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The Common Fixed Point Theorems for Six Self-Mappings with Twice Power Type Φ-Contraction Condition[1]

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Abstract : By using the compatible and weakly compatible conditions of self-mapping pair in metric spaces, we discussed the existence and uniqueness of a common fixed point for six self-mappings with twice power type Φ-contractive condition in complete metric spaces. A new common fixed theorem is obtained, which improves and extends some previous results.

Keywords : compatible mapping pair; weakly compatible mapping pair; twice power type Φ-contractive mapping; common fixed point.
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1 Introduction and Preliminaries

In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems

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in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In [1, 2], Jungck introduced more generalized commuting mappings, called compatible and weakly compatible mappings, which are more general than commuting and weakly commuting mappings. These concept has been useful for obtaining more comprehensive fixed point theorems (see, e.g., [1–21], etc.).

Let (X, d) be a metric space and let f and g be two maps from X into itself. f and g are commuting if $fgx = gfx$ for all x in X. To generalize the notion of commuting maps, Sessa [3] introduced the concept of weakly commuting maps. He defines f and g to be weakly commuting if

$$d(fgx, gfx) \leq d(gx, fx)$$

for all $x \in X$. Obviously, commuting maps are weakly commuting but the converse is not true.

In 1986, Jungck [1] gave more generalized commuting and weakly commuting maps called compatible maps. f and g above are called compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Clearly, weakly commuting maps are compatible, but the implication is not reversible (see [1]).

In 1996, Jungck [2] generalized the compatibility by introducing the concept of weak compatibility. He defines f and g to be weakly compatible if $ft = gt$ for some $t \in X$ implies that $fgt = gft$. It is clear that compatible maps are weakly compatible. The converse is not true.

In 1984, Chang [4] proved a common fixed point theorems for three self-mappings which expansive type commuting condition.

In 2001, Gu [5] proved a common fixed point theorems for four self-mappings which Φ-expansive type compatibility condition.

Motivated by the recent works, in this paper, by using the compatible and weakly compatible conditions of self-mapping pair in metric spaces, we discussed the existence and uniqueness of common fixed point for six self-mappings with twice power type Φ-contractive condition in complete metric spaces. A new common fixed theorem is obtained. The results presented in this paper improves and extends some previous results.

Definition 1.1. Let Φ be a function, we called Φ satisfies the condition (Φ), if the function Φ satisfying the following condition:

$$(\Phi) : \Phi : [0, \infty) \to [0, \infty)$$

and continuous at a point t from the right, nondecreasing and $\Phi(t) < t$, $\forall t > 0$.

In order to prove the main results of this paper, we need the following lemma:

Lemma 1.2 ([4]). *Let the function* Φ *satisfies the condition* (Φ), *then we have*

(i) *For all real number* $t \in [0, \infty)$, *if* $t \leq \Phi(t)$, *then* $t = 0$*;*

(ii) *For all nonnegative sequence* $\{t_n\}$, *if* $t_{n+1} \leq \Phi(t_n)$, $n = 1, 2, 3, \ldots$, *then* $\lim_{n\to\infty} t_n = 0$.

Lemma 1.3 ([4]). *Let* (X, d) *be a complete metric space,* $\{y_n\}$ *is a sequence in* X *which satisfies the condition* $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. *Suppose that* $\{y_n\}$ *is not a Cauchy sequence in* X, *then there must exists an* $\epsilon_0 > 0$, *and the positive integer sequence* $\{m_i\}$, $\{n_i\}$, *such that*

(i) $m_i > n_i + 1$, $n_i \to \infty$ $(i \to \infty)$;

(ii) $d(y_{m_i}, y_{n_i}) \geq \epsilon_0$; $d(y_{m_i-1}, y_{n_i}) < \epsilon_0$, *for* $i = 0, 1, 2, \ldots$.

2 Main Results

In this section, we shall prove our main theorems.

Theorem 2.1. *Let* (X, d) *be a complete metric space and let* S, T, A, B, L *and* M *be six mappings of* X *into itself, satisfying the following conditions:*

(i) $S(X) \subset BM(X)$, $T(X) \subset AL(X)$*;*

(ii) $AL = LA$, $SL = LS$, $BM = MB$, $TM = MT$*;*

(iii) $\forall x, y \in X$,

$$d^2(Sx, Ty) \leq \Phi(\max\{d(ALx, BMy)d(ALx, Sx),\\ d(ALx, Ty)d(BMy, Sx), d(ALx, BMy)d(BMy, Ty)\})$$

where Φ *satisfies the condition* (Φ).

If one of the following conditions is satisfied, then S, T, A, B, L *and* M *have a unique common fixed point in* X.

(1) *Either* S *or* AL *is continuous, the pair* (S, AL) *is compatible, the pair* (T, BM) *is weakly compatible;*

(2) *Either* T *or* BM *is continuous, the pair* (T, BM) *is compatible, the pair* (S, AL) *is weakly compatible;*

(3) *Either* AL *or* BM *is surjection, the pairs* (S, AL)*and* (T, BM) *are weakly compatible.*

Proof. Let x_0 in X be arbitrary, since $S(X) \subset BM(X)$, $T(X) \subset AL(X)$, there exists the sequences $\{x_n\}$ and $\{y_n\}$ in X, such that

$$y_{2n} = Sx_{2n} = BMx_{2n+1},\ y_{2n+1} = Tx_{2n+1} = ALx_{2n+2}, \quad for \ \ n = 1, 2, 3, \ldots.$$

Let $d_n = d(y_n, y_{n+1})$. Now we shall show that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.1}$$

In fact, from condition (iii) and the property of Φ, we have

$$\begin{aligned} d^2(y_{2n-1}, y_{2n}) &= d^2(Tx_{2n-1}, Sx_{2n}) = d^2(Sx_{2n}, Tx_{2n-1}) \\ &\le \Phi(\max\{d(ALx_{2n}, BMx_{2n-1})d(ALx_{2n}, Sx_{2n}), \\ &\quad d(ALx_{2n}, Tx_{2n-1})d(BMx_{2n-1}, Sx_{2n}), \\ &\quad d(ALx_{2n}, BMx_{2n-1})d(BMx_{2n-1}, Tx_{2n-1})\}) \\ &= \Phi(\max\{d(y_{2n-1}, y_{2n-2})d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n-1})d(y_{2n-2}, y_{2n}), \\ &\quad d(y_{2n-1}, y_{2n-2})d(y_{2n-2}, y_{2n-1})\}) \\ &= \Phi(\max\{d(y_{2n-1}, y_{2n-2})d(y_{2n-1}, y_{2n}), d^2(y_{2n-1}, y_{2n-2})\}). \end{aligned} \tag{2.2}$$

Now suppose that $d(y_{2n-1}, y_{2n-2}) < d(y_{2n-1}, y_{2n})$, by the (2.2) and the property of function Φ, we get $d^2(y_{2n-1}, y_{2n}) \le \Phi(d^2(y_{2n-1}, y_{2n}))$. Therefore, by virtue of this and using Lemma 1.1(i), we have $d^2(y_{2n-1}, y_{2n}) = 0$, which implies that $d(y_{2n-1}, y_{2n}) = 0$. Thus $d(y_{2n-1}, y_{2n-2}) < d(y_{2n-1}, y_{2n}) = 0$, which is a contradiction. It follows that, in any event, we have $d(y_{2n-1}, y_{2n-2}) \ge d(y_{2n-1}, y_{2n})$. By the (2.2) we have $d^2(y_{2n-1}, y_{2n}) \le \Phi(d^2(y_{2n-1}, y_{2n-2}))$. Hence by Lemma 1.1(ii) we get

$$d^2(y_{2n-1}, y_{2n}) \to 0 \ (n \to \infty)$$

and so $\lim_{n\to\infty} d(y_{2n-1}, y_{2n}) = 0$.

