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Convolution Properties for Certain Classes of Meromorphic p-Valent Functions Defined by Subordination

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Abstract : In the present paper, we study convolution properties for certain classes of meromophic multivalent functions. Further, with the help of convolution, coefficient estimates and inclusion relationship for these classes are also discussed.

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1 Introduction

Let $\sum_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \qquad (p \in \mathbb{N} := \{1, 2, 3, ...\}) \tag{1.1}$$

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which are analytic and p-valent in the punctured unit disk

$$U^* := \{z : z \in C \ \text{ and } \ 0 <| z |< 1\} =: U \setminus \{0\}.$$

For function $f \in \Sigma_p$, given by (1.1) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \quad (p \in \mathbb{N}) \ , \tag{1.2}$$

the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z). \tag{1.3}$$

For complex parameters

$$a_1, a_2, ..., a_q \quad \text{and} \quad b_1, b_2, ..., b_s \ \ (b_j \neq \mathbb{Z}_-^0 = \{0, -1, -2, -3, ...\}, \ j = 1, 2, 3, ...)$$

we now define the generalized hypergeometric function ${}_qF_s(a_1, ..., a_q; b_1, ..., b_s; z)$ as follows:

$$ {}_qF_s(a_1, ..., a_q; b_1, ..., b_s; z) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!},$$
$$(q \leq s + 1; \ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ z \in U) \ , \tag{1.4}$$

where $(\lambda)_r$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_r := \frac{\Gamma(\lambda + r)}{\Gamma(\lambda)} = \begin{cases} 1, & (r = 0, \ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + r - 1), & (r \in \mathbb{N}, \ \lambda \in \mathbb{C}) \ . \end{cases} \tag{1.5}$$

Corresponding to a function

$$h_p(a_1, ..., a_q; b_1, ..., b_s; z) = z^{-p}_q F_s(a_1, ..., a_q; b_1, ..., b_s; z), \tag{1.6}$$

Liu and Srivatava [1] (see also [2]) consider a linear operator

$$H_p(a_1, ..., a_q; b_1, ..., b_s; z) : \Sigma_p \mapsto \Sigma_p,$$

which is defined by the following Hadamard product (or convolution):

$$H_p(a_1, ..., a_q; b_1, ..., b_s; z)f(z) := h_p(a_1, ..., a_q; b_1, ..., b_s; z) * f(z), \tag{1.7}$$

so that, for a function f defined in (1.1), we have

$$H_p(a_1, ..., a_q; b_1, ..., b_s; z)f(z) := z^{-p} + \sum_{k=1}^{\infty} \Gamma_k[a_1; b_1] a_k z^{k-p}, \tag{1.8}$$

where

$$\Gamma_k[a_1; b_1] = \frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_s)_k k!}. \tag{1.9}$$

For our convenience, we write

$$H_{p,q,s}[a_1; b_1] := H_p(a_1, ..., a_q; b_1, ..., b_s; z) . \tag{1.10}$$

Special cases of the Liu-Srivastava linear operator include the meromorphic analogue of the Carlson-Shaffer linear operator $\mathcal{L}_p(a, c) := H_{p,2,1}(1, a; c)$ $(a, c > 0)$ (studied among others by Liu and Srivastava ([1, 3, 4]), Liu [5], and Yang [6]), the operator $D^{n+p} := \mathcal{L}_p(n + p, 1)$ $(n > -p)$, which is analogous to the Ruscheweyh derivative operator (investigated by Yang [7]) and the operator

$$J_{c,p} := \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) dt := \mathcal{L}_p(c, c + 1) \ \ (c > 0)$$

(studied by Uralegaddi and Somanatha [8]). It is to be noted that the Liu-Srivastava operator investigated in ([9, 10]) is the meromorphic analogue of the Dziok- Srivastava [11] linear operator.

