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A New Iterative Algorithm for Variational Inclusions with H-Monotone Operators[1]

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Abstract : In this paper, a new algorithm for solving a class of variational inclusions involving H-monotone operators is considered in Hilbert spaces. We investigate a general iterative algorithm, which consists of a resolvent operator technique step followed by a suitable projection step. We prove the convergence of the algorithm for a maximal monotone operator without Lipschitz continuity. These results generalize many known results in recent literatures.

Keywords : H-monotone operator; resolvent operator technique; maximal monotone operator; variational inclusion.
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1 Introduction

The variational inequality, as an important subject of current mathematics, has not only stimulated new results dealing with partial differential equations, but

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also has been used in large variety of problems arising from mechanics, physics, optimization and control, economic and transportation equilibrium, and engineering sciences. Because of its wide applications, the classic variational inequality has been well studied and generalized in various directions. Among these generalizations, variational inclusion is of interest and importance. One of the most important and interesting problems in theory of variational inequality is the development of an efficient and implementable algorithm for solving variational inequality and its generalizations. It is known that monotonicity of the underlying operators plays a prominent role in the theory of variational inequality and its generalizations.

In order to study various variational inequalities and variational inclusions, in 2003, Fang and Huang [1] introduced a new class of monotone operators named H-monotone operators. For an H-monotone operator, they gave the definition of its resolvent operator and established the Lipschitz continuity of the resolvent operator. By using the resolvent operator technique, they constructed an iterative algorithm for approximating a solution of a class of variational inclusions involving H-monotone operators. Following the works of Fang and Huang [1], Fang et al. [2], Feng and Ding [3], Lan [4], Lan et al. [5], Verma [6, 7], Xia and Huang [8], Zhang [9] have introduced the concepts of (H, η)-monotone operators, A-monotone operators, (A, η)-monotone operators, M-monotone operators, general H-monotone operators, G-η-monotone operators, respectively. By using the resolvent operator technique, they studied a number of variational inequalities and variational inclusions.

In [1], Fang and Huang considered the following variational inclusions: find an $x \in \mathcal{H}$ such that

$$0 \in A(x) + M(x), \tag{1.1}$$

where $\mathcal{H}$ is a Hilbert space, $A : \mathcal{H} \to \mathcal{H}$ is a single-valued operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ is a multi-valued operator. Under the assumptions that $A, H : \mathcal{H} \to \mathcal{H}$ are strongly monotone and Lipschitz continuous, and $M : \mathcal{H} \to 2^{\mathcal{H}}$ is an H-monotone operator, they proved the convergence of an algorithm for solving problem (1.1). However, the strongly monotonicity and Lipschitz continuity were very strong conditions which would restrict the use of this method. In this paper, we relax the assumptions on operators A and H. This is one of the two main motivations of this paper.

On the other hand, problem (1.1) includes many problems as special cases (see, Fang and Huang [1]). For example, if $M = \partial\varphi$, where $\partial\varphi$ denotes the subdifferential of a proper, convex and lower semi-continuous functional $\varphi : \mathcal{H} \to R\bigcup\{+\infty\}$, then problem (1.1) reduces to the following problem: find an $x \in \mathcal{H}$ such that

$$\langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in \mathcal{H}, \tag{1.2}$$

which is called a general mixed variational inequality. If $A : \mathcal{H} \to 2^{\mathcal{H}}$ is a multi-valued operator, (1.2) is called a generalized mixed variational inequality which has been encountered in many applications, in particular, in mechanical problems (see, e.g., [10, 11]) and equilibrium problems (see, e.g., [12, 13]). Moreover, generalized mixed variational inequalities have been studied by many authors, see, for example, [8, 14–16]. Obviously, the variational inclusions (1.1) can not include generalized mixed variational inequalities as a special case. So it is a significant work that how to construct algorithms for approximating an solution of problem (1.1) involving multi-valued operator A. This is another motivation of this paper.