Similarly, it can be proved that $\lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = 0$. Thus, $\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, so we have the conclusion of (2.1).

Next we will prove $\{y_n\}$ is a Cauchy sequence in X. If not, by Lemma 1.2, there exists an $\epsilon_0 > 0$ and the positive integer sequences $\{m_i\}$, $\{n_i\}$, such that

(a) $m_i > n_i + 1$, $n_i \to \infty$ $(i \to \infty)$.

(b) $d(y_{m_i}, y_{n_i}) \ge \epsilon_0$; $d(y_{m_i-1}, y_{n_i}) < \epsilon_0$, for $i = 0, 1, 2, \ldots$.

Letting $e_i = d(y_{m_i}, y_{n_i})$, then we get

$$\epsilon_0 \le e_i \le d(y_{m_i}, y_{m_i-1}) + d(y_{m_i-1}, y_{n_i}) < \epsilon_0 + d(y_{m_i-1}, y_{m_i}).$$

Letting $i \to \infty$ in the above, and from the (2.1) we have

$$\lim_{i\to\infty} e_i = \epsilon_0 \ \text{(from the right)}. \tag{2.3}$$

On the other hand, we have

$$e_i = d(y_{m_i}, y_{n_i}) \le d(y_{m_i}, y_{m_i+1}) + d(y_{m_i+1}, y_{n_i+1}) + d(y_{n_i+1}, y_{n_i}). \tag{2.4}$$

Now we consider four possible cases for $d(y_{m_i+1}, y_{n_i+1})$.

Case (1): We can assume that n_i is odd and m_i is even. By virtue of condition (iii), we have

$$\begin{aligned}d^2(y_{m_i+1}, y_{n_i+1}) &= d^2(Tx_{m_i+1}, Sx_{n_i+1}) = d^2(Sx_{n_i+1}, Tx_{m_i+1})\\ &\leq \Phi(\max\{d(ALx_{n_i+1}, BMx_{m_i+1})d(ALx_{n_i+1}, Sx_{n_i+1}),\\ &\quad d(ALx_{n_i+1}, Tx_{m_i+1})d(BMx_{m_i+1}, Sx_{n_i+1}),\\ &\quad d(ALx_{n_i+1}, BMx_{m_i+1})d(BMx_{m_i+1}, Tx_{m_i+1})\})\\ &= \Phi(\max\{d(y_{n_i}, y_{m_i})d(y_{n_i}, y_{n_i+1}), d(y_{n_i}, y_{m_i+1})d(y_{m_i}, y_{n_i+1})\\ &\quad d(y_{n_i}, y_{m_i})d(y_{m_i}, y_{m_i+1})\})\\ &\leq \Phi(\max\{e_i d_{n_i}, (e_i + d_{m_i})(e_i + d_{n_i}), e_i d_{m_i}\}). \qquad (2.5)\end{aligned}$$

Letting $i \to \infty$ in (2.5), in view of (2.1), (2.3) and the assumption about $\Phi(t)$ is right-continuous, we have $\lim_{i\to\infty} d^2(y_{m_i+1}, y_{n_i+1}) \leq \Phi(\epsilon_0^2)$, this implies that

$$\lim_{i\to\infty} d(y_{m_i+1}, y_{n_i+1}) \leq [\Phi(\epsilon_0^2)]^{1/2}. \qquad (2.6)$$

Letting $i \to \infty$ in (2.4), using (2.1) and (2.6) we obtain

$$\epsilon_0 \leq e_i \leq 0 + [\Phi(\epsilon_0^2)]^{1/2} + 0 = [\Phi(\epsilon_0^2)]^{1/2}.$$

It follows that $\epsilon_0^2 \leq e_i^2 \leq \Phi(\epsilon_0^2) < \epsilon_0^2$, which is a contradiction.

Case (2): We can assume that n_i and m_i are all even. By virtue of condition(iii), we have

$$d(y_{m_i+1}, y_{n_i+1}) = d(Tx_{m_i+1}, Tx_{n_i+1}) \leq d(Sx_{n_i}, Tx_{m_i+1}) + d(Sx_{n_i}, Tx_{n_i+1}). \qquad (2.7)$$

$$\begin{aligned}d^2(Sx_{n_i},&Tx_{m_i+1})\\ &\leq \Phi(\max\{d(ALx_{n_i}, BMx_{m_i+1})d(ALx_{n_i}, Sx_{n_i}),\\ &\quad d(ALx_{n_i}, Tx_{m_i+1})d(BMx_{m_i+1}, Sx_{n_i}),\\ &\quad d(ALx_{n_i}, BMx_{m_i+1})d(BMx_{m_i+1}, Tx_{m_i+1})\})\\ &= \Phi(\max\{d(y_{n_i-1}, y_{m_i})d(y_{n_i-1}, y_{n_i}), d(y_{n_i-1}, y_{m_i+1})d(y_{m_i}, y_{n_i}),\\ &\quad d(y_{n_i-1}, y_{m_i})d(y_{m_i}, y_{m_i+1})\})\\ &\leq \Phi(\max\{(d_{n_i-1} + e_i)d_{n_i-1}, (d_{n_i-1} + e_i + d_{m_i})e_i, (d_{n_i-1} + e_i)d_{m_i}\}). \qquad (2.8)\end{aligned}$$

On letting $i \to \infty$ in (2.8), using (2.1), (2.3) and the assumption about $\Phi(t)$ is right-continuous, we have $\lim_{i\to\infty} d^2(Sx_{n_i}, Tx_{m_i+1}) \leq \Phi(\epsilon_0^2)$, which implies that

$$\lim_{i\to\infty} d(Sx_{n_i}, Tx_{m_i+1}) \leq [\Phi(\epsilon_0^2)]^{1/2}. \qquad (2.9)$$

It follows from (2.1) that

$$\lim_{i\to\infty} d(Sx_{n_i}, Tx_{n_i+1}) = \lim_{i\to\infty} d(y_{n_i}, y_{n_i+1}) = 0. \qquad (2.10)$$

Letting $i \to \infty$ in (2.7), using (2.9) and (2.10) we obtain

$$\lim_{i\to\infty} d(y_{m_i+1}, y_{n_i+1}) \leq [\Phi(\epsilon_0^2)]^{1/2} + 0 = [\Phi(\epsilon_0^2)]^{1/2}. \tag{2.11}$$

Letting $i \to \infty$ in (2.4), using (2.1) and (2.11) we obtain

$$\epsilon_0 \leq e_i \leq 0 + [\Phi(\epsilon_0^2)]^{1/2} + 0 = [\Phi(\epsilon_0^2)]^{1/2},$$

this implies that $\epsilon_0^2 \leq e_i^2 \leq \Phi(\epsilon_0^2) < \epsilon_0^2$, which is a contradiction.

