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Identities in (x(yz))z with Opposite Loop and Reverse Arc Graph Varieties of Type (2,0)

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Abstract : Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies a term equation $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$. A class of graph algebras $\mathcal{V}$ is called a graph variety if $\mathcal{V} = \overline{Mod\Sigma}$ where Σ is a subset of $T(X) \times T(X)$. A term equation $s \approx t$ is called an identity in a graph variety V if G satisfies $s \approx t$ for all $G \in \mathcal{V}$. A graph variety $\mathcal{V}' = Mod\Sigma^{'}$ is called an $(x(yz))z$ with opposite loop and reverse arc graph variety if $\Sigma^{'}$ is a set of $(x(yz))z$ with opposite loop and reverse arc term equations. In this paper we characterize identities in each (x(yz))z with opposite loop and reverse arc graph variety.

Keywords : varieties; binary algebra; graph algebras; identities in (x(yz))z with opposite loop; reverse arc graph varieties.
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1 Introduction

Graph algebras were invented by Shallon in [1] to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph*

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algebra $\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and with two basic operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if} \quad (u,v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

In a study by Pöschel and Wessel [2], graph varieties were investigated for finite undirected graphs in order to get graph theoretic results (structure theorems) from universal algebra via graph algebras. In [3], these investigations are extended to arbitrary (finite) directed graphs where the authors ask for a graph theoretic characterization of graph varieties, i.e., of classes of graphs which can be defined by term equations for their corresponding graph algebras. The answer is a theorem of **Birkhoff-type**, which uses graph theoretic closure operations. *A class of finite directed graphs is equational* (i.e., a graph variety) *if and only if it is closed with respect to finite restricted pointed subproducts and isomorphic copies.*

In [4], Anantpinitwatna and Poomsa-ard characterized all identities in (x(yz))z with loop graph varieties. In [5, 6], Anantpinitwatna and Poomsa-ard characterized all identities in biregular leftmost and (x(yz))z with reverse arc graph varieties respectively. In [7], Khampakdee and Poomsa-ard characterized identities in the class of $x(yx) \approx x(yy)$ graph algebras. In [8], Poomsa-ard characterized identities in the class of associative graph algebras. In [9, 10], Poomsa-ard et al. characterized identities in the class of idempotent graph algebras and in the class of transitive graph algebras respectively. In [11], Krapeedang and Poomsa-ard characterized all (x(yz))z with opposite loop and reverse arc graph varieties.

In this paper we characterize all identities in each (x(yz))z with opposite loop and reverse arc graph variety.

2 Terms, Identities and Graph Varieties

In [12], Denecke and Wismath gave a basic definiont about universal algebra as the following:

Definition 2.1. Let A be a non-empty set. Let I be some non-empty index set, and let $(f_i^A)_{i\in I}$ be a function which assigns to every element of I an n_i-ary operation f_i^A defined on A. Then the pair $\mathcal{A} = (A; (f_i^A)_{i\in I})$ is called an *(indexed) algebra* (indexed by the set I). The set A is called the *base* or *carrier set* or *universe* of $\mathcal{A}$, and $(f_i^A)_{i\in I}$ is called the *sequence of fundamental operations* of $\mathcal{A}$. For each $i \in I$ the natural number n_i ia called the *arity* of f_i^A. The sequence $\tau = (n_i)_{i\in I}$ of all the arities is called the *type* of the algebra $\mathcal{A}$. We use the name $Alg(\tau)$ for the class of all algebras of a given type τ.

We see that graph algebra is type $\tau = (2, 0)$. In [13], Pöschel introduced terms for graph algebras; The underlying formal language has to contain a binary operation symbol (juxtaposition) and a symbol for the constant ∞ (denoted by ∞ too).

Definition 2.2. The set $T(X)$ of all terms over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...,$ and ∞ are terms;

(ii) if t_1 and t_2 are terms, then $t_1 t_2$ is a term.

$T(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps. Thus terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are binary terms. We denote the set of all binary terms by $T(X_2)$. The leftmost variable of a term t is denoted by $L(t)$. A term in which the symbol ∞ occurs is called a *trivial term.*

Definition 2.3. For each non-trivial term t of type $\tau = (2, 0)$, one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1 t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$$

where $t = t_1 t_2$ is a compound term.

