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On Sapogov's Extension of Čebyšev's Inequality and Related Results

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Abstract : In this paper, we give Sapogov's extension of Čebyšev's inequality, majorization type inequality for double integrals by using continuous convex function and Green function. We also give important applications of generalized Cauchy means.

Keywords : convex function; Green function; Čebyšev's inequality; majorization type inequality; mean value theorems; Cauchy means.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ and (Ω, A, μ) be a measure space. For a $\mu-$ measurable function $w : \Omega \to \mathbb{R}$, consider the Lebesgue space $L_w(\Omega, A, \mu) = \{y : \Omega \to \mathbb{R}, y$ is $\mu-$ measurable and $\int_\Omega w(t)|y(t)|d\mu(t) < \infty\}$. Assume $W = \int_\Omega w(t)d\mu(t) > 0$. If $y, z : \Omega \to \mathbb{R}$ are $\mu-$ measurable functions and $y, z, yz \in L_w(\Omega, \mu)$, then we may consider the Čebyšev functional

$$T_w(y,z) = \frac{1}{W}\int_\Omega w(t)y(t)z(t)d\mu(t) - \frac{1}{W}\int_\Omega w(t)y(t)d\mu(t)\frac{1}{W}\int_\Omega w(t)z(t)d\mu(t). \tag{1.1}$$

Let us note that the following identity is a generalization of Sonin's identity

$$T_w(y,z) = \frac{1}{W}\int_\Omega w(t)(y(t)-\alpha)(z(t)-\overline{z})d\mu(t), \tag{1.2}$$

where α is an arbitrary real number and

$$\overline{z} = \frac{1}{W}\int_\Omega w(t)z(t)d\mu(t). \tag{1.3}$$

Namely, Sonin [1] has given (1.2) for $\Omega = [a, b], \mu(t) = t$. He has used (1.2) in the proof of Čebyšev inequality.

Theorem 1.1 (Čebyšev's inequality)**.** *Let $\Omega = [a, b]$. If y and z are monotonic in same order then*

$$T_w(y,z) \geq 0. \tag{1.4}$$

Moreover, if y and z are monotone in opposite direction, then the reverse inequality in (1.4) is valid.

Proof. (Sonin) Since $\overline{z}$ is the mean value of z, then if z is an nondecreasing function, there exist a number c $(a < c < b)$ such that

$$z(t) - \overline{z} \geq 0 \text{ for } t > c \text{ and } z(t) - \overline{z} \leq 0 \text{ for } t < c.$$

For $\alpha = y(c+0)$ the product $(y(t)-\alpha)(z(t)-\overline{z})$ has constant sign. This product is positive if y is a non decreasing function and negative if y is an non increasing function. □

Remark 1.1. *A related proof was also given by Sapogov [2].*

Let us note that Sonin's proof of Čebyšev's inequality can be modified for the following generalization of Čebyšev's inequality.

Theorem 1.2 (Brunn, [3])**.** *Let $\Omega = [a, b]$. If for $a \leq t \leq b$ we have*

$$\operatorname{sgn}(y(t) - y(t_m)) = \operatorname{sgn}(z(t) - \overline{z}),$$

where $\overline{z}$ is defined by (1.3) and t_m is determined from $z(t_m) = \overline{z}$, then inequality (1.4) holds.

Remark 1.2. *In fact Brunn has proved Theorem 1.2 in case $w(t) = 1, \mu(t) = t$.*

Of course, as direct consequence of Sonin's identity (1.2) we have the following result.

Theorem 1.3. *If*

$$(y(t) - \alpha)(z(t) - \overline{z}) \geq 0 \text{ for } \mu - a.e.\ t \in \Omega, \tag{1.5}$$

then (1.4) is valid. If reverse inequality in (1.5) is valid, the reverse inequality in (1.4) is valid, too.

Theorem 1.4 ([4])**.** *Let us denote by ω the set $\omega = \{t | z(t) - \overline{z} > 0, t \in \Omega\}$ and ω is nonempty set and it is not equal to Ω. Let for all $t_1 \in \omega$ and $t_2 \in \Omega \setminus \omega$ we have*

$$y(t_2) \leq y(t_1). \tag{1.6}$$

Then (1.4) is valid. If the reverse inequality in (1.6) is valid then the reverse inequality in (1.4) is valid, too.

Proof. Set $\alpha = \sup_{t \in \Omega \setminus \omega} y(t)$. It is easy to see that (1.5) is valid. So (1.4) is valid. If the reverse inequality in (1.6) is valid, we have reverse inequality in (1.5), that is in (1.4), too. □

Remark 1.5. *Sapogov proved Theorem 1.4 for the case $w(t) = 1$ for all $t \in \Omega$.*

This paper is organized in the following manner: in Section 2 we give Sapogov's extension of Čebyšev's inequality and using this extension prove majorization type inequality. In Section 3 we prove majorization type inequalities for double integrals by using continuous convex function and Green function. For these inequalities we give two mean value theorems and also introduce generalized Cauchy type means. In Section 4 we prove positive semi-definiteness of matrices generated by the differences deduced from majorization type results for double integrals which implies exponential convexity and log-convexity of these differences and also obtained Lyapunov's and Dresher's type inequalities for these differences.

2 On Sapogov's Extension of Čebyšev's Inequality

In fact the following generalization of the main result of Sapogov is equivalent to Theorem 1.4.

Lemma 2.1. *Let $w, \mu, v, x, y, z : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$ with μ be increasing and $w(t), x(t), v(t) > 0$ for all $t \in [a, b]$. Denote $\lambda = \frac{\int_a^b w(t)z(t)v(t)d\mu(t)}{\int_a^b w(t)x(t)v(t)d\mu(t)}$. Suppose that there exist two intervals I_1 and I_2 with $I_1 \cup I_2 = [a, b]$ such that* *(i)* $\frac{y(t_2)}{v(t_2)} \leq \frac{y(t_1)}{v(t_1)}$ *for* $t_1 \in I_1, t_2 \in I_2$,

(ii) $\frac{z(t_2)}{x(t_2)} \leq \lambda \leq \frac{z(t_1)}{x(t_1)}$ *for* $t_1 \in I_1, t_2 \in I_2$.