Similarly, we can also complete the proof when n_i and m_i are all odd, or n_i is even and m_i is odd. This is the anticipated contradiction. Hence, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, suppose that $y_n \to y^* \in X$ then the sequences $\{y_{2n-1}\}$ and $\{y_{2n}\}$ are said to be convergent to y^*, which implies that

$$Ax_{2n} = y_{2n-1} \to y^*, Sx_{2n} = y_{2n} \to y^* \ (n \to \infty). \tag{2.12}$$

(1) Either S or AL is continuous, the pair (S, AL) is compatible ,the pair (T, BM) is weakly compatible.

As AL is continuous, $\{(AL)^2 x_{2n}\}$ and $\{(AL)Sx_{2n}\}$ are all converge to ALy^*. Since (S, AL) is compatible, by (2.12) we have $d(S(AL)x_{2n}, (AL)Sx_{2n}) \to 0 \ (n \to \infty)$. Thus

$$S(AL)x_{2n} \to ALy^* \ (n \to \infty).$$

Now we will prove that y^* is a common fixed point of S, T, A, B, L and M. We finish the proof by the following six steps.

Step 1. We shall prove $ALy^* = y^*$. In fact, using condition (iii) we have

$$\begin{aligned} d^2(S(AL)x_{2n}, Tx_{2n+1}) &\leq \Phi(\max\{d((AL)^2x_{2n}, BMx_{2n+1})d((AL)^2x_{2n}, S(AL)x_{2n}), \\ &\quad d((AL)^2x_{2n}, Tx_{2n+1})d(BMx_{2n+1}, S(AL)x_{2n}), \\ &\quad d((AL)^2x_{2n}, BMx_{2n+1})d(BMx_{2n+1}, Tx_{2n+1})\}). \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d^2(ALy^*, y^*) &\leq \Phi(\max\{d(ALy^*, y^*)d(ALy^*, ALy^*), \\ &\quad d(ALy^*, y^*)d(y^*, ALy^*), d(ALy^*, y^*)d(y^*, y^*)\}) \\ &= \Phi(d^2(ALy^*, y^*)). \end{aligned} \tag{2.13}$$

From the (2.13) and Lemma1.1 (i) we have $d^2(ALy^*, y^*) = 0$, which implies that $ALy^* = y^*$.

Step 2. We shall prove $Sy^* = y^*$. Using condition (iii) we have

$$\begin{aligned} d^2(Sy^*, Tx_{2n+1}) &\leq \Phi(\max\{d(ALy^*, BMx_{2n+1})d(ALy^*, Sy^*), \\ &\quad d(ALy^*, Tx_{2n+1})d(BMx_{2n+1}, Sy^*), \\ &\quad d(ALy^*, BMx_{2n+1})d(BMx_{2n+1}, Tx_{2n+1})\}). \end{aligned} \tag{2.14}$$

Note that $ALy^* = y^*$, letting $n \to \infty$ in (2.14) we have

$$\begin{aligned} d^2(Sy^*, y^*) &\leq \Phi(\max\{d(ALy^*, y^*)d(ALy^*, Sy^*), \\ &\qquad d(ALy^*, y^*)d(y^*, Sy^*), d(ALy^*, y^*)d(y^*, y^*)\}) \\ &= \Phi(0) \leq \Phi(d^2(Sy^*, y^*)) \ \text{(since } \Phi(t) \text{ is nondecreasing).} \end{aligned} \tag{2.15}$$

By the (2.15) and Lemma1.1 (i) we have $d^2(Sy^*, y^*) = 0$, this implies that $Sy^* = y^*$.

Step 3. We shall prove $Ty^* = y^*$. Since $Sy^* = y^*$ and $S(X) \subset BM(X)$, $\exists u \in X$, such that $y^* = ALy^* = Sy^* = BMu$. Using condition (iii) we have

$$\begin{aligned} d^2(BMu, Tu) &= d^2(Sy^*, Tu) \\ &\leq \Phi(\max\{d(ALy^*, BMu)d(ALy^*, Sy^*), \\ &\qquad d(AL)y^*, Tu)d(BMu, Sy^*), d(ALy^*, BMu)d(BMu, Tu)\}) \\ &= \Phi(0) \leq \Phi(d^2(BMu, Tu)) \ \text{(since } \Phi(t) \text{ is nondecreasing).} \end{aligned}$$

Therefore, by Lemma 1.1 (i)we have $d^2(BMu, Tu) = 0$, and so $BMu = Tu$. Since (T, BM) is weakly compatible, we have $Ty^* = T(BM)u = (BM)Tu = BMy^*$. Using condition (iii) we have

$$\begin{aligned} d^2(y^*, Ty^*) &= d^2(Sy^*, Ty^*) \\ &\leq \Phi(\max\{d(ALy^*, BMy^*)d(ALy^*, Sy^*), \\ &\qquad d(ALy^*, Ty^*)d(BMy^*, Sy^*), d(ALy^*, BMy^*)d(BMy^*, Ty^*)\}) \\ &= \Phi(d^2(y^*, Ty^*)). \end{aligned}$$

Therefore, by Lemma 1.1 (i), we have $d^2(y^*, Ty^*) = 0$, this implies that $Ty^* = y^*$.

Step 4. We prove that $BMy^* = y^*$. Using condition (iii) we have

$$\begin{aligned} d^2(Sx_{2n}, Tu) &\leq \Phi(\max\{d(ALx_{2n}, BMu)d(ALx_{2n}, Sx_{2n}), \\ &\qquad d(ALx_{2n}, Tu)d(BMu, Sx_{2n}), d(ALx_{2n}, BMu)d(BMu, Tu)\}). \end{aligned}$$

Letting $n \to \infty$, and note that $BMu = Tu$, we have

$$\begin{aligned} d^2(y^*, Tu) &\leq \Phi(\max\{d(y^*, Tu)d(y^*, y^*), d(y^*, Tu)d(Tu, y^*), d(y^*, Tu)d(Tu, Tu)\}) \\ &= \Phi(d^2(y^*, Tu)). \end{aligned}$$

Therefore, by Lemma 1.1(i), we have $d^2(y^*, Tu) = 0$, which implies that $Tu = y^*$. Hence $Tu = y^* = BMu$. Since (T, BM) is weakly compatible, we have

$$y^* = Ty^* = T(BM)u = (BM)Tu = BMy^*.$$

Thus, $BMy^* = y^*$.