$L(t)$ is called the *root* of the graph $G(t)$, and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, we assign the empty graph ϕ to every trivial term t.

Definition 2.4. A non-trivial term t of type $\tau = (2, 0)$ is called $(x(yz))z$ with opposite loop and reverse arc term if and only if $G(t)$ is a graph with $V(t) = \{x, y, z\}$ and $E(t) = E \cup (\cup_{X \in E'} X)$, where $E = \{(x, y), (x, z), (y, z)\}$, $E' \subseteq \{U, V, W\}$, , $E' \neq \phi$ and $U = \{(x, x), (z, y)\}$, $V = \{(y, y), (z, x)\}$, $W = \{(z, z), (y, x)\}$. A term equation $s \approx t$ is called an $(x(yz))z$ with opposite loop and reverse arc term equation if s and t are $(x(yz))z$ with opposite loop and reverse arc terms.

Definition 2.5. We say that a graph $G = (V, E)$ satisfies a term equation $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$ (i.e., we have $s = t$ for every assignment $V(s) \cup V(t) \to V \cup \{\infty\}$), and in this case, we write $G \models s \approx t$. Given a class $\mathcal{G}$ of graphs and a set Σ of term equations (i.e., $\Sigma \subset T(X) \times T(X)$) we introduce the following notation:

$G \models \Sigma$ if $G \models s \approx t$ for all $s \approx t \in \Sigma$, $\mathcal{G} \models s \approx t$ if $G \models s \approx t$ for all $G \in \mathcal{G}$,
$\mathcal{G} \models \Sigma$ if $G \models \Sigma$ for all $G \in \mathcal{G}$, $Id\mathcal{G} = \{s \approx t \mid s, t \in T(X),\ \mathcal{G} \models s \approx t\}$,
$Mod\Sigma = \{G \mid G$ is a graph and $G \models \Sigma\}$, $\mathcal{V}(\mathcal{G}) = ModId\mathcal{G}$.

$\mathcal{V}(\mathcal{G})$ is called the *graph variety generated by* $\mathcal{G}$ and $\mathcal{G}$ is called an *graph variety* if $\mathcal{V}(\mathcal{G}) = \mathcal{G}$. $\mathcal{G}$ is called *equational* if there exists a set Σ' of term equations such that $\mathcal{G} = \text{Mod}\Sigma'$. Obviously $\mathcal{V}(\mathcal{G}) = \mathcal{G}$ if and only if $\mathcal{G}$ is an equational class.

Definition 2.6. Let $G = (V, E)$ and $G^{'} = (V^{'}, E^{'})$ be graphs. A *homomorphism* h from G into G' is a mapping $h : V \to V^{'}$ carrying edges to edges, that is, for which $(u, v) \in E$ implies $(h(u), h(v)) \in E^{'}$.

3 Identities in (x(yz))z with Opposite Loop and Reverse Arc Graph Varieties

Graph identities were characterized in [14] by the following proposition:

Proposition 3.1. *A non-trivial equation $s \approx t$ is an identity in the class of all graph algebras if and only if either both terms s and t are trivial or none of them is trivial, $G(s) = G(t)$ and $L(s) = L(t)$.*

Further, the following propositions were proven in [14]:

Proposition 3.2. *Let $G = (V, E)$ be a graph and let $h : X \cup \{\infty\} \longrightarrow V \cup \{\infty\}$ be an evaluation of the variables such that $h(\infty) = \infty$. Consider the canonical extension of h to the set of all terms. Then there holds: if t is a trivial term then $h(t) = \infty$. Otherwise, if $h : G(t) \longrightarrow G$ is a homomorphism of graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of graphs, then $h(t) = \infty$.*

Proposition 3.3. *Let s and t be non-trivial terms from $T(X)$ with variables $V(s) = V(t) = \{x_0, x_1, ..., x_n\}$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if the graph algebra $\underline{A(G)}$ has the following property:*

A mapping $h : V(s) \longrightarrow V$ is a homomorphism from $G(s)$ into G if and only if it is a homomorphism from $G(t)$ into G.

All (x(yz))z with opposite loop and reverse arc graph varieties were characterized in [11]. Here we will give some detail about this as the following:

Theorem 3.4. *Let $G = (V, E)$ be a graph. Then we have*

$$G \in \mathcal{K}_1 = Mod\{((xx)(y(zy)))z \approx (x((yy)(zx)))z\}$$

if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(a, a), (c, b) \in E$ if and only if $(b, b), (c, a) \in E$.