Then the following inequality holds

$$\int_a^b w(t)x(t)y(t)d\mu(t) \int_a^b w(t)z(t)v(t)d\mu(t) \leq \int_a^b w(t)z(t)y(t)d\mu(t) \int_a^b w(t)x(t)v(t)d\mu(t). \quad (2.1)$$

Proof. From (i) we can say that there exists some $\alpha \in \mathbb{R}$ such that

$$\frac{y(t_2)}{v(t_2)} \leq \alpha \leq \frac{y(t_1)}{v(t_1)}, \text{ for } t_1 \in I_1, t_2 \in I_2. \quad (2.2)$$

Let us consider $g(t) = y(t) - \alpha v(t)$ and

$$h(t) = z(t)\left(\int_a^b w(t)x(t)v(t)d\mu(t)\right) - x(t)\left(\int_a^b w(t)z(t)v(t)d\mu(t)\right), \; t \in [a,b].$$

Now from (2.2) we may write

$$g(t_1) \geq 0, \; g(t_2) \leq 0 \text{ for } t_1 \in I_1, \; t_2 \in I_2, \quad (2.3)$$

and similarly from (ii) we may write

$$z(t_1) - \lambda x(t_1) \geq 0, \; z(t_2) - \lambda x(t_2) \leq 0 \text{ for } t_1 \in I_1, t_2 \in I_2. \quad (2.4)$$

Since $\int_a^b w(t)x(t)v(t)d\mu(t) > 0$, so multiplying this with (2.4) we obtain

$$h(t_1) \geq 0, \text{ and } h(t_2) \leq 0 \text{ for } t_1 \in I_1, t_2 \in I_2. \quad (2.5)$$

By using (2.3) and (2.5) we have $g(t)h(t) \geq 0$ for all $t \in [a,b]$, so we may write

$$\int_a^b w(t)g(t)h(t)d\mu(t) \geq 0. \quad (2.6)$$

From (2.6) we obtain (2.1). □

Remark 2.2. *In [5], Otachel proved inequality (2.1) using the relation of synchronicity between vectors with respect to dual bases in Banach spaces V and its dual V^*.*

Remark 2.3. *If we set in Lemma 2.1: $v(t) = x(t) = 1$ for every $t \in [a,b]$ we will get Čebyšev's result. On the other hand if we set in the corresponding Čebyšev's result: $z(t) \to \frac{z(t)}{v(t)}$ and $y(t) \to \frac{y(t)}{x(t)}$, we will get Lemma 2.1.*

In the following theorem we prove majorization type inequality by using Lemma 2.1.

Theorem 2.4. *Let $\phi : I \to \mathbb{R}$ be a continuous convex function on the interval I. If $\varphi \in \partial\phi$ ($\partial\phi$ is the subdifferential of ϕ) and $u, v, w, x, y, \mu : [a, b] \to \mathbb{R}$ are continuous functions such that μ is increasing, $w(t), u(t), v(t) > 0$ and $x(t), y(t) \in I$ for all $t \in [a, b]$. Denote $\lambda = \frac{\int_a^b w(t)(x(t)-y(t))v(t)d\mu(t)}{\int_a^b w(t)u(t)v(t)d\mu(t)}$. Suppose that there exist two intervals I_1 and I_2 with $I_1 \cup I_2 = [a, b]$ such that*

(i) $\frac{\varphi(y(t_2))}{v(t_2)} \le \frac{\varphi(y(t_1))}{v(t_1)}$ *for* $t_1 \in I_1, t_2 \in I_2$,

(ii) $\frac{x(t_2)-y(t_2)}{u(t_2)} \le \lambda \le \frac{x(t_1)-y(t_1)}{u(t_1)}$ *for* $t_1 \in I_1, t_2 \in I_2$.

Under the above assumptions, the following assertions hold.

$$(A)\ \text{If}\ \int_a^b w(t)(x(t)-y(t))v(t)d\mu(t) = 0, \tag{2.7}$$

$$\text{then}\ \int_a^b w(t)\phi(y(t))d\mu(t) \le \int_a^b w(t)\phi(x(t))d\mu(t). \tag{2.8}$$

(B) If $\int_a^b w(t)(x(t)-y(t))v(t)d\mu(t) \int_a^b w(t)\varphi(y(t))u(t)d\mu(t) \ge 0$, *then (2.8) holds.*

Proof. It follows from [6, Theorem 5] that

$$\int_a^b w(t)(\phi(x(t)) - \phi(y(t))d\mu(t)) \ge \int_a^b w(t)(x(t)-y(t))\varphi(y(t))d\mu(t). \tag{2.9}$$

Utilizing Lemma 2.1, we get

$$\begin{aligned} &\int_a^b w(t)(x(t)-y(t))\varphi(y(t))d\mu(t) \\ &\qquad \ge \frac{\int_a^b w(t)(x(t)-y(t))v(t)d\mu(t) \int_a^b w(t)\varphi(y(t))u(t)d\mu(t)}{\int_a^b w(t)u(t)v(t)d\mu(t)}, \end{aligned} \tag{2.10}$$

since $\int_a^b w(t)u(t)v(t)d\mu(t) > 0$. So, if $\int_a^b w(t)(x(t)-y(t))v(t)d\mu(t) = 0$ then (2.8) follows from (2.9) and (2.10).

Similarly, if the condition $\int_a^b w(t)(x(t)-y(t))v(t)d\mu(t) \int_a^b w(t)\varphi(y(t))u(t)d\mu(t) \ge 0$ are fulfilled, then (2.8) holds by virtue of (2.9) and (2.10). This completes the proof. □

In fact in the following corollary we prove majorization type inequality by using Sapogov's result.