Combining step (1)-(4), we get: $y^* = Sy^* = Ty^* = ALy^* = BMy^*$.

Step 5. We shall prove $Ly^* = y^*$, $Ay^* = y^*$. Using condition (iii) we have

$$\begin{aligned} d^2(SLy^*, Tx_{2n+1}) \leq \Phi(\max\{&d((AL)Ly^*, BMx_{2n+1})d((AL)Ly^*, SLy^*),\\ &d(AL)Ly^*, Tx_{2n+1})d(BMx_{2n+1}, SLy^*),\\ &d((AL)Ly^*, BMx_{2n+1})d(BMx_{2n+1}, Tx_{2n+1})\}). \end{aligned} \tag{2.16}$$

Since $LS = SL, AL = LA$, we have $LSy^* = SLy^* = Ly^*$, $(AL)Ly^* = L(AL)y^* = Ly^*$. Thus, letting $n \to \infty$ in (2.16), and note that $ALy^* = y^*$ we obtain

$$\begin{aligned} d^2(Ly^*, y^*) &\leq \Phi(\max\{d(Ly^*, y^*)d(Ly^*, Ly^*), d(Ly^*, y^*)d(y^*, Ly^*),\\ &\qquad d(Ly^*, y^*)d(y^*, y^*)\})\\ &= \Phi(d^2(Ly^*, y^*)). \end{aligned}$$

Therefore, by Lemma 1.1 (i), we have $d^2(Ly^*, y^*) = 0$, this implies that $Ly^* = y^*$. Hence from $ALy^* = y^*$ we get $Ay^* = y^*$.

Step 6. *We shall prove $My^* = y^*$, $By^* = y^*$. Using condition (iii), we have*

$$\begin{aligned} d^2(Sx_{2n}, TMy^*) \leq \Phi(\max\{&d(ALx_{2n}, (BM)My^*)d(ALx_{2n}, Sx_{2n}),\\ &d(ALx_{2n}, TMy^*)d((BM)My^*, Sx_{2n}),\\ &d(ALx_{2n}, (BM)My^*)d((BM)My^*, TMy^*)\}). \end{aligned} \tag{2.17}$$

Since $BM = MB$, $TM = MT$, $BMy^* = y^*$, we have $TMy^* = MTy^* = My^*$ and $(BM)My^* = M(BM)y^* = My^*$. Letting $n \to \infty$ in (2.17), and note that $Ty^* = y^*$ and $BMy^* = Ty^*$ we have

$$\begin{aligned} d^2(y^*, My^*) &\leq \Phi(\max\{d(y^*, My^*)d(y^*, y^*), d(y^*, My^*)d(My^*, y^*),\\ &\qquad d(y^*, My^*)d(My^*, My^*)\})\\ &= \Phi(d^2(y^*, My^*)). \end{aligned}$$

Therefore, by Lemma 1.1 (i), we have $d^2(y^*, My^*) = 0$, which implies that $My^* = y^*$. From $BMy^* = y^*$ we get $By^* = y^*$.

By the above, we have $y^* = Sy^* = Ty^* = Ay^* = By^* = Ly^* = My^*$. Hence, we prove that y^* is a common fixed point of $S, T, A, B, L$and M in this case. As S is continuous, $\{S^2x_{2n}\}$ and $\{S(AL)x_{2n}\}$ all converge to Sy^*. As (S, AL) is compatible and by (2.12), we have $d(S(AL)x_{2n}, (AL)Sx_{2n}) \to 0 (n \to \infty)$. Thus

$$(AL)Sx_{2n} \to Sy^* (n \to \infty).$$

Using condition (iii), we have

$$\begin{aligned} d^2(S^2x_{2n}, Tx_{2n+1}) \leq \Phi(\max\{&d((AL)Sx_{2n}, BMx_{2n+1})d((AL)Sx_{2n}, S^2x_{2n}),\\ &d((AL)Sx_{2n}, Tx_{2n+1})d(BMx_{2n+1}, S^2x_{2n}),\\ &d((AL)Sx_{2n}, BMx_{2n+1})d(BMx_{2n+1}, Tx_{2n+1})\}). \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d^2(Sy^*, y^*) &\le \Phi(\max\{d(Sy^*, y^*)d(Sy^*, Sy^*), d(Sy^*, y^*)d(y^*, Sy^*), \\ &\qquad d(Sy^*, y^*)d(y^*, y^*)\}) \\ &= \Phi(d^2(Sy^*, y^*)). \end{aligned}$$

Therefore, by Lemma 1.1 (i), we have $d^2(Sy^*, y^*) = 0$, which implies that $Sy^* = y^*$. Since $y^* = Sy^* \in S(X) \subset BM(X)$, then exists $v \in X$ such that $y^* = Sy^* = BMv$. Using condition (iii) we have

$$\begin{aligned} &d^2(S^2x_{2n}, Tv) \\ &\quad \le \Phi(\max\{d((AL)Sx_{2n}, BMv)d((AL)Sx_{2n}, S^2x_{2n}), \\ &\qquad d((AL)Sx_{2n}, Tv)d(BMv, S^2x_{2n}), d((AL)Sx_{2n}, BMv)d(BMv, Tv)\}). \end{aligned}$$

Letting $n \to \infty$, and note that $Sy^* = BMv$ we have

$$\begin{aligned} d^2(Sy^*, Tv) &\le \Phi(\max\{d(Sy^*, BMv)d(Sy^*, Sy^*), d(Sy^*, Tv)d(BMv, Sy^*), \\ &\qquad d(Sy^*, BMv)d(BMv, Tv)\}) \\ &= \Phi(0) \le \Phi(d^2(Sy^*, Tv)). \end{aligned}$$

Therefore, by Lemma 1.1 (i) we have $d^2(Sy^*, Tv) = 0$, which implies that $Sy^* = Tv$. Thus, $y^* = Sy^* = BMv = Tv$. Since (T, BM) is weakly compatible, we have $Ty^* = T(BM)v = (BM)Tv = BMy^*$. Again using condition (iii), we have

$$\begin{aligned} &d^2(Sx_{2n}, Ty^*) \\ &\quad \le \Phi(\max\{d(ALx_{2n}, BMy^*)d(ALx_{2n}, Sx_{2n}), \\ &\qquad d(ALx_{2n}, Ty^*)d(BMy^*, Sx_{2n}), d(ALx_{2n}, BMy^*)d(BMy^*, Ty^*)\}). \end{aligned}$$

