in (2.1) and simplify as above to get (3.1).

It is remarkable to note that if $r_m > r_{m-1}$ for some integer $m > 1$, then (3.1) would give a contradiction that $0 < r_m \le \alpha r_m < r_m$, since $0 < \alpha < 1$.

Hence $r_n \le r_{n-1}$ for all n, and from (3.1) we get $r_n \le \alpha r_{n-1}$ for all n. In other words, $\langle x_n \rangle_{n=1}^{\infty}$ is a contractive sequence and hence is a Cauchy sequence in X [4, Theorem 3.1.7].

In view of the convergence of the subsequence, assertion (i) follows, and

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} Tx_{2n-1} = z \text{ for some } z \in X. \quad (3.2)$$

Writing $x = z$ and $y = x_{2n}$ in (2.1),

$$\begin{aligned}&[1+\beta d(z, x_{2n+1})]\, d(Sz, x_{2n+2})\\ &\le \alpha \max\left\{d(z,Sz), d(z,x_{2n+1}), d(x_{2n+1},x_{2n+2}), \frac{1}{2}[d(z,x_{2n+2}) + d(x_{2n+1},Sz)]\right\}\\ &\quad + \beta[d(z,Sz)d(x_{2n+1},TSx_{2n+2}) + d(z,x_{2n+2})d(x_{2n+1},Sz)].\end{aligned}$$

Proceeding the limit as $n \to \infty$ in this, and using (3.2), we obtain that

$$(1+\beta.0)\, d(Sz,z) \le \alpha \max\left\{d(z,Sz), 0, 0, \frac{1}{2}[0 + d(z,Sz)]\right\} + \beta\,[0 + 0.d(z,Sz)]$$

or $d(Sz,z) \le \alpha d(z,Sz)$ so that $d(Sz,z) = 0$ or $z \in Fix(S)$.

Now we observe that $Fix(S) \subset Fix(T)$. In fact, if $p \in Fix(S)$, then writing $x = y = p$ in (2.1), we get

$$\begin{aligned}&[1+\beta d(p,Sp)]\, d(Sp,TSp)\\ &\qquad \le \alpha \max\left\{d(p,Sp), d(p,Sp), d(Sp,TSp), \frac{1}{2}[d(p,TSp) + d(Sp,Sp)]\right\}\\ &\qquad\quad + \beta[d(p,Sp)d(Sp,TSp) + d(p,TSp)d(Sp,Sq)].\end{aligned}$$

or $d(p,Tp) \le \alpha d(p,Tp)$ so that $d(p,Tp) = 0$ or $p \in Fix(T)$. Thus p is a fixed point of T whenever it is a fixed point of S, and (ii) immediately follows.

To prove (iii), in view of (ii), it suffices to prove that $Fix(T) \subset Fix(S)$. In fact, let p be a fixed point of T. Since $T(X) \subset S(X)$, we get $p = Tp = Sq$ for some $q \in X$. Now (2.1) with $x = p$, and $y = q$ gives

$$\begin{aligned}&[1+\beta d(p,Sq)]\, d(Sp,TSq)\\ &\qquad \le \alpha \max\left\{d(p,Sp), d(p,Sq), d(Sq,TSq), \frac{1}{2}[d(p,TSq) + d(Sp,Sq)]\right\}\\ &\qquad\quad + \beta[d(p,Sp)d(Sq,TSq) + d(p,TSq)d(Sp,Sq)],\end{aligned}$$

which on routine simplification gives $d(Sp, p) \leq \alpha\ d(p, Sp)$ so that $p = Sp$, that is $p \in Fix(S)$. The other way of stating (iii) is that z is a common fixed point of S and T.

The uniqueness of the common fixed point follows directly from (2.1), proving (iv). To get the continuity of S at z, we see from (2.1) with $y = z$ that

$$\begin{aligned}
&[1 + \beta d(x, Sz)]\, d(Sx, TSz) \\
&\qquad \leq \alpha\ \max\left\{d(x, Sx), d(x, Sz), d(Sz, TSz), \frac{1}{2}[d(x, TSz) + d(Sx, Sz)]\right\} \\
&\qquad\quad + \beta[d(x, Sx)d(Sz, TSz) + d(x, TSz)d(Sx, Sz)],
\end{aligned}$$

or

$$\begin{aligned}
[1 + \beta d(x, z)]\, d(Sx, z) \leq \alpha\ \max\left\{d(x, Sx), d(x, z), 0, \frac{1}{2}[d(x, z) + d(Sx, z)]\right\} \\
+ \beta[0 + d(x, z)d(Sx, z)].
\end{aligned}$$