Corollary 2.5. *Under the assumptions of Theorem 2.4, let $u(t) = v(t) = 1$ for all $t \in [a, b]$. Denote $\lambda = \frac{1}{W}\int_a^b w(t)(x(t)-y(t))d\mu(t)$, where $W = \int_a^b w(t)d\mu(t) > 0$. If there exist two intervals I_1 and I_2 with $I_1 \cup I_2 = [a, b]$ such that*

$$(i)\ y(t_2) \le y(t_1)\ \text{for}\ t_1 \in I_1,\ t_2 \in I_2, \tag{2.11}$$

$$(ii)\ x(t_2) - y(t_2) \le \lambda \le x(t_1) - y(t_1)\ \text{for}\ t_1 \in I_1, t_2 \in I_2, \tag{2.12}$$

then assertions (A) and (B) of Theorem 2.4 hold.

Proof. It is sufficient to show that condition (i) of Theorem 2.4 is satisfied for $v(t) = 1, t \in [a, b]$. Since ϕ is convex function and $\varphi \in \partial\phi$, φ is nondecreasing function, so (2.11) implies (2.4), for $v(t) = 1, t \in [a, b]$. □

Conditions (2.11) and (2.12) are fulfilled for $I_1 = [a, c], I_2 = [c, b]$ where $a < c < b$, if both y and $x - y$ are monotonic nonincreasing functions. Likewise, if both y and $x - y$ are monotonic nondecreasing functions, then (2.11) and (2.12) hold for $I_1 = [c, b]$ and $I_2 = [a, c]$. In these cases, Corollary 2.5, assertion (A) of Theorem 2.4, reduces to a result [6, Theorem 6].

Corollary 2.6. *Under the assumptions of Theorem 2.4, let $u(t) = v(t) = t$ for all $t \in [a, b] \subset \mathbb{R}^+$. Denote $\lambda = \frac{1}{\tilde{W}} \int_a^b tw(t)(x(t)-y(t))d\mu(t)$, $\tilde{W} = \int_a^b t^2 w(t)d\mu(t) > 0$. If there exist two intervals I_1 and I_2 with $I_1 \cup I_2 = [a, b]$ such that*

$$(i) \quad \frac{\varphi(y(t_2))}{t_2} \leq \frac{\varphi(y(t_1))}{t_1} \ \text{for } t_1 \in I_1,\ t_2 \in I_2, \tag{2.13}$$

$$(ii) \quad \frac{x(t_2) - y(t_2)}{t_2} \leq \lambda \leq \frac{x(t_1) - y(t_1)}{t_1} \ \text{for } t_1 \in I_1, t_2 \in I_2, \tag{2.14}$$

then assertions (A) and (B) of Theorem 2.4 hold.

Proof. Apply Theorem 2.4. □

Remark 2.7. *For related discrete version of Lemma 2.1, Theorem 2.4, Corollary 2.5 and Corollary 2.6 see [7, 8].*

3 Majorization Inequalities for Double Integrals

The following theorem is a simple consequence of Theorem 12.14 in [9] (see also [10, p. 328], [11, p. 583]):

Theorem 3.1. *Let $\phi : I \to \mathbb{R}$ be a continuous convex function on the interval I and $x, y, w : [a, b] \to I$ be continuous functions such that x, y are decreasing and let $\mu : [a, b] \to \mathbb{R}$ be a function of bounded variation.*

(a) If

$$\int_a^\nu w(t)y(t)\, d\mu(t) \ \leq \ \int_a^\nu w(t)x(t)\, d\mu(t) \ \text{ for every } \ \nu \in [a, b], \tag{3.1}$$

and

$$\int_a^b w(t)x(t)\, d\mu(t) \ = \ \int_a^b w(t)y(t)\, d\mu(t) \tag{3.2}$$

hold, then

$$\int_a^b w(t)\phi\left(y(t)\right)\, d\mu(t) \ \leq \ \int_a^b w(t)\phi\left(x(t)\right)\, d\mu(t). \tag{3.3}$$

(b) *If $\phi : I \to \mathbb{R}$ is continuous increasing convex function and (3.1) holds, then (3.3) holds.*

In [12], the following theorem is proved by Maligranda et al. (1995):

Theorem 3.2. *Suppose that $\phi : [0, \infty) \to \mathbb{R}$ is a convex function and $x, y, w : [a, b] \to \mathbb{R}^+$ be integrable functions. Let $\mu : [a, b] \to \mathbb{R}$ be an increasing and satisfying (3.1) and (3.2).*

(i) If y is a decreasing function on $[a, b]$, then (3.3) holds.

(ii) If x is an increasing function on $[a, b]$, then reverse inequality in (3.3) holds.

Theorem 3.3 ([6, p.419])**.** *Let $\phi : I \to \mathbb{R}$ be continuous convex function on I and $x, y, w, \mu : [a, b] \to I$ be real continuous functions such that μ is monotonic nondecreasing, $w(t) > 0$ is of bounded variation on $[a, b]$. If y, $x - y$ are increasing (decreasing) on $[a, b]$ and satisfying (3.2), then (3.3) holds.*

We give the above results for double integrals.

Theorem 3.4.

(a) *Let $\phi : I \to \mathbb{R}$ be a continuous convex function on the interval I and $w, x, y : [a, b] \times [c, d] \to I$ be continuous functions such that $x(t, s), y(t, s)$ are decreasing in $t \in [a, b]$ and let $\mu : [a, b] \to \mathbb{R}$ be a function of bounded variation, $u : [c, d] \to \mathbb{R}$ be an increasing function.*

(a$_1$) *If for each $s \in [c, d]$*

$$\int_a^\nu w(t,s)y(t,s)\,d\mu(t) \le \int_a^\nu w(t,s)x(t,s)\,d\mu(t), \quad \nu \in [a,b] \tag{3.4}$$

and

$$\int_a^b w(t,s)x(t,s)\,d\mu(t) = \int_a^b w(t,s)y(t,s)\,d\mu(t) \tag{3.5}$$

hold, then

$$\int_c^d \int_a^b w(t,s)\phi\left(y(t,s)\right)\,d\mu(t)du(s) \le \int_c^d \int_a^b w(t,s)\phi\left(x(t,s)\right)d\mu(t)du(s). \tag{3.6}$$

(a$_2$) If for each $s \in [c, d]$, (3.4) holds, then for continuous increasing convex function $\phi : I \to \mathbb{R}$, (3.6) holds.