Letting $n \to \infty$, and note that $BMy^* = Ty*$ we obtain

$$\begin{aligned} d^2(y^*, Ty^*) &\le \Phi(\max\{d(y^*, Ty^*)d(y^*, y^*), d(y^*, Ty^*)d(Ty^*, y^*), \\ &\qquad d(y^*, Ty^*)d(Ty^*, Ty^*)\}) \\ &= \Phi(d^2(y^*, Ty^*)). \end{aligned}$$

Therefore, by Lemma 1.1 (i), we have $d^2(y^*, Ty^*) = 0$, this implies that $y^* = Ty^*$. Since $y^* = Ty^* \in T(X) \subset AL(X)$, then exists $w \in X$ such that $y^* = Ty^* = ALw$. Using condition (iii), and note that $BMy^* = Ty* = ALw$ we have

$$\begin{aligned} d^2(Sw, y^*) &= d^2(Sw, Ty^*) \\ &\le \Phi(\max\{d(ALw, BMy^*)d(ALw, Sw), d(ALw, Ty^*)d(BMy^*, Sw), \\ &\qquad d(ALw, BMy^*)d(BMy^*, Ty^*)\}) \\ &= \Phi(0) \le \Phi(d^2(Sw, y^*)). \end{aligned}$$

Therefore, by Lemma 1.1 (i), we have $d^2(Sw, y^*) = 0$, which implies that $y^* = Sw$, thus $y^* = ALw = Sw$. Since (S, AL) is compatible, we have $Sy^* = S(AL)w = (AL)Sw = ALy^*$.

By the above, we have $y^* = Sy^* = Ty^* = ALy^* = BMy^*$.

The fact that $y^* = Ay^* = By^* = Ly^* = My^*$ can be proved similar above the step 5 and the step 6. Thus $y^* = Sy^* = Ty^* = Ay^* = By^* = Ly^* = My^*$. Hence, we prove that y^* is a common fixed point of $S, T, A, B, L$and M in this case.

Next we prove y^* is the unique common fixed point of S, T, A, B, L and M. Let us suppose that z be another common fixed point of S, T, A, B, L and M, then from condition (iii) we get

$$\begin{aligned} d^2(y^*, z) &= d^2(Sy^*, Tz) \\ &\leq \Phi(\max\{d(ALy^*, BMz)d(ALy^*, Sy^*), d(ALy^*, Tz)d(BMz, Sy^*), \\ &\quad d(ALy^*, BMz)d(BMz, Tz)\}) \\ &= \Phi(d^2(y^*, z)). \end{aligned}$$

By virtue of this and using Lemma 1.1 (i), we obtain that $d^2(y^*, z) = 0$, which implies that $y^* = z$. Therefore, y^* is a unique common fixed point of S, T, A, B, L and M.

(2) Either T or BM is continuous, the pair (T, BM) is compatible ,the pair (S, AL) is weakly compatible. The proof is similar (1).

(3) Either AL or BM is surjection, the pairs (S, AL) and (T, BM) are weakly compatible.

Suppose that AL is surjection, then $y^* \in X$, $\exists u \in X$, such that $ALu = y^*$. By condition(ii), we get

$$\begin{aligned} &d^2(Su, Tx_{2n+1}) \\ &\leq \Phi(\max\{d(ALu, BMx_{2n+1})d(ALu, Su), \\ &\quad d(ALu, Tx_{2n+1})d(BMx_{2n+1}, Su), d(ALu, BMx_{2n+1})d(BMx_{2n+1}, Tx_{2n+1})\}). \end{aligned}$$

Letting $n \to \infty$, we get

$$d^2(Su, y^*) \leq \Phi(0) \leq \Phi(d^2(Su, y^*)).$$

Therefore, by Lemma 1.1 (i) we have $d^2(Su, y^*) = 0$, which implies that $Su = y^*$. Thus $Su = ALu = y^*$. Since (S, AL) is weakly compatible, we have $ALy^* = (AL)Su = S(AL)u = Sy^*$. Putting $u = y^*$ in (2.18), we get $Sy^* = y^*$. Hence, $ALy^* = Sy^* = y^*$. Now, using step 3- step 6 and continuing step 6 we can prove that y^* is a common fixed point of S, T, A, B, L and M. Thus it follows easily from condition (iii) that y^* is a unique common fixed point of S, T, A, B, L and M.

Suppose that BM is surjection, similarly, we can prove y^* is the unique common fixed point of S, T, A, B, L and M. This completes the proof of Theorem 2.1. □

Remark 2.2. *In Theorem 2.1, we taken: 1) $S = T$; 2) $A = B$; 3) $L = M$; 4) $S = T$ and $A = B$; 5) $S = T$, $A = B$ and $L = M$; 6)$S = T$ and $A = B = I$; 7) $S = T$, $A = B$ and $L = M = I$, several new result can be obtain.*

If we take $L = M = I$ (I is the identity map) in Theorem 2.1, then we can obtain following results.

Theorem 2.3. *Let (X, d) be a complete metric space and let S, T, A and B be four mappings of X into itself, satisfying the following conditions:*

(i) $S(X) \subset B(X)$, $T(X) \subset A(X)$;

(ii) $\forall x, y \in X$,

$$d^2(Sx, Ty) \le \Phi(\max\{d(Ax, By)d(Ax, Sx), d(Ax, Ty)d(By, Sx), d(Ax, By)d(By, Ty)\})$$

where Φ satisfies the condition (Φ).

If one of the following conditions is satisfied, then S, T, A and B have a unique common fixed point in X.

(1) Either S or A is continuous, the pair (S, A) is compatible, the pair (T, B) is weakly compatible;

(2) Either T or B is continuous, the pair (T, B) is compatible, the pair (S, A) is weakly compatible;

(3) Either AL or B is surjection, the pairs $(S, A)$and (T, B) are weakly compatible.

Now, we give an example to support the Theorem 2.3.

Example 2.4. *Let $X = [0, 1]$ be a metric space with the usual metric $d(x, y) = |x - y|$, $\forall\ x, y \in X$. Define $\Phi(t) = \frac{t}{2}$, $\forall\ t \in [0, \infty)$, the maps A, B, S and T as follows:*

$$Ax = x\ \ \forall\ x \in X,\ Tx = \begin{cases} \frac{4}{5},\ x \in [0, \frac{1}{2}], \\ \frac{5}{6},\ x \in (\frac{1}{2}, 1]. \end{cases},$$

$$Sx = \begin{cases} 1,\ x \in [0, \frac{1}{2}], \\ \frac{5}{6},\ x \in (\frac{1}{2}, 1]. \end{cases} \quad and \quad Bx = \begin{cases} 1,\ x \in [0, \frac{1}{2}], \\ \frac{5}{6},\ x \in (\frac{1}{2}, 1), \\ 0, \qquad x = 1. \end{cases}$$

We know A is continuous, but S, T and B are are discontinuous. Since $AX = X$, $TX = \{\frac{4}{5}, \frac{5}{6}\}$, $SX = \{\frac{5}{6}, 1\}$ and $BX = \{0, \frac{5}{6}, 1\}$, we have $SX \subset BX$ and $TX \subset AX$. Therefore, the condition (i) of Theorem 2.3 is satisfied.