Proof. Let $G = (V, E)$ be a graph. Suppose that $G \in \mathcal{K}_1$ and for any $a, b, c \in V$, suppose that $(a, b), (b, c), (a, c), (a, a), (c, b) \in E$. Let s and t be non-trivial terms such that $s = ((xx)(y(zy)))z$ and $t = (x((yy)(zx)))z$ and let $h : V(s) \to V$ be a function such that $h(x) = a$, $h(y) = b$ and $h(z) = c$. We see that h is a homomorphism from $G(s)$ into G. By Proposition 3.3, we have h is a homomorphism from $G(t)$ into G. Since $(y, y), (z, x) \in E(t)$, we have $(h(y), h(y)) = (b, b) \in E$ and $(h(z), h(x)) = (c, a) \in E$. In the same way, we can prove that if $(a, b), (b, c), (a, c), (b, b), (c, a) \in E$, then $(a, a), (c, b) \in E$.

Conversely, suppose that $G = (V, E)$ be a graph which has property that, for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(a, a), (c, b) \in E$ if and only if $(b, b), (c, a) \in E$. Let s and t be non-trivial terms such that $s = ((xx)(y(zy)))z$ and $t = (x((yy)(zx)))z$ and let $h : V(s) \to V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G. Since $(x, y), (y, z), (x, z), (x, x), (z, y) \in E(s)$, we have $(h(x), h(y)), (h(y), h(z)), (h(x), h(z)), (h(x), h(x)), (h(z), h(y)) \in E$. By assumption, we get $(h(y), h(y)), (h(z), h(x)) \in E$. Hence, h is a homomorphism from $G(t)$ into G. In the same way, we can prove that if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Then, by Proposition 3.3 we get $\underline{A(G)}$ satisfies $s \approx t$. □

By the similar way, we can prove the other $(x(yz))z$ with opposite loop and reverse arc graph varieties and we get the following table:

Table 1. $(x(yz))z$ with opposite loop and reverse arc graph varieties and the property of graphs.

Graph variety	Properties of graphs, for any $a, b, c \in V$ if $(a,b),(b,c),(a,c) \in E$,
$\mathcal{K}_1 = Mod\{((xx)(y(zy)))z \approx (x((yy)(zx)))z\}$	then $(a,a),(c,b) \in E$ if and only if $(b,b),(c,a) \in E$.
$\mathcal{K}_2 = Mod\{((xx)(y(zy)))z \approx (x(yx)(zz)))z\}$	then $(a,a),(c,b) \in E$ if and only if $(c,c),(b,a) \in E$.
$\mathcal{K}_3 = Mod\{(x((yy)(zx)))z \approx (x((yx)(zz)))z\}$	then $(b,b),(c,a) \in E$ if and only if $(c,c),(b,a) \in E$.
$\mathcal{K}_4 = Mod_g\{((xx)(y(zy)))z \approx ((xx)((yy)((zx)y)))z\}$	and $(a,a),(c,b) \in E$, then $(b,b),(c,a) \in E$.
$\mathcal{K}_5 = Mod\{(x((yy)(zx)))z \approx ((xx)((yy)((zx)y)))z\}$	and $(b,b),(c,a) \in E$, then $(a,a),(c,b) \in E$.
$\mathcal{K}_6 = Mod\{(x((yx)(zz)))z \approx ((xx)((yx)((zy)z)))z\}$	and $(c,c),(b,a) \in E$, then $(a,a),(c,b) \in E$.
$\mathcal{K}_7 = Mod\{((xx)((yy)((zx)y)))z \approx ((xx)(((yx)y)(((zx)y)z)))z\}$	and $(a,a),(c,b),(b,b),(c,a) \in E$, then $(c,c),(b,a) \in E$.
$\mathcal{K}_8 = Mod\{((xx)(y(zy)))z \approx (x((yy)(zx)))z,$ $((xx)(y(zy)))z \approx (x(yx)(zz)))z\}$	then (i) $(a,a),(c,b) \in E$ if and only if $(b,b),(c,a) \in E$, (ii) $(a,a),(c,b) \in E$ if and only if $(c,c),(b,a) \in E$.