In this paper, we consider the following variational inclusions problem: find an $x \in \mathcal{H}$ such that

$$0 \in A(x) + M(x), \tag{1.3}$$

where $\mathcal{H}$ is a Hilbert space, $A, M : \mathcal{H} \to 2^{\mathcal{H}}$ are two multi-valued operators. We provide a new iterative algorithm for solving problem (1.3) in Hilbert spaces. We first show that how to generate the sequences $\{x^k\}$ and $\{z^k\}$ by an algorithm, which consists of a resolvent operator technique step followed by a suitable orthogonal projection onto a hyperplane. Under the assumption that A, H are maximal monotone, we prove that the sequences $\{x^k\}$ and $\{z^k\}$ are both weakly convergent and the weak limit point of $\{x^k\}$ is the same as that of $\{z^k\}$. We also prove that the weak limit point of these sequences is a solution to problem (1.3).

2 Preliminaries

Suppose that $X \subset \mathcal{H}$ is a nonempty closed convex subset and

$$\mathrm{dist}(z, X) := \inf_{x \in X} \|z - x\|$$

is the distance from z to X. Let $P_X[z]$ denote the projection of z onto X, that is, $P_X[z]$ satisfies the condition

$$\|z - P_X[z]\| = \mathrm{dist}(z, X).$$

The following well-known properties of the projection operator will be used in this paper.

Proposition 2.1 ([17]). *Let X be a nonempty closed convex subset in $\mathcal{H}$. Then the following properties hold:*

(i) $\langle x - y, x - P_X[x] \rangle \geq 0$, *for all $x \in \mathcal{H}$ and $y \in X$;*

(ii) $\langle P_X[x] - x, y - P_X[x] \rangle \geq 0$, *for all $x \in \mathcal{H}$ and $y \in X$;*

(iii) $\|P_X[x] - P_X[y]\| \leq \|x - y\|$, *for all $x, y \in \mathcal{H}$.*

Definition 2.2. Let $T : \mathcal{H} \to \mathcal{H}$ be a single-valued operator. T is said to be

(i) monotone if
$$\langle Tx - Ty, x - y \rangle \geq 0, \ \ \forall x, y \in \mathcal{H};$$

(ii) strictly monotone if, T is monotone and $\langle Tx - Ty, x - y \rangle = 0$ if and only if $x = y$;

(iii) strongly monotone, if there exists constant $\rho > 0$ such that

$$\langle Ty - Tx, y - x\rangle \geq \rho\|x - y\|^2, \ \ \forall x, y \in \mathcal{H}.$$

Definition 2.3. Let $H : \mathcal{H} \to \mathcal{H}$ be a single-valued operator. A multi-valued operator M is said to be

(i) monotone if

$$\langle u - v, x - y\rangle \geq 0, \ \ \forall x, y \in \mathcal{H}, u \in Mx, v \in My;$$

(ii) monotone with respect to H if

$$\langle u - v, Hx - Hy\rangle \geq 0, \ \ \forall x, y \in \mathcal{H}, u \in Mx, v \in My;$$

(iii) maximal monotone if, T is monotone and $(I + \lambda M)(\mathcal{H}) = \mathcal{H}$ for all $\lambda > 0$, where I denotes the identity mapping on $\mathcal{H}$;

(iv) H-monotone [1] if M is monotone and $(H + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$.

Definition 2.4 ([1])**.** Let $H : \mathcal{H} \to \mathcal{H}$ be a strictly monotone operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an H-monotone operator. The resolvent operator $R^H_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ is defined by

$$R^H_{M,\lambda}(x) = (H + \lambda M)^{-1}(x), \ \ \forall x \in \mathcal{H}. \tag{2.1}$$

We will use the following lemma.

Lemma 2.5 ([18])**.** *Let $\sigma \in [0, 1)$ and $\mu = \sqrt{1 - (1 - \sigma^2)^2}$. If $v = u + \xi$ with $\|\xi\|^2 \leq \sigma^2(\|u\|^2 + \|v\|^2)$, then*

(i) $\langle u, v\rangle \geq (\|u\|^2 + \|v\|^2)(1 - \sigma^2)/2$;

(ii) $(1 - \mu)\|v\| \leq (1 - \sigma^2)\|u\| \leq (1 + \mu)\|v\|$.

3 Iterative Algorithm and Convergence Analysis

Choose a positive sequence $\{\lambda_k\}$. Suppose that a single-valued operator $H : \mathcal{H} \to \mathcal{H}$ is onto, that is, $H(\mathcal{H}) = \mathcal{H}$. We describe a new iterative algorithm for variational inclusion problem (1.3).