On the other hand, by definition we know,

$$d(Sx, Ty) = \begin{cases} \frac{1}{5}, & x, y \in [0, \frac{1}{2}], \\ \frac{1}{6}, & x \in [0, \frac{1}{2}],\ y \in (\frac{1}{2}, 1], \\ \frac{1}{30}, & x \in (\frac{1}{2}, 1],\ y \in [0, \frac{1}{2}], \\ 0, & x, y \in (\frac{1}{2}, 1]. \end{cases} ;\ d(Ax, By) = \begin{cases} |x - 1|,\ y \in [0, \frac{1}{2}], \\ |x - \frac{5}{6}|,\ y \in (\frac{1}{2}, 1), \\ x, \qquad y = 1. \end{cases}$$

$$d(Ax,Sx)=\begin{cases}|x-1|, & x\in[0,\frac{1}{2}],\\ |x-\frac{5}{6}|, & x\in(\frac{1}{2},1].\end{cases};\ d(By,Sx)=\begin{cases}0, & x,y\in[0,\frac{1}{2}],\\ \frac{1}{6}, & x\in[0,\frac{1}{2}],\ y\in(\frac{1}{2},1),\\ 1, & x\in[0,\frac{1}{2}],\ y=1,\\ \frac{1}{6}, & x\in(\frac{1}{2},1],\ y\in[0,\frac{1}{2}],\\ 0, & x\in(\frac{1}{2},1],\ y\in(\frac{1}{2},1),\\ \frac{5}{6}, & x\in(\frac{1}{2},1],\ y=1.\end{cases}$$

$$d(By,Ty)=\begin{cases}\frac{1}{5}, & y\in[0,\frac{1}{2}],\\ 0, & y\in(\frac{1}{2},1),\\ \frac{5}{6}, & y=1.\end{cases};\ d(Ax,Ty)=\begin{cases}|x-\frac{4}{5}|, & y\in[0,\frac{1}{2}],\\ |x-\frac{5}{6}|, & y\in(\frac{1}{2},1].\end{cases}$$

Thus we can obtain that

$$d(SAx,ASx)=d(1,1)=0\le d(Sx,Ax),\ \ \forall\, x\in\left[0,\frac{1}{2}\right];$$

$$d(TBx,BTx)=d\left(\frac{5}{6},\frac{5}{6}\right)=0\le d(Tx,Bx),\ \ \forall\, x\in\left[0,\frac{1}{2}\right];$$

$$d(SAx,ASx)=d\left(\frac{5}{6},\frac{5}{6}\right)=0\le d(Sx,Ax),\ \ \forall\, x\in\left(\frac{1}{2},1\right];$$

$$d(TBx,BTx)=d\left(\frac{5}{6},\frac{5}{6}\right)=0=d(Tx,Bx),\ \ \forall\, x\in\left(\frac{1}{2},1\right);$$

$$d(TBx,BTx)=d\left(\frac{4}{5},\frac{5}{6}\right)=\frac{1}{30}<\frac{5}{6}=d(Tx,Bx),\ \ x=1.$$

This implies that

$$d(SAx,ASx)\le d(Sx,Ax)\ \ \textit{and}\ \ d(TBx,BTx)\le d(Tx,Bx)\ \forall\, x\in[0,1].$$

Therefore, the pairs (S,A) and (T,B) are satisfying the condition (1) of the Theorem 2.3.

Now we prove the contraction condition (ii) of Theorem 2.3 is satisfied. In fact, for any $x,y\in(\frac{1}{2},1]$, we have $d(Sx,Ty)=0$. Hence the condition (ii) is satisfied. Next we will prove the contractive condition (ii) is hold by the four steps.

Step 1. *For all $x,y\in[0,\frac{1}{2}]$, we have $|x-1|\in[\frac{1}{2},1]$, this shows that*

$$\begin{aligned}\max\{d(Ax,By)d(Ax,Sx),d(Ax,Ty)d(By,Sx),d(Ax,By)d(By,Ty)\}\\ &=\max\left\{(x-1)^2,\left|x-\frac{4}{5}\right|\cdot 0,|x-1|\cdot\frac{1}{5}\right\}\\ &\ge\max\left\{\left(\frac{1}{2}\right)^2,0,\frac{1}{2}\cdot\frac{1}{5}\right\}=\frac{1}{4}.\end{aligned}$$

Hence we have

$$\begin{aligned}d^2(Sx,Ty) &= \left(\frac{1}{5}\right)^2 < \frac{1}{2}\cdot\frac{1}{4} = \Phi\left(\frac{1}{4}\right)\\ &\le \Phi\left(\max\left\{(x-1)^2,\ \left|x-\frac{4}{5}\right|\cdot 0,\ |x-1|\cdot\frac{1}{5}\right\}\right)\\ &= \max\{d(Ax,By)d(Ax,Sx),\ d(Ax,Ty)d(By,Sx),\ d(Ax,By)d(By,Ty)\}.\end{aligned}$$

Which implies that the contraction condition (ii) of Theorem 2.3 is satisfied for all $x, y \in [0, \frac{1}{2}]$.

Step 2. *For all* $x \in [0, \frac{1}{2}]$ *and* $y \in (\frac{1}{2}, 1]$, *we have* $|x - \frac{5}{6}| \in [\frac{1}{3}, \frac{5}{6}]$ *and* $|x-1| \in [\frac{1}{2}, 1]$, *thus we have*

$$\begin{aligned}&\max\{d(Ax,By)d(Ax,Sx),\ d(Ax,Ty)d(By,Sx),\ d(Ax,By)d(By,Ty)\}\\ &\qquad= \max\left\{\left|x-\frac{5}{6}\right|\cdot|x-1|,\ \left|x-\frac{5}{6}\right|\cdot\frac{1}{6},\ \left|x-\frac{5}{6}\right|\cdot 0\right\}\\ &\qquad\ge \max\left\{\frac{1}{3}\cdot\frac{1}{2},\ \frac{1}{3}\cdot\frac{1}{6},\ 0\right\} = \frac{1}{6}.\end{aligned}$$

Hence we have

$$\begin{aligned}d^2(Sx,Ty) &= \left(\frac{1}{6}\right)^2 < \frac{1}{2}\cdot\frac{1}{6} = \Phi\left(\frac{1}{6}\right)\\ &\le \Phi\left(\max\left\{\left|x-\frac{5}{6}\right|\cdot|x-1|,\ \left|x-\frac{5}{6}\right|\cdot\frac{1}{6},\ \left|x-\frac{5}{6}\right|\cdot 0\right\}\right)\\ &= \max\{d(Ax,By)d(Ax,Sx),\ d(Ax,Ty)d(By,Sx),\ d(Ax,By)d(By,Ty)\}.\end{aligned}$$

Which implies that the contraction condition (ii) of Theorem 2.3 is satisfied for all $x \in [0, \frac{1}{2}]$ *and* $y \in (\frac{1}{2}, 1]$.