(b) *Suppose that $\phi : [0, \infty) \to \mathbb{R}$ is a convex function and $x, y, w : [a, b] \times [c, d] \to \mathbb{R}^+$ are integrable functions. Let $\mu : [a, b] \to \mathbb{R}$, $u : [c, d] \to \mathbb{R}$ be increasing functions and satisfying conditions (3.4) and (3.5).*

(b$_1$) *If for each $s \in [c,d]$, $y(t,s)$ is a decreasing function in $t \in [a,b]$, then (3.6) holds.*

(b$_2$) *If for each $s \in [c,d]$, $x(t,s)$ is an increasing function in $t \in [a,b]$, then the reverse inequality in (3.6) holds.*

(c) *Let $\phi : I \to \mathbb{R}$ be continuous convex function on the interval I, $w, x, y : [a,b] \times [c,d] \to I$ be continuous functions with $w(t,s) > 0$ is of bounded variation and let $\mu : [a,b] \to \mathbb{R}$, $u : [c,d] \to \mathbb{R}$ be increasing functions. If $y(t,s)$ and $x(t,s) - y(t,s)$ are increasing (decreasing) in $t \in [a,b]$ and satisfying condition (3.5), then (3.6) holds.*

(d) *Let $\phi : I \to \mathbb{R}$ be a continuous convex function on the interval I, $\varphi \in \partial\phi$ ($\partial\phi$ is the subdifferential of ϕ), $w, x, y, g, h : [a,b] \times [c,d] \to \mathbb{R}$ be continuous functions with $x(t,s), y(t,s) \in I$, $w(t,s), g(t,s), h(t,s) > 0$ and $\mu : [a,b] \to \mathbb{R}$, $u : [c,d] \to \mathbb{R}$ be increasing functions. Denote $\lambda = \frac{\int_a^b w(t,s)(x(t,s)-y(t,s))d\mu(t)}{\int_a^b w(t,s)g(t,s)h(t,s)d\mu(t)}$. Suppose that there exist two intervals I_1 and I_2 with $I_1 \cup I_2 = [a,b]$ such that for each $s \in [c,d]$*

$$(i) \quad \frac{\varphi(y(t_2,s))}{h(t_2,s)} \le \frac{\varphi(y(t_1,s))}{h(t_1,s)} \quad \text{for} \quad t_1 \in I_1, t_2 \in I_2, \tag{3.7}$$

$$(ii) \quad \frac{x(t_2,s) - y(t_2,s)}{g(t_2,s)} \le \lambda \le \frac{x(t_1,s) - y(t_1,s)}{g(t_1,s)} \quad \text{for } t_1 \in I_1, t_2 \in I_2. \tag{3.8}$$

If $\int_a^b w(t,s)(x(t,s) - y(t,s))h(t,s)d\mu(t) \int_a^b w(t,s)\varphi(y(t,s))w(t,s)d\mu(t) \ge 0$, then (3.6) holds.

Proof. (a) By using Theorem 3.1 we may write

$$\int_a^b w(t,s)\phi\left(y(t,s)\right) d\mu(t) \le \int_a^b w(t,s)\phi\left(x(t,s)\right) d\mu(t), \quad \text{for each } s \in [c,d]. \tag{3.9}$$

Integrating both hand sides with respect to $u(s)$, we deduce the desire result (3.6).
In similar way we can prove (b), (c) and (d). □

Now, we give majorization type result by using the Green function.
Consider the Green function G defined on $[\alpha,\beta] \times [\alpha,\beta]$ by

$$G(t,s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha \le s \le t; \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t \le s \le \beta. \end{cases} \tag{3.10}$$

The function G is convex in s, it is symmetric, so it is also convex in t. The function G is continuous in s and continuous in t. For any function $\phi : [\alpha,\beta] \to \mathbb{R}$, $\phi \in C^2([\alpha,\beta])$, we can easily show by integrating by parts that the following is valid

$$\phi(x) = \frac{\beta - x}{\beta - \alpha}\phi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\phi(\beta) + \int_\alpha^\beta G(x,s)\phi''(s)ds, \tag{3.11}$$

where the function G is defined as above in (3.10) ([13]).

Theorem 3.5. *Let $w, x, y : [a,b] \times [c,d] \to \mathbb{R}$, $\mu : [a,b] \to \mathbb{R}$ and $u : [c,d] \to \mathbb{R}$ be continuous functions and $[\alpha, \beta]$ be an interval such that $x(t,s), y(t,s) \in [\alpha, \beta]$ for $(t,s) \in [a,b] \times [c,d]$. Also let (3.5) holds. Then the following are equivalent.*

(i) For every continuous convex function $\phi : [\alpha, \beta] \to \mathbb{R}$, (3.6) holds.

(ii) For all $\tau \in [\alpha, \beta]$ holds

$$\int_c^d \int_a^b w(t,s) G(y(t,s), \tau) d\mu(t) du(s) \leq \int_c^d \int_a^b w(t,s) G(x(t,s), \tau) d\mu(t) du(s). \tag{3.12}$$

Moreover, the statements (i) and (ii) are also equivalent if we change the sign of inequality in both inequalities, in (3.6) and in (3.12).

Proof. $(i) \Rightarrow (ii)$: Let (i) holds. As the function $G(\cdot, \tau)$ $(\tau \in [\alpha, \beta])$ is also continuous and convex, it follows that also for this function (3.6) holds, i.e., (3.12) holds .