ALGORITHM 3.1.

Step 0. (Initialization) Select initial $z^0 \in \mathcal{H}$ and set $k = 0$.

Step 1. (Resolvent operator step) Find an $x^k \in \mathcal{H}$ such that

$$x^k = R^H_{M,\lambda_k}[H(z^k) - \lambda_k g^k], \ \ g^k \in A(x^k). \tag{3.1}$$

Step 2. (Projection step) Set $w^k = \frac{1}{\lambda_k}[H(z^k) - H(x^k) - \lambda_k g^k]$. If $g^k + w^k = 0$, then stop; otherwise, set

$$z^{k+1} = H^{-1}[H(z^k) - \beta_k(g^k + w^k)] \ \text{ with } \beta_k = \frac{\langle g^k + w^k, H(z^k) - H(x^k)\rangle}{\|g^k + w^k\|^2}. \tag{3.2}$$

Step 3. Let $k = k+1$ and return to Step 1.

In this paper, we focus our attention on obtaining general conditions ensuring the convergence of $\{z^k\}$ and $\{x^k\}$ toward a solution of problem (1.3), under the following hypothesises:

(A1) A positive sequence $\{\lambda_k\}$ satisfies

$$\alpha_1 := \inf_{k\geq 0} \lambda_k > 0, \quad \alpha_2 := \sup_{k\geq 0} \lambda_k < \infty. \tag{3.3}$$

(A2) The set of solutions of problem (1.3), denoted by S, is nonempty.

(A3) A strongly monotone operator $H : \mathcal{H} \to \mathcal{H}$ is onto, that is, $H(\mathcal{H}) = \mathcal{H}$ and weakly continuous on $\mathcal{H}$.

(A4) A multi-valued operator $M : \mathcal{H} \to 2^{\mathcal{H}}$ is H-monotone and monotone with respect to H.

(A5) A multi-valued operator $A : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal monotone and monotone with respect to H.

By the definition of R^H_{M,λ_k}, we note that subproblem (3.1) is equivalent to the following problem: find an $x^k \in \mathcal{H}$ such that

$$H(z^k) \in H(x^k) + \lambda_k[M(x^k) + g^k]. \tag{3.4}$$

By (3.4) and the definition of w^k in Algorithm 3.1, we have $w^k \in M(z^k)$. If $H : \mathcal{H} \to \mathcal{H}$ is strictly monotone, $M : \mathcal{H} \to 2^{\mathcal{H}}$ is H-monotone, then M is maximal monotone (see proposition 2.1 in [1]). Furthermore, if $A : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal monotone, then $H + \lambda_k[M + A]$ must be a maximal monotone operator, and then $H + \lambda_k[M + A]$ must be onto (see [19]). So the sequence $\{x^k\}$ is well defined. Since $H : \mathcal{H} \to \mathcal{H}$ is onto, by (3.2), we know that the sequence $\{z^k\}$ is also well defined.

It is easy to see that (3.2) is a projection step because it can be written as $H(z^{k+1}) = P_K(H(z^k))$, where $P_K : \mathcal{H} \to K$ is the orthogonal projection operator onto the hyperplane $K = \{z \in \mathcal{H} : \langle g^k + w^k, z - H(x^k)\rangle \leq 0\}$. In fact, by (3.2) we have $H(z^{k+1}) = H(z^k) - \beta_k(g^k + w^k)$. Hence, for each $y \in K$, we deduce that

$$\begin{aligned}
&\langle H(z^k) - H(z^{k+1}), y - H(z^k)\rangle \\
&\qquad = \beta_k\langle g^k + w^k, y - H(z^k)\rangle \\
&\qquad = \beta_k\langle g^k + w^k, y - H(x^k)\rangle + \beta_k\langle g^k + w^k, H(x^k) - H(z^k)\rangle \\
&\qquad = \beta_k\langle g^k + w^k, H(x^k) - H(z^k)\rangle \quad (\text{since } \langle g^k + w^k, y - H(x^k)\rangle \leq 0) \\
&\qquad \leq 0 \quad (\text{since } \beta_k = \langle g^k + w^k, H(z^k) - H(x^k)\rangle / \|g^k + w^k\|^2).
\end{aligned}$$

By Proposition 2.1, we know that $H(z^{k+1}) = P_K(H(z^k))$. By monotonicity of M, A and Theorem 3.1(ii) below, the hyperplane K separates the current iterate $H(z^k)$ from the set H(S), where $S = \{x \in H : 0 \in M(x) + A(x)\}$ (see Remark 3.1 below). Thus, in Algorithm 3.1, the resolvent operator iteration step (3.1) is used to construct this separation hyperplane, the next iterate $H(z^{k+1})$ is then obtained by a trivial projection of $H(z^k)$, which is not expensive at all from a numerical point of view.