Further, let T be the set of all (x(yz))z with opposite loop and reverse arc term equations. Since for any $\Sigma \subset T$ the (x(yz))z with opposite loop and reverse arc graph variety $Mod\Sigma = \bigcap_{s\approx t\in\Sigma} Mod\{s \approx t\}$, we have $\mathcal{K}_8 = \mathcal{K}_1 \cap \mathcal{K}_2$. Then, we can check that $\mathcal{K} = \{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, ..., \mathcal{K}_8\}$ is the set of all (x(yz))z with opposite loop and reverse arc graph varieties, where $\mathcal{K}_0 = Mod\{(x((yx)z))z \approx (x((yx)z))z\}$ is the class of all graph algebras.

Now we characterize all identities in each $(x(yz))z$ with opposite loop and reverse arc graph variety. Clearly, if $s \approx t$ is a trivial equation (s, t are trivial or $G(s) = G(t)$ and, $L(s) = L(t)$), then $s \approx t$ is an identity in each $(x(yz))z$ with opposite loop and reverse arc graph variety. Further, if s is a trivial term and t is a non-trivial term or both of them are non-trivial with $L(s) \neq L(t)$ or $V(s) \neq V(t)$, then $s \approx t$ is not an identity in every $(x(yz))z$ with opposite loop and reverse arc graph variety, since for a complete graph G we have an evaluation of the variables h such that $h(s) = \infty$ and $h(t) \neq \infty$. So we consider the case s

and t are non-trivial with $L(s) = L(t)$, $V(s) = V(t)$ and $G(s) \neq G(t)$. Before we do this let us introduce some notation. For any non-trivial term t and $x \in V(t)$ let

$A_x(t) = \{x' \in V(t) \mid x' \text{ is an in-neighbor of } x \text{ in } G(t)\}$,

$A_x^{'}(t) = \{x' \in V(t) \mid x' \text{ is an out-neighbor of } x \text{ in } G(t)\}$,

$A_x^{''}(t) = \{x' \in V(t) \mid x' \text{ is an in-neighbor and an out-neighbor of } x \text{ in } G(t)\}$,

$A_x^0(t) = \{x\}$, $A_x^1(t) = \{x' \in V(t) \mid x' \in A_x^{''}(t)$ or $x' \in A_x'(t)$ which has z such that $(x,z),(z,x),(x',z) \in E(t)$ or $x' \in A_x(t)$ which has z', z'' such that $(x,z'),(z',x),(x',z') \in E(t)$ or $(x,z''),(z'',x),(z'',x') \in E(t)\}$,

$A_x^2(t) = \bigcup_{y \in A_x^1(t)} A_y^1(t)$,..., $A_x^n(t) = \bigcup_{y \in A_x^{n-1}(t)} A_y^1(t)$, $A_x^*(t) = \bigcup_{i=0}^{\infty} A_x^i(t)$,

$B_x^0(t) = \{x\}$, $B_x^1(t) = \{x' \in V(t) \mid x' \in A_x^{''}(t)$ or $x' \in A_x(t)$ which has z such that $(x,z),(z,x),(x',z) \in E(t)$ or $x' \in A_x'(t)$ which has z' such that $(x,z'),(z',x),(z',x') \in E(t)\}$,

$B_x^2(t) = \bigcup_{y \in B_x^1(t)} B_y^1(t)$,..., $B_x^n(t) = \bigcup_{y \in B_x^{n-1}(t)} B_y^1(t)$, $B_x^*(t) = \bigcup_{i=0}^{\infty} B_x^i(t)$,

$C_x^0(t) = \{x\}$, $C_x^1(t) = \{x' \in V(t) \mid x' \in A_x^{''}(t)$ or $x' \in A_x'(t)$ which has z, z' such that $(x,z),(z,x),(x',z) \in E(t)$ or $(x,z'),(z',x),(z',x') \in E(t)$ or $x' \in A_x(t)$ which has z'' such that $(x,z''),(z'',x),(z'',x') \in E(t)\}$,

$C_x^2(t) = \bigcup_{y \in C_x^1(t)} C_y^1(t)$,..., $C_x^n(t) = \bigcup_{y \in C_x^{n-1}(t)} C_y^1(t)$, $C_x^*(t) = \bigcup_{i=0}^{\infty} C_x^i(t)$,