$(ii) \Rightarrow (i)$: Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a convex function, $\phi \in C^2([\alpha, \beta])$ and (ii) holds. Then, we can represent the function ϕ in the form (3.11), where the function G is defined in (3.10). By easy calculation, using (3.11), we can easily get that

$$\begin{aligned}
&\int_c^d \int_a^b w(t,s) \phi(x(t,s)) d\mu(t) du(s) - \int_c^d \int_a^b w(t,s) \phi(y(t,s)) d\mu(t) du(s) \\
&\qquad = \int_\alpha^\beta \left[\int_c^d \int_a^b w(t,s) G(x(t,s), \tau) d\mu(t) du(s) \right. \\
&\qquad\qquad \left. - \int_c^d \int_a^b w(t,s) G(y(t,s), \tau) d\mu(t) du(s) \right] \phi''(\tau)\, d\tau.
\end{aligned}$$

Since ϕ is a convex function, then $\phi''(\tau) \geq 0$ for all $\tau \in [\alpha, \beta]$. So, if for every $\tau \in [\alpha, \beta]$ the inequality (3.12) holds , then it follows that for every convex function $\phi : [\alpha, \beta] \to \mathbb{R}$, with $\phi \in C^2([\alpha, \beta])$, inequality (3.6) holds.

At the end, note that it is not necessary to demand the existence of the second derivative of the function ϕ ([10, p.172]). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials. $\square$

Remark 3.6. *Under the assumptions of Theorem 3.5, if for all $\tau \in [\alpha, \beta]$, the inequality (3.12) holds then by setting $\phi(x) = x^2, x \in [\alpha, \beta]$, in (3.6) we get*

$$\int_c^d \int_a^b w(t,s) y^2(t,s) d\mu(t) du(s) \leq \int_c^d w(t,s) x^2(t,s) d\mu(t) du(s). \tag{3.13}$$

Theorem 3.7. *Let $\phi \in C^2([\alpha, \beta])$ and $w, x, y : [a,b] \times [c,d] \to \mathbb{R}$, $\mu : [a,b] \to \mathbb{R}$, $u : [a,b] \to \mathbb{R}$ be continuous functions such that $x(t,s), y(t,s) \in [\alpha, \beta]$ for $(t,s) \in [a,b] \times [c,d]$. Let (3.5) holds. If for all $\tau \in [\alpha, \beta]$, the inequality (3.12) holds or if* *for all $\tau \in [\alpha,\beta]$, the reverse inequality in (3.12) holds, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\int_c^d \int_a^b w(t,s)\phi(x(t,s))d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)\phi(y(t,s))d\mu(t)du(s)$$
$$= \frac{\psi''(\xi)}{2}\left(\int_c^d \int_a^b w(t,s)x^2(t,s)d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)y^2(t,s)d\mu(t)du(s)\right). \tag{3.14}$$

Proof. The proof is analogous to the proof of Theorem 8 in [14]. □

Theorem 3.8. *Let $\phi,\psi \in C^2([\alpha,\beta])$ and w,x,y,u,μ be defined as in Theorem 3.7. Also let (3.5) holds. If for all $\tau \in [\alpha,\beta]$, the inequality (3.12) holds or if for all $\tau \in [\alpha,\beta]$, the reverse inequality in (3.12) holds, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\frac{\phi''(\xi)}{\psi''(\xi)} = \frac{\int_c^d \int_a^b w(t,s)\phi(x(t,s))d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)\phi(y(t,s))d\mu(t)du(s)}{\int_c^d \int_a^b w(t,s)\psi(x(t,s))d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)\psi(y(t,s))d\mu(t)du(s)}, \tag{3.15}$$

provided that the denominators are non zero.

Proof. The proof is analogous to the proof of Theorem 9 in [14]. □

Corollary 3.9. *Under the assumptions of Theorem 3.8, set $\phi(x) = x^l$ and $\psi(x) = x^m$, for $l,m \in \mathbb{R}\setminus\{0,1\}$, $l \neq m$ with $[\alpha,\beta] \subset \mathbb{R}^+$, then there exists $\xi \in [\alpha,\beta]$ such that*

$$\xi^{l-m} = \frac{m(m-1)\int_c^d \int_a^b w(t,s)x^l(t,s)\,d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)y^l(t,s)\,d\mu(t)du(s)}{l(l-1)\int_c^d \int_a^b w(t,s)x^m(t,s)\,d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)y^m(t,s)\,d\mu(t)du(s)}. \tag{3.16}$$

Proof. Apply Theorem 3.8. □

Now we are able to introduce generalized Cauchy means from (3.15). Namely, suppose that $\frac{\phi''}{\psi''}$ has inverse function, then from (3.15) we have

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\int_c^d \int_a^b w(t,s)\phi(x(t,s))d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)\phi(y(t,s))d\mu(t)du(s)}{\int_c^d \int_a^b w(t,s)\psi(x(t,s))d\mu(t)du(s) - \int_c^d \int_a^b w(t,s)\psi(y(t,s))d\mu(t)du(s)}\right). \tag{3.17}$$

Remark 3.10. *Since the function $\xi \to \xi^{l-m}$ with $l \neq m$ is invertible, then from (3.16) we have*

$$\alpha \leq \left\{\frac{m(m-1)\int_c^d \int_a^b w(t,s)x^l(t,s)d\mu(t)du(s) - A_1}{l(l-1)\int_c^d \int_a^b w(t,s)x^m(t,s)d\mu(t)du(s) - A_2}\right\}^{\frac{1}{l-m}} \leq \beta. \tag{3.18}$$

where $A_1 = \int_c^d \int_a^b w(t,s)y^l(t,s)d\mu(t)du(s)$ *and* $A_2 = \int_c^d \int_a^b w(t,s)y^m(t,s)d\mu(t)u(s)$. *We shall say that the expression in the middle defines a class of means.*

4 Applications of Generalized Cauchy Means

In this section, we give some very important applications of generalized Cauchy means i.e., monotonicity of these means.