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in U such that

$$w(0) = 0, \ \ | w(z) | < 1 \ \ (z \in U) \ \text{ and } \ g(z) = f(w(z)).$$

For this subordination, the symbol $g(z) \prec f(z)$ is used. In case f(z) is univalent in U, the subordination $g(z) \prec f(z)$ is equivalent to

$$g(0) = f(0) \ \text{ and } \ g(U) \subset f(U).$$

Now we define two subclasses $S_p^*[A, B]$ and $K_p[A, B]$ of the class Σ_p, for $-1 \leq B < A \leq 1$ and $p \in \mathbb{N}$ as follows:

$$S_p^*[A, B] = \left\{ f \in \Sigma_p : \frac{zf'(z)}{f(z)} \prec -p\frac{1 + Az}{1 + Bz} \ \ (z \in U^*) \right\} \tag{1.11}$$

and

$$K_p[A, B] = \left\{ f \in \Sigma_p : 1 + \frac{zf''(z)}{f'(z)} \prec -p\frac{1 + Az}{1 + Bz} \ \ (z \in U^*) \right\}. \tag{1.12}$$

Clearly

$$f(z) \in S_p^*[A, B] \Leftrightarrow \frac{-zf'(z)}{p} \in K_p[A, B].$$

We note that the class $S_p^*[A, B]$ was studied by Mogra [12] and the class $K_p[A, B]$ was studied by Srivastava et al. [8]. Also

$$S_1^*[1 - 2\alpha, -1] = S^*(\alpha), \ \ K_1[1 - 2\alpha, -1] = K(\alpha) \ (0 \leq \alpha < 1),$$

and

$$S_p^*\left[1-\frac{2\alpha}{p},-1\right]=S_p^*(\alpha)\ ,\quad K_p\left[1-\frac{2\alpha}{p},-1\right]=K_p(\alpha)\ (0\leq\alpha<p).$$

Next, using the Liu-Srivastava operator $H_{p,q,s}[a_1;b_1]$, we intruduce the following classes of analytic functions for $q,s\in N$ and $-1\leq B<A\leq 1$

$$\begin{aligned}S^*_{p,q,s}[a_1;A,B]&=S^*_p[a_1,...,a_q;b_1,...,b_s;A,B]\\&=\left[f\in\Sigma_p:H_{p,q,s}[a_1;b_1]f(z)\in S^*_p[A,B]\right]\omega\end{aligned}\tag{1.13}$$

and

$$\begin{aligned}K_{p,q,s}[a_1;A,B]&=K_p[a_1,...,a_q;b_1,...,b_s;A,B]\\&=\left[f\in\Sigma_p:H_{p,q,s}[a_1;b_1]f(z)\in K_p[A,B]\right].\end{aligned}\tag{1.14}$$

We also note that

$$f(z)\in K_{p,q,s}[a_1;A,B]\Leftrightarrow\frac{-zf^{'}(z)}{p}\in S^*_{p,q,s}[a_1;A,B]$$

Many important properties of certain subclasses of meromorphic p-valent functions were studied by several authors including Aouf and Srivastava [13], Cho and Kim [14] Joshi and Srivastava [15], Liu and Owa [16], Liu and Srivastava [1], Owa et al. [17] and Srivastava et al. [8].

2 Main Results

We assume throughout this section that $0<\theta<2\pi$, $-1\leq B<A\leq 1, p\in N$ and $\Gamma_k[a_1;b_1]$ is defined by (1.9).

Theorem 2.1. *The function $f(z)$ defined by (1.1) is in the class $S_p^*[A,B]$ if and only if*

$$z^p\left[f(z)*\frac{1+(D-1)z}{z^p(1-z)^2}\right]\neq 0\quad(z\in U^*),\tag{2.1}$$

where $D=\dfrac{e^{-i\theta}+B}{p(A-B)}$.