$D_x^0(t) = \{x\}$, $D_x^1(t) = \{x' \in V(t) \mid x' \in A_x^{''}(t)$ or $x' \in A_x(t)$ which has z such that $(x,z),(z,x),(x',z) \in E(t)\}$,

$D_x^2(t) = \bigcup_{y \in D_x^1(t)} D_y^1(t)$,..., $D_x^n(t) = \bigcup_{y \in D_x^{n-1}(t)} D_y^1(t)$, $D_x^*(t) = \bigcup_{i=0}^{\infty} D_x^i(t)$,

$F_x^0(t) = \{x\}$, $F_x^1(t) = \{x' \in V(t) \mid x' \in A_x^{''}(t)$ or $x' \in A_x(t)$ which has z such that $(x,z),(z,x),(z,x') \in E(t)$ or $x' \in A_x'(t)$ which has z' such that $(x,z'),(z',x),(x',z') \in E(t)\}$,

$F_x^2(t) = \bigcup_{y \in F_x^1(t)} F_y^1(t)$,..., $F_x^n(t) = \bigcup_{y \in F_x^{n-1}(t)} F_y^1(t)$, $F_x^*(t) = \bigcup_{i=0}^{\infty} F_x^i(t)$,

$H_x^0(t) = \{x\}$, $H_x^1(t) = \{x' \in V(t) \mid x' \in A_x^{''}(t)$ or $x' \in A_x'(t)$ which has z such that $(x,z),(z,x),(z,x') \in E(t)\}$,

$H_x^2(t) = \bigcup_{y \in H_x^1(t)} H_y^1(t)$,..., $H_x^n(t) = \bigcup_{y \in H_x^{n-1}(t)} H_y^1(t)$, $H_x^*(t) = \bigcup_{i=0}^{\infty} H_x^i(t)$,

$I_x^0(t) = \{x\}$, $I_x^1(t) = \{x' \in V(t) \mid x' \in A_x^{''}(t)\}$,

$I_x^2(t) = \bigcup_{y \in I_x^1(t)} I_y^1(t)$,..., $I_x^n(t) = \bigcup_{y \in I_x^{n-1}(t)} I_y^1(t)$, $I_x^*(t) = \bigcup_{i=0}^{\infty} I_x^i(t)$.

Then, all identities in each $(x(yz))z$ with opposite loop and reverse arc graph variety are characterized by the following theorems:

Theorem 3.5. *Let $s \approx t$ be a non-trivial equation, $L(s) = L(t)$, $V(s) = V(t)$. Then, $s \approx t \in Id\mathcal{K}_1$ if and only if the following conditions are satisfied:*

(i) *for any* $x \in V(s)$*, there exists* $y \in A_x^*(s)$ *such that* $(y, y) \in E(s)$ *if and only if there exists* $y' \in A_x^*(t)$ *such that* $(y', y') \in E(t)$*;*

(ii) *for any* $x, y \in V(s)$ *with* $x \neq y$*,* $(x, y) \in E(s)$ *or* $(y, x) \in E(s)$ *and, there exists* $y' \in A_y^*(s)$*,* $x' \in A_x^*(s)$ *such that* $(y', y') \in E(s)$ *and* $(x', x') \in E(s)$ *if and only if* $(x, y) \in E(t)$ *or* $(y, x) \in E(t)$ *and, there exists* $y'' \in A_y^*(t)$*,* $x'' \in A_x^*(t)$ *such that* $(y'', y'') \in E(t)$ *and* $(x'', x'') \in E(t)$*.*

Proof. Suppose that there exists $x \in V(s)$ which there is $y \in A_x^*(s)$ such that $(y, y) \in E(s)$ but $(y', y') \notin E(t)$ for all $y' \in A_x^*(t)$. Consider the graph $G = (V, E)$ which obtains from $G(t)$ by adding minimum edges to $G(t)$ until $G \in \mathcal{K}_1$. Let $h : V(s) \to V$ be an identity evaluation of the variables. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence, $s \approx t \notin Id\mathcal{K}_1$. Similarly, we prove the converse. Suppose that there exist $x, y \in V(s)$ with $x \neq y$, such that (ii) true for $G(s)$ but it is not true for $G(t)$. If $(x, y) \notin E(t)$ and $(y, x) \notin E(t)$, then consider the graph $G = (V, E)$ such that $V = \{0, 1, 2\}$, $E = \{(0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (2, 0), (2, 2)\}$. By Table 1., we see that $G \in \mathcal{K}_1$. Let $h : V(s) \longrightarrow V$ such that $h(x) = 1$, $h(y) = 2$ and $h(z) = 0$ for all other $z \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence, $s \approx t \notin Id\mathcal{K}_1$. Otherwise, consider the graph $G = (V, E)$ which obtains from $G(t)$ by adding minimum edges to $G(t)$ until $G \in \mathcal{K}_1$. Let $h : V(s) \to V$ be an identity evaluation of the variables. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence, $s \approx t \notin Id\mathcal{K}_1$. Similarly, we prove the converse.