To define Cauchy type means for majorization type result, the following families of functions will be useful.

Lemma 4.1. *Consider the functions* $\eta_r : [0,\infty) \to \mathbb{R}$

$$\eta_r(x) = \begin{cases} \frac{x^r}{r(r-1)}, & r \neq 1; \\ x\log x, & r = 1. \end{cases} \tag{4.1}$$

Then $\eta_r''(x) = x^{r-2}$, *that is* η_r *is convex for* $x \geq 0, r > 0$, *with the convention that* $0\log 0 = 0$.

Lemma 4.2. *Consider the functions* $\varphi_r : (0,\infty) \to \mathbb{R}$

$$\varphi_r(x) = \begin{cases} \frac{x^r}{r(r-1)}\ , & r \neq 0,1; \\ -\log x\ , & r = 0; \\ x\log x\ , & r = 1. \end{cases} \tag{4.2}$$

Then $\varphi_r''(x) = x^{r-2}$, *that is* φ_r *is convex for* $x > 0, r \in \mathbb{R}$.

Lemma 4.3. *Consider the functions* $\delta_r : \mathbb{R} \to \mathbb{R}$

$$\delta_r(x) = \begin{cases} \frac{1}{r^2}e^{rx}\ , & r \neq 0; \\ \frac{1}{2}x^2, & r = 0. \end{cases} \tag{4.3}$$

Then $\delta_r''(x) = e^{rx}$, *that is* δ_r *is convex for* $x \in \mathbb{R}, r \in \mathbb{R}$.

Definition 4.4 ([10, p. 2])**.** A function $\phi : I \to \mathbb{R}$ is convex if

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \geq 0, \tag{4.4}$$

holds for every $s_1 < s_2 < s_3$, $s_1, s_2, s_3 \in I$.

The following important subclass i.e., the class of exponentially convex functions, introduced by Bernstein [15], will be crucial importance in studying the properties of Cauchy type means (for example monotonicity). Also our method can give a method of producing families of exponentially convex functions.

Definition 4.5 ([16]). A function $\phi : I \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{k,l=1}^{n} a_k a_l \phi(x_k + x_l) \geq 0,$$

for all $n \in \mathbb{N}$, $a_k \in \mathbb{R}$ and $x_k \in I$, $k = 1, 2, ..., n$ such that $x_k + x_l \in I, 1 \leq k, l \leq n$, or equivalently

$$\sum_{k,l=1}^{n} a_k a_l \phi(\frac{x_k + x_l}{2}) \geq 0.$$

Corollary 4.6 ([16]). *If $\phi : I \to \mathbb{R}$ is exponentially convex function, then*

$$\det\left[\phi(\frac{x_k + x_l}{2})\right]_{k,l=1}^{n} \geq 0$$

for every $n \in \mathbb{N}$ $x_k \in I$, $k = 1, 2, ..., n$.

Corollary 4.7 ([16]). *If $\phi : I \to (0, \infty)$ is exponentially convex function, then ϕ is a* log-*convex function that is*

$$\phi(\lambda x + (1 - \lambda)y) \leq \phi^{\lambda}(x)\phi^{1-\lambda}(y), \ for\ all\ x, y \in I,\ \lambda \in [0, 1].$$

Let w, x, y, u, μ, ϕ be defined as in Theorem 3.5. We define the functional $A(x, y, w; \phi)$ by

$$A(x, y, w; \phi) = \int_c^d \int_a^b w(t, s)\phi(x(t, s))d\mu(t)du(s) - \int_c^d \int_a^b w(t, s)\phi(y(t, s))\, d\mu(t)du(s). \tag{4.5}$$

We begin with defining Cauchy type means for the family of functions η_r.

Theorem 4.8. *Let w, x, y, u, μ be defined as in Theorem 3.5 with $[\alpha, \beta] \subset \mathbb{R}^+ \cup \{0\}$. Also let (3.5) holds. Consider $\Upsilon_r^1 = A(x, y, w; \eta_r)$, $r \in \mathbb{R}^+$, if (3.12) holds for every $\tau \in [\alpha, \beta]$ and $\Upsilon_r^2 = -A(x, y, w; \eta_r)$, if (3.12) holds in the opposite direction for every $\tau \in [\alpha, \beta]$. Then the following statements are valid for Υ_r^i(i=1,2).*

(i) For every $n \in \mathbb{N}$ and for every $r_k \in \mathbb{R}^+$, $k \in \{1, 2, 3, ..., n\}$, the matrix $[\Upsilon^i_{\frac{r_k + r_l}{2}}]_{k,l=1}^{n}$ is positive semi-definite. Particularly

$$\det[\Upsilon^i_{\frac{r_k + r_l}{2}}]_{k,l=1}^{n} \geq 0. \tag{4.6}$$

(ii) The function $r \to \Upsilon_r^i$ is exponentially convex.

(iii) If $\Upsilon_r^i > 0$, then the function $r \to \Upsilon_r^i$ is log-*convex, i.e for $0 < r < s < t < \infty$, we have*

$$(\Upsilon_s^i)^{t-r} \leq (\Upsilon_r^i)^{t-s}(\Upsilon_t^i)^{s-r}. \tag{4.7}$$

Proof. (i) As in [14] we can show that the function defined by

$$\mu(x) = \sum_{k,l=1}^{n} a_k a_l \eta_{r_{kl}}(x),$$

where $r_{kl} = \frac{r_k + r_l}{2} > 0, a_k \in \mathbb{R}$ for all $k \in \{1, 2, 3, ..., n\},\ x \geq 0$ is convex. Therefore we have

$$\sum_{k,l=1}^{n} a_k a_l \Upsilon^i_{r_{kl}} \geq 0, \tag{4.8}$$

hence the matrix $[\Upsilon^i_{\frac{r_k+r_l}{2}}]^n_{k,l=1}$ is positive semi-definite.