Step 3. *For all* $x \in [0, \frac{1}{2}]$ *and* $y = 1$, *we have* $|x-1| \in [\frac{1}{2}, 1]$ *and* $|x - \frac{4}{5}| \in [\frac{3}{10}, \frac{4}{5}]$, *thus we have that*

$$\begin{aligned}&\max\{d(Ax,By)d(Ax,Sx),\ d(Ax,Ty)d(By,Sx),\ d(Ax,By)d(By,Ty)\}\\ &\qquad= \max\left\{x\cdot|x-1|,\ \left|x-\frac{4}{5}\right|\cdot 1,\ x\cdot\frac{5}{6}\right\}\\ &\qquad\ge \max\left\{0\cdot\frac{1}{2},\ \frac{3}{10}\cdot 1,\ 0\cdot\frac{5}{6}\right\} = \frac{3}{10}.\end{aligned}$$

Hence we have

$$\begin{aligned}d^2(Sx,Ty) &= \left(\frac{1}{6}\right)^2 < \frac{1}{2}\cdot\frac{3}{10} = \Phi\left(\frac{3}{10}\right)\\ &\le \Phi\left(\max\left\{x\cdot|x-1|,\ \left|x-\frac{4}{5}\right|\cdot 1,\ x\cdot\frac{5}{6}\right\}\right)\\ &= \max\{d(Ax,By)d(Ax,Sx),\ d(Ax,Ty)d(By,Sx),\ d(Ax,By)d(By,Ty)\}.\end{aligned}$$

Which implies that the contraction condition (ii) of Theorem 2.3 is satisfied for all $x \in [0, \frac{1}{2}]$ *and* $y = 1$.

Step 4. *For all* $x \in (\frac{1}{2}, 1]$ *and* $y \in [0, \frac{1}{2}]$, *we have* $|x-1| \in [\frac{1}{2}, 1]$, $|x-\frac{5}{6}| \in [\frac{1}{3}, \frac{5}{6}]$ *and* $|x - \frac{4}{5}| \in [\frac{3}{10}, \frac{4}{5}]$, *thus we have*

$$\begin{aligned}\max\{&d(Ax,By)d(Ax,Sx),\ d(Ax,Ty)d(By,Sx),\ d(Ax,By)d(By,Ty)\}\\ &= \max\left\{|x-1|\cdot\left|x-\frac{5}{6}\right|,\ \left|x-\frac{4}{5}\right|\cdot\frac{1}{6},\ |x-1|\cdot\frac{1}{5}\right\}\\ &\geq \max\left\{\frac{1}{2}\cdot\frac{1}{3},\ \frac{3}{10}\cdot\frac{1}{6},\ \frac{1}{2}\cdot\frac{1}{5}\right\} = \frac{1}{6}.\end{aligned}$$

Hence we have

$$\begin{aligned}d^2(Sx,Ty) &= \left(\frac{1}{30}\right)^2 < \frac{1}{2}\cdot\frac{1}{6} = \Phi\left(\frac{1}{6}\right)\\ &\leq \Phi\left(\max\left\{|x-1|\cdot\left|x-\frac{5}{6}\right|,\ \left|x-\frac{4}{5}\right|\cdot\frac{1}{6},\ |x-1|\cdot\frac{1}{5}\right\}\right)\\ &= \max\{d(Ax,By)d(Ax,Sx),\ d(Ax,Ty)d(By,Sx),\ d(Ax,By)d(By,Ty)\}.\end{aligned}$$

Which implies that the contraction condition (ii) of Theorem 2.3 is satisfied for all $x \in (\frac{1}{2}, 1]$ *and* $y \in [0, \frac{1}{2}]$.

By the above, the contractive conditions (ii) of Theorem 2.3 are all satisfied for all $x, y \in [0, 1]$. *It is easy to show that the* $x = \frac{5}{6}$ *be a unique common fixed point of maps* S, T, A *and* B.

Theorem 2.5. *Let* (X, d) *be a complete metric space,* $\{T_i\}_{i\in\Lambda}$ *is a family of self mappings on* X, *where* Λ *is the index set of the family, the power set of* Λ *is greater than or equal to 2, and let* A, B, L *and* M *be four mappings of* X *into itself, satisfying the following conditions:*

(i) $T_i(X) \subset BM(X)$, $T_i(X) \subset AL(X)$ $(\forall i \in \Lambda)$;

(ii) $AL = LA$, $T_iL = LT_i$, $BM = MB$, $T_iM = MT_i$ $(\forall i \in \Lambda)$;

(iii) $\forall x, y \in X,\ \ i, j \in \Lambda(i \neq j)$,

$$\begin{aligned}d^2(T_ix, T_jy) \leq \Phi(\max\{&d(ALx, BMy)d(ALx, T_ix), d(ALx, T_jy)d(BMy, T_ix),\\ &d(ALx, BMy)d(BMy, T_jy)\}),\end{aligned}$$

where Φ *satisfies the condition* (Φ).

If one of the following conditions is satisfied then A, B, L, M*,and* $\{T_i\}_{i\in\Lambda}$ *have a unique common fixed point in* X.

(1) Either $T_i(\forall i \in \Lambda)$ *or* A *is continuous, the pair* (T_i, AL) *is compatible, the pair* (T_i, BM) *is weakly compatible;*

(2) *Either* $T_i(\forall i \in \Lambda)$ *or* B *is continuous, the pair* (T_i, AL) *is weakly compatible, the pair* (T_i, BM) *is compatible;*

(3) *Either* AL *or* BM *is surjection, the pairs* (T_i, AL) *and* $(T_i, BM)(\forall i \in \Lambda)$ *are all weakly compatible.*

Proof. Let $i, j, m \in \Lambda$, $i \neq j, i \neq m$, by Theorem 2.1 we see there exists a unique common fixed point x_{ij} of A, B, L, M, T_i, T_j, and there exists a unique common fixed point x_{im} of A, B, L, M, T_i, T_m. Next we proved $x_{ij} = x_{im}$, in fact, from condition (iii) and the property of Φ, we have

$$\begin{aligned} &d^2(x_{ij}, x_{im}) = d^2(T_i x_{ij}, T_m x_{im}) \\ &\leq \Phi(\max\{d(ALx_{ij}, BMx_{im})d(ALx_{ij}, T_i x_{ij}), d(ALx_{ij}, T_m x_{im})d(BMx_{im}, T_i x_{ij}) \\ &\quad d(ALx_{ij}, BMx_{im})d(BMx_{im}, T_m x_{im})\}) \\ &= \Phi(d^2(x_{ij}, x_{im})). \end{aligned}$$

By virtue of this and using Lemma 1.1 (i), we obtain that $d^2(x_{ij}, x_{im}) = 0$, which gives $x_{ij} = x_{im}$. Since i, j, m be arbitrary in Λ, hence there exists a unique common fixed point of A, B, L, M and $T_i(\forall i \in \Lambda)$ in X. □