Conversely, suppose that $s \approx t$ are non-trivial equation, $L(s) = L(t)$, $V(s) = V(t)$ satisfying (i) and (ii). Let $G = (V, E)$ be a graph in $\mathcal{K}_1$ and let $h : V(s) \longrightarrow V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G and let $(x, y) \in E(t)$. If $x = y$, then by (i), there exists $z \in A_x^*(s)$ such that $(z, z) \in E(s)$. By Table 1. and h is a homomorphism, we have $(h(x), h(x)) \in E$. If $x \neq y$, then (ii) true for the graph $G(s)$. If $(x, y) \in E(s)$, then $(h(x), h(y)) \in E$. Suppose that $(x, y) \notin E(s)$. Then by (ii), we get that $(y, x) \in E(s)$, there exists $y' \in A_y^*(s)$ such that $(y', y') \in E(s)$ and there exists $x' \in A_x^*(s)$ such that $(x', x') \in E(s)$. By Table 1. and h is a homomorphism from $G(s)$ into G, we have $(h(y), h(y)), (h(x), h(x)), (h(y), h(x)) \in E$. Thus, by Table 1. again, we get $(h(x), h(y)) \in E$. Therefore, h is a homomorphism from $G(t)$ into G. In the same way, we can prove that if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Hence, by Proposition 3.3, we get $s \approx t \in Id\mathcal{K}_1$. □

Theorem 3.6. *Let* $s \approx t$ *be a non-trivial equation,* $L(s) = L(t)$*,* $V(s) = V(t)$*. Then,* $s \approx t \in Id\mathcal{K}_2$ *if and only if the following conditions are satisfied:*

(i) *for any* $x \in V(s)$*, there exists* $y \in B_x^*(s)$ *such that* $(y, y) \in E(s)$ *if and only if there exists* $y' \in B_x^*(t)$ *such that* $(y', y') \in E(t)$*;*

(ii) *for any* $x, y \in V(s)$ *with* $x \neq y$*,* $(x, y) \in E(s)$ *or* $(y, x) \in E(s)$ *and, there exists* $y' \in B_y^*(s)$*,* $x' \in B_x^*(s)$ *such that* $(y', y') \in E(s)$ *and* $(x', x') \in E(s)$ *if and only if* $(x, y) \in E(t)$ *or* $(y, x) \in E(t)$ *and, there exists* $y'' \in B_y^*(t)$*,* $x'' \in B_x^*(t)$ *such that* $(y'', y'') \in E(t)$ *and* $(x'', x'') \in E(t)$*.*

Proof. The proof is similar to the proof of Theorem 3.1. □

Theorem 3.7. *Let $s \approx t$ be a non-trivial equation, $L(s) = L(t)$, $V(s) = V(t)$. Then, $s \approx t \in Id\mathcal{K}_3$ if and only if the following conditions are satisfied:*

(i) for any $x \in V(s)$, there exists $y \in C^*_x(s)$ such that $(y, y) \in E(s)$ if and only if there exists $y' \in C^*_x(t)$ such that $(y', y') \in E(t)$;

(ii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$, there exists $y' \in C^*_y(s)$ such that $(y', y') \in E(s)$ and there exists $x' \in C^*_x(s)$ such that $(x', x') \in E(s)$ if and only if $(x, y) \in E(t)$ or $(y, x) \in E(t)$, there exists $y'' \in C^*_y(t)$ such that $(y'', y'') \in E(t)$ and there exists $x'' \in C^*_x(t)$ such that $(x'', x'') \in E(t)$.