(ii) Since $\lim_{r\to 1} \Upsilon^i_r = \Upsilon^i_1$ and $0 \log 0 = 0$, we conclude that Υ^i_r is continuous for all $r > 0, x \geq 0$ and $[\Upsilon^i_{\frac{r_k+r_l}{2}}]^n_{k,l=1}$ is positive semi-definite matrix, so using Definition 4.5 we have that exponential convexity of the function $r \to \Upsilon^i_r$.

(iii) Assume that $\Upsilon^i_r > 0$, then by Corollary 4.7 we have that Υ^i_r is log-convex. □

Let w, x, y, u, μ be stated as in Theorem 3.5 such that (3.5) holds and $[\alpha, \beta] \subset \mathbb{R}^+ \cup \{0\}$. If for all $\tau \in [\alpha, \beta]$, the inequality (3.12) holds or if for all $\tau \in [\alpha, \beta]$, the reverse inequality in (3.12) holds. We also assume that $\Upsilon^i_r > 0$ (i=1,2) for $r \in \mathbb{R}^+$. We give the following definition.

$$M_{l,m} := \left(\frac{\Upsilon^1_l}{\Upsilon^1_m}\right)^{\frac{1}{l-m}} = \left(\frac{\Upsilon^2_l}{\Upsilon^2_m}\right)^{\frac{1}{l-m}} \quad \text{for } l, m \in \mathbb{R}^+ \text{ such that } l \neq m, \tag{4.9}$$

where Υ^i_r is defined in Theorem 4.8. By Remark 3.10 these expressions define a class of means. We can extend these means to the other cases, by limit we have

$$M_{l,l} = \exp\left(\frac{\int_c^d \int_a^b w(t,s)\left[x^l(t,s)\log x(t,s) - y^l(t,s)\log y(t,s)\right] d\mu(t) du(s)}{\int_c^d \int_a^b w(t,s)[x^l(t,s) - y^l(t,s)]\, d\mu(t) du(s)} - \frac{2l-1}{l(l-1)}\right),$$

$$M_{1,1} = \exp\left(\frac{\int_c^d \int_a^b w(t,s)\left[x(t,s)\log^2 x(t,s) - y(t,s)\log^2 y(t,s)\right] d\mu(t) du(s)}{2\int_c^d \int_a^b w(t,s)[x(t,s)\log x(t,s) - y(t,s)\log y(t,s)]\, d\mu(t) du(s)} - 1\right).$$

Theorem 4.9. *Let $c, d, l, m \in \mathbb{R}^+$ such that $c \leq l$ and $d \leq m$, then the following inequality is valid.*

$$M_{c,d} \leq M_{l,m} \quad \textit{for } c, d, l, m \in \mathbb{R}^+ \ . \tag{4.10}$$

Proof. Since Υ^i_r (i=1,2) is log-convex. Therefore it holds ([10, p.2]) that

$$\frac{\log \Upsilon^i_c - \log \Upsilon^i_d}{c-d} \leq \frac{\log \Upsilon^i_l - \log \Upsilon^i_m}{l-m} \tag{4.11}$$

with $c \leq l$, $d \leq m$, $c \neq d$, $l \neq m$. Consequently

$$\left(\frac{\Upsilon^i_c}{\Upsilon^i_d}\right)^{c-d} \leq \left(\frac{\Upsilon^i_l}{\Upsilon^i_m}\right)^{l-m}. \tag{4.12}$$

And for $c = l$ and/ or $d = m$ we consider limiting cases. □

Now, we define Cauchy type means for the family of functions φ_r.

Theorem 4.10. *Let w, x, y, u, μ be defined as in Theorem 3.5 with $[\alpha, \beta] \subset \mathbb{R}^+$. Also let (3.5) holds. Consider $\tilde{\Upsilon}^1_r = A(x, y, w; \varphi_r)$, $r \in \mathbb{R}$, if (3.12) holds for every $\tau \in [\alpha, \beta]$ and $\tilde{\Upsilon}^2_r = -A(x, y, w; \varphi_r)$, $r \in \mathbb{R}$, if (3.12) holds in the opposite direction for every $\tau \in [\alpha, \beta]$. Then the following statements are valid for $\tilde{\Upsilon}^i_r$ (i=1,2).*

(i) For every $n \in \mathbb{N}$ and for every $r_k \in \mathbb{R}$, $k \in \{1, 2, 3, ..., n\}$, the matrix $[\tilde{\Upsilon}^i_{\frac{r_k+r_l}{2}}]^n_{k,l=1}$ is positive semi-definite. Particularly

$$\det[\tilde{\Upsilon}^i_{\frac{r_k+r_l}{2}}]^n_{k,l=1} \geq 0. \tag{4.13}$$

(ii) The function $r \to \tilde{\Upsilon}^i_r$ is exponentially convex.

(iii) If $\tilde{\Upsilon}^i_r > 0$, then the function $r \to \tilde{\Upsilon}^i_r$ is log*-convex, i.e for $-\infty < r < s < t < \infty$, we have*

$$(\tilde{\Upsilon}^i_s)^{t-r} \leq (\tilde{\Upsilon}^i_r)^{t-s} (\tilde{\Upsilon}^i_t)^{s-r}. \tag{4.14}$$

Proof. The proof is similar to the proof of Theorem 4.8. □

Let w, x, y, u, μ be stated as in Theorem 3.5 such that (3.5) holds and $[\alpha, \beta] \subset \mathbb{R}^+$. If for all $\tau \in [\alpha, \beta]$, the inequality (3.12) holds or if for all $\tau \in [\alpha, \beta]$, the reverse inequality in (3.12) holds. We also assume that $\tilde{\Upsilon}^i_r > 0$ for $r \in \mathbb{R}$. We give the following definition.