Now we give some properties for the iterative sequence generated by Algorithm 3.1 in Hilbert space $\mathcal{H}$. First, we state some useful estimates that are direct consequences of Lemma 2.5.

Theorem 3.1. *Under (3.1)-(3.2), we have*

(i) $\lambda_k\|g^k + w^k\| = \|H(x^k) - H(z^k)\|$;

(ii) $(\lambda_k^2\|g^k + w^k\|^2 + \|H(x^k) - H(z^k)\|^2)/(2\lambda_k) \leq \langle g^k + w^k, H(z^k) - H(x^k)\rangle$.

Proof. Since $w^k = \frac{1}{\lambda_k}[H(z^k) - H(x^k) - \lambda_k g^k]$, we apply Lemma 2.1 to $\sigma = 0$, $\xi = 0$, $v = \lambda_k(g^k + w^k)$ and $u = H(z^k) - H(x^k)$ to get (i) and (ii). □

Remark 3.2. *Suppose that* $g^k + w^k = 0$ *in* **Step 2**, *we have*

$$0 \in M(x^k) + A(x^k),$$

that is, x^k *is a solution of problem (1.3). On the other hand, assuming* $g^k+w^k \neq 0$, *Theorem 3.1 (ii) yields* $\langle g^k + w^k, H(z^k) - H(x^k)\rangle > 0$. *By the monotonicity with respect to* H *of* M *and* A, *it is easy to see that, for all* $x^* \in S$ *(*S *denotes the solution set of problem (1.3)),*

$$\langle 0 - (w^k + g^k), H(x^*) - H(x^k)\rangle \geq 0, \ \ \forall w^k \in M(x^k), g^k \in A(x^k),$$

which leads to

$$\langle w^k + g^k, H(x^*) - H(x^k)\rangle \leq 0, \ \ \forall w^k \in M(x^k), g^k \in A(x^k).$$

Thus, the hyperplane $\{z \in H : \langle g^k + w^k, z - H(x^k)\rangle \leq 0\}$ *strictly separates* $H(z^k)$ *from* $H(S)$. *The latter is the geometric motivation for the projection step (3.2).*

Theorem 3.3. *Suppose that* $x^* \in \mathcal{H}$ *is a solution of problem (1.3). Then*

$$\|H(x^*) - H(z^{k+1})\|^2 \leq \|H(x^*) - H(z^k)\|^2 - \|H(z^{k+1}) - H(z^k)\|^2 \tag{3.5}$$

and so the sequence $\{\|H(x^*) - H(z^k)\|^2\}$ *is convergent (not necessarily to* 0*). Moreover,*

$$\sum_{k=0}^{\infty} \|H(z^{k+1}) - H(z^k)\|^2 < \infty. \tag{3.6}$$

Proof. By Step 2, we have

$$\begin{aligned}
&\|H(x^*) - H(z^{k+1})\|^2 \\
&\quad = \|H(x^*) - H(z^k) - (H(z^{k+1}) - H(z^k))\|^2 \\
&\quad = \|H(x^*) - H(z^k)\|^2 - 2\langle H(x^*) - H(z^k), H(z^{k+1}) - H(z^k)\rangle \\
&\qquad + \|H(z^{k+1}) - H(z^k)\|^2 \\
&\quad = \|H(x^*) - H(z^k)\|^2 - 2\langle H(z^{k+1}) - H(z^k), H(z^{k+1}) - H(z^k)\rangle \\
&\qquad - 2\langle H(x^*) - H(z^{k+1}), H(z^{k+1}) - H(z^k)\rangle + \|H(z^{k+1}) - H(z^k)\|^2 \\
&\quad = \|H(x^*) - H(z^k)\|^2 - 2\langle H(x^*) - H(z^{k+1}), H(z^{k+1}) - H(z^k)\rangle \\
&\qquad - \|H(z^{k+1}) - H(z^k)\|^2.
\end{aligned}$$