So that

$$d(Sx, z) \leq \alpha \max\left\{d(x, Sx), d(x, z), 0, \frac{1}{2}[d(x, z) + d(Sx, z)]\right\},$$

which on using the triangle inequality of the meteric d gives

$$\begin{aligned}
d(Sx, z) &\leq \alpha\ \max\left\{[d(x, z) + d(z, Sx)], d(x, z), \frac{1}{2}[d(x, z) + d(Sx, z)]\right\} \\
&= \alpha\ [d(x, z) + d(Sx, z)].
\end{aligned}$$

So that $(1 - \alpha)d(Sx, z) \leq \alpha\ d(x, z)$. Thus

$$d(Sx, Sz) = d(Sx, z) \leq \left(\frac{\alpha}{1 - \alpha}\right) d(x, z). \tag{3.3}$$

Given $\epsilon > 0$, choose

$$\delta = \left(\frac{1 - \alpha}{\alpha}\right) \epsilon.$$

Then (3.3) would imply that $d(Sx, Sz) < \epsilon$ whenever $d(x, z) < \delta$, showing that S is continuous at z.

Again writing $x = z$ in (2.1), using $Sz = z$ and then simplifying, we get

$$\begin{aligned}
d(z, TSy) &\leq \alpha\ \max\left\{d(z, Sy), \frac{1}{2}[d(z, TSy) + d(z, Sy)]\right\} \\
&= \alpha \max\{d(z, Sy), d(z, TSy)\}
\end{aligned}$$

or

$$d(z, TSy) \leq \alpha \max\{d(Sy, z), d(TSy, z)\} = \alpha d(Sy, z) = \left(\frac{\alpha^2}{1 - \alpha}\right) d(y, z)$$

for all $y \in X$, in view of (3.3).

This reveals that TS is continuous at z, proving the last part of (v).

In the remainder of the proof, we suppose that $0 < \alpha < \frac{1}{2}$ and $y_0 \in X$ is arbitrary. Then the iterates:

$$y_n = s^n y_0, n = 1, 2, 3, \ldots \tag{3.4}$$

will describe the S-orbit at y_0. From again (2.1) with $x = y_n$, and $y = z$, we observe that

$$\begin{aligned} &[1 + \beta d(y_n, z)]\, d(Sy_n, z) \\ &\quad \leq \alpha \max\left\{d(y_n, Sy_n), d(y_n, z), \frac{1}{2}[d(y_n, z) + d(Sy_n, z)]\right\} + \beta d(y_n, z) d(Sy_n, z). \end{aligned}$$

Setting $u_n = d(y_n, z)$, this can be written as

$$u_{n+1} \leq \alpha \max\left\{d(y_n, y_{n+1}), u_n, \frac{1}{2}[u_n + u_{n+1}]\right\} + \beta u_n u_{n+1} \text{ for all } n. \tag{3.5}$$

If $u_{m+1} > u_m$ for some positive integer m, we would obtain that

$$d(y_m, y_{m+1}) \leq d(y_m, z) + d(z, y_{m+1}) = u_m + u_{m+1} < 2u_{m+1}$$

which together with (3.5) implies a contradiction that $u_{m+1} \leq 2\alpha u_{m+1} < u_{m+1}$.

Hence we must have $u_{n+1} > u_n$ for all n so that agin from (3.5), we have $u_{n+1} \leq 2\alpha u_n$ for all n. Proceeding the limit as $n \to \infty$ in this and using the choice of α, we find that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} d(y_n, z) = 0.$$

That is, the $S-$orbit at y_0 converges to z. Since y_0 is arbitrary point in X, it follows that the common fixed point z is a contractive fixed point of S. □

Remark 3.2. *Writing* $\beta = 0$ *in Theorem* 3.1, *we get Theorem* 1.2, *in view of Remark* 2.1.

Remark 3.3. *When* $\beta = 0$ *and* X *is complete in Theorem* 3.1, *we get Theorem 1.1 of Akkouchi [5].*

Acknowledgement : The authors are highly thankful to the referees for their valuable suggestions in improving the manuscript of the paper in favour of the publication.

References

[1] B. Fisher, Result and Conjecture on Fixed Points, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis Mat Nat. 62 (1977) 769–775.

[2] T. Phaneendra, A Generalization of Brian Frisher's Theorem, The Math. Edu. 38 (2004) 92–94.

[3] S. Leader, Fixed Points for a General Contraction in Metric Space, Math. Japon. 24 (1) (1979) 17–24.

[4] R.A. Bartle, Introduction to Real Analysis, JWE, 1996.

[5] M. Akkuuchi, A Note on a Common Fixed Point Theorem of Brian Fisher, J. De Mat. 8 (2003) 26–30.

(Received 24 June 2011)
(Accepted 17 January 2012)