$$\tilde{M}_{l,m} := \left(\frac{\tilde{\Upsilon}^i_l}{\tilde{\Upsilon}^i_m} \right)^{\frac{1}{l-m}} \quad \text{for } l, m \in \mathbb{R} \text{ such that } l \neq m, \text{ i=1,2}, \tag{4.15}$$

where $\tilde{\Upsilon}^i_r$ is defined in Theorem 4.10 . By Remark 3.10 these expressions define a class of means. We can extend these means to the other cases. Namely, for $l \neq 0, 1$, by limit we have

$$\tilde{M}_{l,l} = \exp\left(\frac{\int_c^d \int_a^b w(t,s)\big(x^l(t,s) \log x(t,s) - y^l(t,s) \log y(t,s)\big)\, d\mu(t) du(s)}{\int_a^b \int_a^b w(t,s)(x^l(t,s) - y^l(t,s))\, d\mu(t) du(s)} - \frac{2l-1}{l(l-1)} \right),$$

$$\tilde{M}_{0,0} = \exp\left(\frac{\int_c^d \int_a^b w(t,s)\big(\log^2 x(t,s) - \log^2 y(t,s)\big)\, d\mu(t) du(s)}{2 \int_c^d \int_a^b w(t,s)(\log x(t,s) - \log y(t,s))\, d\mu(t) du(s)} + 1 \right),$$

and $\tilde{M}_{1,1} = M_{1,1}$.

Theorem 4.11. *Let $c, d, l, m \in \mathbb{R}$ such that $c \leq l$ and $d \leq m$, then the following inequality is valid.*

$$\tilde{M}_{c,d} \leq \tilde{M}_{l,m} \quad \textit{for } c, d, l, m \in \mathbb{R}, \tag{4.16}$$

Proof. The proof is similar to the proof of Theorem 4.9. □

Finally, we define Cauchy type means for the family of functions δ_r.

Theorem 4.12. *Let w, x, y, u, μ be defined as in Theorem 3.5 such that (3.5) holds. Consider $\overline{\Upsilon}^1_r = A(x, y, w; \delta_r)$, $r \in \mathbb{R}$, if (3.12) holds for every $\tau \in [\alpha, \beta]$ or $\overline{\Upsilon}^2_r = -A(x, y, w; \delta_r)$, $r \in \mathbb{R}$, if (3.12) holds in the opposite direction for every $\tau \in [\alpha, \beta]$. Then the following statements are valid for $\overline{\Upsilon}^i_r$ (i=1,2).*

(i) For every $n \in \mathbb{N}$ and for every $r_k \in \mathbb{R}$, $k \in \{1, 2, 3, ..., n\}$, the matrix $[\overline{\Upsilon}^i_{\frac{r_k+r_l}{2}}]^n_{k,l=1}$ is a positive semi-definite. Particularly

$$\det[\overline{\Upsilon}^i_{\frac{r_k+r_l}{2}}]^n_{k,l=1} \geq 0. \tag{4.17}$$

(ii) The function $r \to \overline{\Upsilon}^i_r$ is exponentially convex.

(iii) If $\overline{\Upsilon}^i_r > 0$, then the function $r \to \overline{\Upsilon}^i_r$ is log*-convex, i.e for $-\infty < r < s < t < \infty$, we have*

$$(\overline{\Upsilon}^i_s)^{t-r} \leq (\overline{\Upsilon}^i_r)^{t-s}(\overline{\Upsilon}^i_t)^{s-r}. \tag{4.18}$$

Proof. The proof is similar to the proof of Theorem 4.8. □

Let w, x, y, u, μ be stated as in Theorem 3.5 such that (3.5) holds. If for all $\tau \in [\alpha, \beta]$, the inequality (3.12) holds or if for all $\tau \in [\alpha, \beta]$, the reverse inequality in (3.12) holds and let $\overline{\Upsilon}^i_r > 0$ for $r \in \mathbb{R}$,

$$\overline{M}_{l,m} = \frac{1}{l-m} \log\left(\frac{m^2}{l^2} . \frac{\int_c^d \int_a^b w(t)\left(e^{lx(t,s)} - e^{ly(t,s)}\right) d\mu(t) du(s)}{\int_c^d \int_a^b w(t)\left(e^{mx(t,s)} - e^{my(t,s)}\right) d\mu(t) du(s)}\right) \tag{4.19}$$

for $l, m \in \mathbb{R} \setminus \{0\}, l \neq m$ define a class of means. Moreover we can extend these means to the other cases.
So by limit we have

$$\overline{M}_{l,l} = \frac{\int_c^d \int_a^b w(t,s)\left(x(t,s)e^{lx(t,s)} - y(t,s)e^{ly(t,s)}\right) d\mu(t) du(s)}{\int_c^d \int_a^b w(t,s)\left(e^{lx(t,s)} - e^{ly(t)}\right) d\mu(t) du(s)} - \frac{2}{l}, \ \ l \neq 0,$$

$$\overline{M}_{0,0} = \frac{\int_c^d \int_a^b w(t,s)\left(x^3(t,s) - y^3(t,s)\right) d\mu(t) du(s)}{3\left(\int_c^d \int_a^b w(t,s)\left(x^2(t,s) - y^2(t,s)\right) d\mu(t) du(s)\right)}.$$

Theorem 4.13. *Let $c, d, l, m \in \mathbb{R}$ such that $c \leq l$ and $d \leq m$, then the following inequality is valid.*

$$\overline{M}_{c,d} \leq \overline{M}_{l,m} \quad \textit{for } c, d, l, m \in \mathbb{R}. \tag{4.20}$$

Proof. The proof is similar to the proof of Theorem 4.9. □

Remark 4.14. *We can give Theorem 3.7, Theorem 3.8, Corollary 3.9, Remark 3.10, Theorem 4.8, Cauchy type means (4.9), Theorem 4.9, Theorem 4.10, Cauchy type means (4.15), Theorem (4.11), Theorem (4.12), Cauchy type means (4.19) and Theorem (4.13) in a similar way for Theorem 3.4(a), (b), (c) and (d).*

Remark 4.15. *For related discrete version of this paper see [14] and for some interesting results related to log convexity see [17].*

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