Since $x^* \in \mathcal{H}$ is a solution of problem (1.3), we have $0 \in M(x^*) + A(x^*)$. By the monotonicity with respect to H of A and M, we deduce that

$$\langle 0 - (g^k + w^k), H(x^*) - H(x^k)\rangle \geq 0, \ \ \forall g^k \in A(x^k), w^k \in M(x^k),$$

which leads to

$$H(x^*) \in K = \{z \in H : \langle g^k + w^k, z - H(x^k)\rangle \leq 0\}.$$

Since $H(z^{k+1}) = P_K(H(z^k))$, by Proposition 2.1(ii), we have

$$\langle H(z^{k+1}) - H(x^*), H(z^k) - H(z^{k+1})\rangle \geq 0,$$

and so

$$\|H(x^*) - H(z^{k+1})\|^2 \leq \|H(x^*) - H(z^k)\|^2 - \|H(z^{k+1}) - H(z^k)\|^2,$$

which implies that (3.5) holds. Thus,

$$0 \leq \|H(x^*) - H(z^{k+1})\|^2 \leq \|H(x^*) - H(z^k)\|^2, \ \ \forall k \geq 0,$$

which yields that the sequence $\{\|H(x^*) - H(z^k)\|^2\}$ is convergent. Let L_∞ be the limit of $\{\|H(x^*) - H(z^k)\|^2\}$.

Now we prove that (3.6) holds. It follows from (3.5) that

$$0 \leq \|H(z^{k+1}) - H(z^k)\|^2 \leq \|H(x^*) - H(z^k)\|^2 - \|H(x^*) - H(z^{k+1})\|^2 \tag{3.7}$$

and so (3.7) implies that

$$\begin{aligned}
\sum_{k=0}^{\infty} \|H(z^{k+1}) - H(z^k)\|^2 &\leq \sum_{k=0}^{\infty} [\|H(x^*) - H(z^k)\|^2 - \|H(x^*) - H(z^{k+1})\|^2] \\
&= \|H(x^*) - H(z_0)\|^2 - L_\infty.
\end{aligned}$$

Thus, we know that $\sum_{k=0}^{\infty} \|H(z^{k+1}) - H(z^k)\|^2 < \infty$ holds. This completes the proof. □

Theorem 3.4. *Suppose that the positive sequence* $\{\lambda_k\}$ *satisfies (3.3), then there exists a constant* $\zeta > 0$ *such that*

$$\langle g^k + w^k, H(z^k) - H(x^k)\rangle \geq \zeta \|g^k + w^k\|^2. \tag{3.8}$$

Proof. If $g^k + w^k = 0$, then (3.8) holds. Now we assume that $g^k + w^k \neq 0$. By Theorem 3.1 (ii), we have

$$\begin{aligned}\langle g^k + w^k, H(z^k) - H(x^k)\rangle &\geq \frac{\lambda_k^2\|g^k + w^k\|^2 + \|H(x^k) - H(z^k)\|^2}{2\lambda_k}\\ &\geq \frac{\lambda_k\|g^k + w^k\|^2}{2}.\end{aligned}$$

Since $\lambda_k \in [\alpha_1, \alpha_2]$,

$$\langle g^k + w^k, H(z^k) - H(x^k)\rangle \geq \frac{\alpha_1}{2}\|g^k + w^k\|^2.$$

This completes the proof. ☐

Theorem 3.5. *If the positive sequence* $\{\lambda_k\}$ *satisfies (3.4), then*

$$\lim_{k\to\infty} \|g^k + w^k\| = 0. \tag{3.9}$$

Proof. If $g^k + w^k \neq 0$, then it follows from (3.2) and (3.8) that, for all k,

$$\begin{aligned}\|H(z^{k+1}) - H(z^k)\| &= \|\beta_k(g^k + w^k)\|\\ &= \langle g^k + w^k, H(z^k) - H(x^k)\rangle / \|g^k + w^k\|\\ &\geq \zeta\|g^k + w^k\|, \end{aligned}\tag{3.10}$$

which clearly also holds for k satisfying $g^k + w^k = 0$. By (3.6) and (3.10), we have

$$\lim_{k\to\infty} \|g^k + w^k\| = 0.$$

This completes the proof. ☐

Theorem 3.6. *Let* $\{x^k\}$ *and* $\{z^k\}$ *be infinite sequences generated by Algorithm 3.1 and let the positive sequence* $\{\lambda_k\}$ *satisfy (3.3). Then* $\{x^k\}$ *and* $\{z^k\}$ *are both bounded. Moreover,* $\{x^k\}$ *and* $\{z^k\}$ *have the same weak accumulation points.*