Proof. First, suppose $f(z)$ is in the class $S_p^*[A,B]$. Then from (1.11), we have

$$-\frac{zf'(z)}{f(z)}\prec p\frac{1+Az}{1+Bz}\qquad(z\in U^*),\tag{2.2}$$

so that by subordination of two functions we say that there exists a function $w(z)$ analytic in U^* with $w(0)=0$, $\mid w(z)\mid<1$ such that

$$-\frac{zf'(z)}{f(z)}=p\frac{1+Aw(z)}{1+Bw(z)}$$

which is equivalent to

$$-\frac{zf'(z)}{f(z)} \neq p\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}, \qquad (z \in U^*, 0 < \theta < 2\pi). \tag{2.3}$$

It can be noted that

$$f(z) * \frac{1}{z^p(1-z)} = f(z) \tag{2.4}$$

and

$$f(z) * \left[\frac{1-\left(1+\frac{1}{p}\right)z}{z^p(1-z)^2}\right] = -\frac{zf'(z)}{p}. \tag{2.5}$$

Now using above (2.4) and (2.5) in (2.3), we can easily obtain (2.1). □

Theorem 2.2. *The function $f(z)$ defined by (1.1) is in the class $K_p[A,B]$ if and only if*

$$z^p\left\{f(z) * \left[\frac{p-\{2+p-(p-1)(D-1)\}z-(p+1)(D-1)z^2}{pz^p(1-z)^3}\right]\right\} \neq 0 \quad (z \in U^*) \tag{2.6}$$

Proof. Choose $g(z) = \dfrac{1+(D-1)z}{z^p(1-z)^2}$ and we note that

$$zg'(z) = \left[\frac{-p+\{2+p-(p-1)(D-1)\}z+(p+1)(D-1)z^2}{z^p(1-z)^3}\right]. \tag{2.7}$$

From the identity $\frac{-zf'(z)}{p} * g(z) = f(z) * \frac{-zg'(z)}{p}$ and the fact that

$$f(z) \in K_p[A,B] \Leftrightarrow \frac{-zf'(z)}{p} \in S_P^*[A,B] \tag{2.8}$$

the result follows from Theorem 2.1. □

Theorem 2.3. *A necessary and sufficient condition for the function $f(z)$ defined by (1.1) to lie in the class $S^*_{p,q,s}[a_1;A,B]$ is that*

$$1+\sum_{k=1}^{\infty}\left[\frac{ke^{-i\theta}+pA+(k-p)B}{p(A-B)}\right]\Gamma_k[a_1;b_1]a_kz^k \neq 0 \ (z \in U^*).$$

Proof. From Theorem 2.1, we find that $f(z) \in S^*_{p,q,s}[A,B]$ if and only if

$$z^p\left[H_{p,q,s}[a_1;b_1]f(z) * \frac{1+(D-1)z}{z^p(1-z)^2}\right] \neq 0 \ (z \in U^*). \tag{2.9}$$

Using series expansion of $\frac{1+(D-1)z}{z^p(1-z)^2}$ and equation (1.3), it gives our desire result. □

Theorem 2.4. *A necessary and sufficient condition for the function $f(z)$ defined by (1.1) to lie in the class $K_{p,q,s}[a_1;A,B]$ is that*

$$1-\sum_{k=1}^{\infty}\frac{(k-p)[ke^{-i\theta}+pA+(k-p)B]}{p^2(A-B)}\Gamma_k[a_1;b_1]a_kz^k\neq 0 \quad (z\in U^*). \perp \tag{2.10}$$

Proof. From Theorem 2.2, we find that $f(z)\in K_{p,q,s}[A,B]$ if and only if

$$z^p\left\{H_{p,q,s}[a_1;b_1]f(z)*\left[\frac{p-\{2+p-(p-1)(D-1)\}z-(p+1)(D-1)z^2}{pz^p(1-z)^3}\right]\right\}\neq 0 \tag{2.11}$$

where $z\in U^*$. Now it can be easily shown that

$$\frac{1}{z^p(1-z)^3}=\frac{1}{z^p}+\sum_{k=1}^{\infty}\frac{(k+1)(k+2)}{2}z^{k-p}, \tag{2.12}$$

$$\frac{z}{z^p(1-z)^3}=\sum_{k=1}^{\infty}\frac{k(k+1)}{2}z^{k-p}, \tag{2.13}$$

and

$$\frac{z^2}{z^p(1-z)^3}=\sum_{k=1}^{\infty}\frac{k(k-1)}{2}z^{k-p}. \tag{2.14}$$

Using (2.12)-(2.14) in (2.11), it gives our desired result and the proof of Theorem 2.4 is completed. □

Unless otherwise mentioned, we assume throughout the reminder of this section that $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ are positive real parameters.