Theorem 2.6. *Let* (X, d) *be a complete metric space,* A, B, L, M *and* $\{T_i\}_{i\in\Lambda}$ *are self-mappings and a family of self-mappings on* X*, respectively, where* Λ *is the index set, the power set of* Λ *is greater than or equal to* 2*, satisfying* $\forall i \in \Lambda$, $T_i(X) \subset BM(X)$, $T_i(X) \subset AL(X)$*, the pairs* (T_i, AL) *and* (T_i, BM) *are commuting. If there exists the positive integer* n *such that* A, B, L, M *and* $\{T_i\}_{i\in I}$ *satisfying the following conditions:*

(i) *One of* AL, BM, $\{T_i\}_{i\in\Lambda}$ *is continuous;*

(ii) $AL = LA$, $T_iL = LT_i$, $BM = MB$, $T_iM = MT_i$ $(\forall i \in \Lambda)$*;*

(iii) $\forall x, y \in X, i, j \in \Lambda, (i \neq j)$,

$$\begin{aligned} d^2({T_i}^n x, {T_j}^n y) \leq \Phi(\max\{&d(ALx, BMy)d(ALx, {T_i}^n x), \\ &d(ALx, {T_j}^n y)d(BMy, {T_i}^n x), d(ALx, BMy)d(BMy, {T_j}^n y)\}) \end{aligned}$$

where ϕ *satisfies the condition* (Φ).

Then A, B, L, M *and* $\{T_i\}_{i\in\Lambda}$ *have a unique common fixed point in* X.

Proof. Since for all $i, j \in \Lambda$, the pairs (T_i, AL) and (T_i, BM) are commuting, then the pairs $({T_i}^n x, AL), ({T_j}^n y, BM)$ are commuting. Therefore, they are compatible. By Theorem 2.1 we know that ${T_i}^n$, ${T_j}^n$, A, B, L and M have a unique common fixed point $z \in X$. This is, ${T_i}^n z = {T_j}^n z = Az = Bz = Lz = Mz = z$. We now prove that z is the common fixed point of T_i $(i \in \Lambda)$. In fact, since ${T_i}^n z = z$, the pairs (T_i, AL) and (T_i, BM) are commuting. Then $(AL)T_i z = T_i(AL)z = T_i z$,

$(BM)T_iz = T_i(BM)z = T_iz$, $T_i{}^nT_iz = T_iT_i{}^nz = T_iz$. Thus, T_iz is the unique common fixed point of AL, BM and $T_i{}^n$. Using condition(iii), we obtain

$$\begin{aligned} d^2(T_iz,z) &= d^2(T_i{}^nT_iz, T_j{}^nz) \\ &\le \Phi(\max\{d(AL)T_iz, BMz)d((AL)T_iz, T_i{}^nT_iz), \\ &\quad d((AL)T_iz, T_j{}^nz)d(BMz, T_i{}^nT_iz), d((AL)T_iz, BMz)d(BMz, T_j{}^nz)\}) \\ &= \Phi(d^2(T_iz,z)). \end{aligned}$$

By virtue of this and using Lemma 1.1 (i), we obtain that $d^2(T_iz, z) = 0$, which gives $T_iz = z$. Thus, z is a common fixed point of A, B, L, M and $\{T_i\}_{i\in\Lambda}$. It follows easily from the condition (iii) that z is a unique common fixed point of A, B, L, M and $\{T_i\}_{i\in\Lambda}$. This completes the proof of Theorem 2.6. □

References

[1] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (4) (1986) 771–779.

[2] G. Jungck, Common fixed points for noncontinuous nonself maps on non-metric spaces, Far East J. Math. Sci. 4 (2) (1996) 199–215.

[3] S. Sessa, On a weak commutativity condition in fixed point considerations. Publ. Inst. Math. (Beograd) (N.S.) 32 (46) (1982) 149–153.

[4] S.S. Chang, Fixed Point Theory with Applications, Chongqing Publishing House, Chongqing, 1984.

[5] F. Gu, Common fixed point theorems for Φ expansive compatible mappings, Journal of BaoJi College of Arts And Science 21 (3) (2001) 176–179.

[6] M.A. Ahmed, Common fixed point theorems for weakly compatible mappings, Rocky Mountain J. Math. 33 (4) (2003) 1189–1203.

[7] A. Aliouche, A common fixed point theorem for weakly compatible mappings in compact metric spaces satisfying an implicit relation, Sarajevo J. Math. 3 (15) (2007) 123–130.

[8] J.M. Chen, F. Gu, The common fixed point theorems for a class of new Φ-contraction mapping, Journal of Hangzhou Teachers College 6 (2007) 414–418.

[9] R. Chugh, S. Kumar, Common fixed points for weakly compatible maps, Proc. Indian Acad. Sci. Math. Sci. 111 (2) (2001) 241–247.

[10] Lj.B. Ćirić, N.T. Nikolić, J.S. Ume, Some common fixed point theorems for weakly compatible mappings, J. Math. Anal. Appl. 314 (2) (2006) 488–499.

[11] A. Djoudi, L. Nisse, Gregus type fixed points for weakly compatible mappings, Bull. Belg. Math. Soc. Simon Stevin 10 (3) (2003) 369–378.

[12] M. Elamrani, B. Mehdaoui, Common fixed point theorems for compatible and weakly compatible mappings, Rev. Colombiana Mat. 34 (1) (2000) 25–33.

[13] X.W. Ju, F. Gu, The common fixed point theorems for a class of contraction mapping. Journal of Hangzhou Normal University 5 (2006) 382–386.

[14] G. Junck, Common fixed points for commuting and compatible maps on compacta, Proc. Amer. Math. Soc. 103 (1988) 977–983.

[15] G. Jungck, Compatible mappings and common fixed points (2), Internat. J. Math. and Math. Scl. 11 (1988) 285–288.

[16] G. Jungck, N. Hussain, Compatible maps and invariant approximations, J. Math. Anal. Appl. 325 (2007) 1003–1012.

[17] V. Popa, A general fixed point theorem for four weakly compatible mappings satisfying an implicit relation, Filomat 19 (2005) 45–51.

[18] V. Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation, Demonstratio Math. 32 (1) (1999) 157–163.

[19] S. Radenović, Z. Kadelburg, D. Jandrlić, A. Jandrlić, Some results on weakly contractive maps, Bull. Iran. Math. Soc. (in press).

[20] N. Hussain, M. Hussain Shah, S. Radenović, Fixed points of weakly contractions through occasionally weak compatibility, J. Comput. Anal. Appl. 13 (3) (2011) 532–543.

[21] Lj.B. Ćirić, A. Razani, S. Radenović, J.S. Ume, Common fixed point theorems for families of weakly compatible maps, Comput. Math. Appl. 55 (2008) 2533–2543.

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