Proof. It follows from Theorem 3.3 that the sequence $\{H(z^k)\}$ is bounded. Using Theorem 3.5 and Theorem 3.1 (i), we have

$$\lim_{k\to\infty} \|H(z^k) - H(x^k)\| = 0. \tag{3.11}$$

Since H is a strongly monotone operator, there exists constant $r > 0$ such that

$$r\|x^k - z^k\|^2 \leq \langle H(x^k) - H(z^k), x^k - z^k\rangle \leq \|H(x^k) - H(z^k)\|\|x^k - z^k\|,$$

which leads to

$$r\|x^k - z^k\| \leq \|H(x^k) - H(z^k)\|. \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\lim_{k\to\infty} \|z^k - x^k\| = 0.$$

We deduce

$$\lim_{k\to\infty} (z^k - x^k) = 0. \tag{3.13}$$

By the boundedness of the sequence $\{z^k\}$, we obtain that the sequence $\{x^k\}$ is bounded. Moreover, (3.13) implies that $\{x^k\}$ and $\{z^k\}$ have the same weak accumulation points. This completes the proof. □

We now study the convergence of the sequences $\{x^k\}$ and $\{z^k\}$ generated by Algorithm 3.1.

Theorem 3.7. *Suppose that the sequence $\{x^k\}$ generated by Algorithm 3.1 is finite. Then the last term is a solution of problem (1.3).*

Proof. If the sequence $\{x^k\}$ is finite, then it must stop at **Step 2** for some x^k. In this case, we have $g^k + w^k = 0$, that is, $0 \in M(x^k) + A(x^k)$. So $x^k \in X$ is a solution of problem (1.3). This completes the proof. □

Now, we assume that the sequence $\{x^k\}$ generated by Algorithm 3.1 is infinite and so is the sequence $\{z^k\}$.

Theorem 3.8. *Every weak accumulation point of the sequence $\{x^k\}$ generated by Algorithm 3.1 is a solution of problem (1.3) and so does the sequence $\{z^k\}$.*

Proof. Existence of weak accumulation points of $\{x^k\}$ follows from Theorem 3.6. Let $\hat{x}$ be a weak accumulation point of $\{x^k\}$. We can extract a subsequence that weakly converges to $\hat{x}$. Without loss of generality, let us suppose that $\lim_{k\to\infty} x^k = \hat{x}$ (weakly). By Theorem 3.6, we have $\lim_{k\to\infty} z^k = \hat{x}$ (weakly).

Now we prove each weak accumulation point of $\{x^k\}$ is a solution of problem (1.3). For all $v \in H$, take an arbitrary $u \in M(v) + A(v)$. Then, there exist points $w' \in M(v)$ and $g' \in A(v)$ such that $w' + g' = u$. Therefore,

$$\langle x^k - v, w^k - w'\rangle \geq 0, \;\; \langle x^k - v, g^k - g'\rangle \geq 0.$$

Adding these inequalities, we have

$$\langle x^k - v, w^k + g^k - (w' + g')\rangle \geq 0.$$

Since $w' + g' = u$,

$$\langle x^k - v, -u\rangle \geq -\langle x^k - v, w^k + g^k\rangle. \tag{3.14}$$

Since $\|w^k + g^k\| \to 0$ (by Theorem 3.5) and $\{x^k\}$ is bounded, we have

$$\langle x^k - v, w^k + g^k\rangle \to 0.$$

Taking limits in (3.14),

$$\langle \hat{x} - v, 0 - u\rangle = \lim_{k\to\infty} \langle x^k - v, 0 - u\rangle \geq 0.$$