Theorem 2.5. *If the function $f(z)$ defined by (1.1) belongs to $S^*_{p,q,s}[a_1;A,B]$, then*

$$\sum_{k=1}^{\infty}[k+pA+(k-p)B]\Gamma_k[a_1;b_1]\mid a_k\mid\leq p(A-B). \tag{2.15}$$

Proof. Since

$$|1+\sum_{k=1}^{\infty}[\frac{ke^{-i\theta}+pA+(k-p)B}{p(A-B)}]\Gamma_k[a_1;b_1]a_kz^k|$$
$$>1-\sum_{k=1}^{\infty}|\frac{ke^{-i\theta}+pA+(k-p)B}{p(A-B)}|\Gamma_k[a_1;b_1]|a_k|$$

and

$$|\frac{ke^{-i\theta}+pA+(k-p)B}{p(A-B)}|=\frac{|ke^{-i\theta}+pA+(k-p)B|}{p(A-B)}\leq\frac{k+pA+(k-p)B}{p(A-B)}$$

the results follows from Theorem 2.3. □

In the same way, we can also prove the following theorem.

Theorem 2.6. *If the function $f(z)$ defined by (1.1) belongs to $K_{p,q,s}[a_1; A, B]$, then*

$$\sum_{k=1}^{\infty}(k-p)[k+pA+(k-p)B]\Gamma_k[a_1;b_1] \mid a_k \mid \leq p^2(A-B). \tag{2.16}$$

Now using the method due to Ahuja [18] (see also [19]), we will prove following theorem

Theorem 2.7. *For $a_1 > 0$, we have $S^*_{p,q,s}[a_1+1; A, B] \subset S^*_{p,q,s}[a_1; A, B]$.*

Proof. If $f(z) \in S^*_{p,q,s}[a_1+1; A, B]$, then from Theorem 2.3 we can write

$$1+\sum_{k=1}^{\infty}\left[\frac{ke^{-i\theta}+pA+(k-p)B}{p(A-B)}]\Gamma_k[a_1+1;b_1]\right] a_k z^k \neq 0 \ (z \in U^*; 0<\theta<2\pi). \tag{2.17}$$

Note that (2.17) can be written as

$$\left[1+\sum_{k=1}^{\infty}\frac{a_1+k}{a_1}z^k\right] * [1+\sum_{k=1}^{\infty}\left[\frac{ke^{-i\theta}+pA+(k-p)B}{p(A-B)}\right]\Gamma_k[a_1;b_1]a_k z^k] \neq 0. \tag{2.18}$$

But

$$\left[1+\sum_{k=1}^{\infty}\frac{a_1+k}{a_1}z^k\right] * [1+\sum_{k=1}^{\infty}\frac{a_1}{a_1+k}z^k] = 1+\sum_{k=1}^{\infty}z^k \quad (z \in U^*) \tag{2.19}$$

and using the property, if $f \neq 0$ and $g * h \neq 0$, then $f * (g * h) \neq 0$. It follows from (2.18) and (2.19) that

$$\left[1+\sum_{k=1}^{\infty}[\frac{ke^{-i\theta}+pA+(k-p)B}{p(A-B)}]\right]\Gamma_k[a_1;b_1]a_k z^k] \neq 0 \quad (z \in U^*; 0<\theta<2\pi). \tag{2.20}$$

In view of Theorem 2.3, we conclude that $f \in S^*_{p,q,s}[a_1; A, B]$ which proves Theorem 2.7. □

In the same way, we can also prove the following theorem.

Theorem 2.8. *For $a_1 > 0$, we have $K_{p,q,s}[a_1+1; A, B] \subset K_{p,q,s}[a_1; A, B]$.*

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