As M is a H-monotone operator and H is a strictly monotone operator, by Proposition 2.1 of Fang and Huang [1], we know that $M : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator. Since $A : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator, $M + A$ must be maximal monotone because the domain of M intersects the interior of the domain of A (see Rockafellar [20]). Since (v, u) is an arbitrary point in $\mathrm{Gph}(M + A)$, and $M + A$ is maximal monotone, we conclude that $(\hat{x}, 0) \in \mathrm{Gph}(M + A)$ and so $0 \in M(\hat{x}) + A(\hat{x})$. This shows that $\hat{x} \in X$ is a solution of problem (1.3). This completes the proof. □

Theorem 3.9. *The sequence $\{z^k\}$ generated by Algorithm 3.1 has a unique weak accumulation point. Thus, $\{z^k\}$ is weakly convergent and so does the sequence $\{x^k\}$.*

Proof. For each $x^* \in S$, it follows from Theorem 3.3 that the sequence $\{\|H(z^k) - H(x^*)\|^2\}$ converges (not necessarily to 0). Now we prove that the sequence $\{z^k\}$ has a unique weak accumulation point and so does the sequence $\{x^k\}$. Let $\hat{z}$ and $\bar{z}$ be two weak accumulation points of $\{z^k\}$, and $\{z^{k_j}\}$ and $\{z^{k_i}\}$ be two subsequences of $\{z^k\}$ that weakly converge to $\hat{z}$ and $\bar{z}$, respectively. By Theorem 3.8, we know that $\hat{z}$, $\bar{z} \in S$. Then the sequences $\{\|H(z^k) - H(\hat{z})\|^2\}$ and $\{\|H(z^k) - H(\bar{z})\|^2\}$ are convergent. Let $\xi = \lim_{k\to\infty} \|H(z^k) - H(\hat{z})\|^2$, $\eta = \lim_{k\to\infty} \|H(z^k) - H(\bar{z})\|^2$ and $\gamma = \|H(\hat{z}) - H(\bar{z})\|^2$. Then

$$\begin{aligned}\|H(z^{k_j}) - H(\bar{z})\|^2 = \|H(z^{k_j}) - H(\hat{z})\|^2 + \|H(\hat{z}) - H(\bar{z})\|^2 \\ + 2\langle H(z^{k_j}) - H(\hat{z}), H(\hat{z}) - H(\bar{z})\rangle \end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\|H(z^{k_i}) - H(\hat{z})\|^2 = \|H(z^{k_i}) - H(\bar{z})\|^2 + \|H(\hat{z}) - H(\bar{z})\|^2 \\ + 2\langle H(z^{k_i}) - H(\bar{z}), H(\bar{z}) - H(\hat{z})\rangle. \end{aligned} \tag{3.16}$$

Taking limit in (3.15) as $j \to \infty$ and (3.16) as $i \to \infty$, observing that the inner products in the right hand sides of (3.15) and (3.16) converge to 0 because H is weakly continuous and $\hat{z}$, $\bar{z}$ are weak limits of $\{z^{k_j}\}$, $\{z^{k_i}\}$ respectively, and get, using the definitions of ξ, η, γ,

$$\xi = \eta + \gamma, \tag{3.17}$$

$$\eta = \xi + \gamma. \tag{3.18}$$

From (3.17) and (3.18), we get $\xi - \eta = \gamma = \eta - \xi$, which implies $\gamma = 0$, i.e. $H(\hat{z}) = H(\bar{z})$. Since H is strongly monotone, there exists a constant $r > 0$ such that

$$r\|\hat{z} - \bar{z}\|^2 \leq \langle H(\hat{z}) - H(\bar{z}), \hat{z} - \bar{z}\rangle \leq \|H(\hat{z}) - H(\bar{z})\|\|\hat{z} - \bar{z}\|. \tag{3.19}$$

It follows from (3.19) and $H(\hat{z}) = H(\bar{z})$ that

$$\hat{z} = \bar{z}.$$

We conclude that all weak accumulation points of $\{z^k\}$ coincide, i.e., $\{z^k\}$ is weakly convergent. This completes the proof. $\square$

Remark 3.10. *The results in this paper generalize the Theorem 3.1 of Fang and Huang [1] in the following aspects (a) A is a multi-valued operator; (b) H is not necessarily a Lipschitz continuous operator; (c) A is not strongly monotone and Lipschitz continuous.*

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