Proof. The proof is similar to the proof of Theorem 3.1. □

Theorem 3.8. *Let $s \approx t$ be a non-trivial equation, $L(s) = L(t)$, $V(s) = V(t)$. Then, $s \approx t \in Id\mathcal{K}_4$ if and only if the following conditions are satisfied:*

(i) for any $x \in V(s)$, there exists $y \in D^*_x(s)$ such that $(y, y) \in E(s)$ if and only if there exists $y' \in D^*_x(t)$ such that $(y', y') \in E(t)$;

(ii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and, there exists $y' \in D^*_y(s)$, $x' \in D^*_x(s)$ such that $(y', y') \in E(s)$ and $(x', x') \in E(s)$ if and only if $(x, y) \in E(t)$ or $(y, x) \in E(t)$ and, there exists $y'' \in D^*_y(t)$, $x'' \in D^*_x(t)$ such that $(y'', y'') \in E(t)$ and $(x'', x'') \in E(t)$.

Proof. The proof is similar to the proof of Theorem 3.1. □

Theorem 3.9. *Let $s \approx t$ be a non-trivial equation , $L(s) = L(t)$, $V(s) = V(t)$. Then, $s \approx t \in Id\mathcal{K}_5$ if and only if the following are satisfied:*

(i) for any $x \in V(s)$, there exists $y \in F^*_x(s)$ such that $(y, y) \in E(s)$ if and only if there exists $y' \in F^*_x(t)$ such that $(y', y') \in E(s)$;

(ii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and, there exists $S \subseteq E(s)$ such that if $G \in \mathcal{K}_5$ and h is a homomorphism from $G(s)$ into G, then $(h(x), h(y)) \in E$ if and only if $(x, y) \in E(t)$ or $(y, x) \in E(t)$ and, there exists $S' \subseteq E(t)$ such that if $G' \in \mathcal{K}_5$ and h' is a homomorphism from $G(t)$ into G', then $(h'(x), h'(y)) \in E'$.

Proof. Suppose that there exists $x \in V(s)$ which there is $y \in F^*_x(s)$ such that $(y, y) \in E(s)$ but $(y', y') \notin E(t)$ for all $y' \in F^*_x(t)$. Consider the graph $G = (V, E)$ which obtains from $G(t)$ by adding minimum edges to $G(t)$ until $G \in \mathcal{K}_5$. Let $h : V(s) \longrightarrow V$ such that $h(x) = x$ for all $x \in V(s)$. We have that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $s \approx t \notin Id\mathcal{K}_5$. Suppose that there exist $x, y \in V(s)$ with $x \neq y$, such that (ii) true for $G(s)$ but it is not true for $G(t)$. If $(x, y) \notin E(t)$ and $(y, x) \notin E(t)$, then consider the graph $G = (V, E)$ such that $V = \{0, 1, 2\}$, $E = \{(0, 0), ((0, 1), (1, 0), (1, 1), (0, 2), (2, 0), (2, 2)\}$. By Table 1, we see that $G \in \mathcal{K}_5$. Let $h : V(s) \longrightarrow V$ such that $h(x) = 1$, $h(y) = 2$ and $h(z) = 0$ for all other

$z \in V(s)$. We see that $h(s) = \infty$, $h(t) = h(L(t))$. Hence, $s \approx t \notin Id\mathcal{K}_5$. Otherwise, consider the graph $G = (V, E)$ which obtains from $G(t)$ by adding minimum edges to $G(t)$ until $G \in \mathcal{K}_5$. Let $h : V(s) \longrightarrow V$ such that $h(x) = x$ for all $x \in V(s)$. We have that $h(s) = \infty$, $h(t) = h(L(t))$. Hence $s \approx t \notin Id\mathcal{K}_5$.

Conversely, suppose that $s \approx t$ is a non-trivial equation, $L(s) = L(t)$, $V(s) = V(t)$ satisfying (i) and (ii). Let $G = (V, E)$ be a graph in $\mathcal{K}_5$ and let $h : V(s) \longrightarrow V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G and let $(x, y) \in E(t)$. If $x = y$, then by (i) there exists $y' \in F^*_x(s)$ such that $(y', y') \in E(s)$. By Table 1. and h is a homomorphism, we have $(h(x), h(x)) \in E$. If $x \neq y$, then (ii) true for the graph $G(s)$. If $(x, y) \in E(s)$, then $(h(x), h(y)) \in E$. Suppose $(x, y) \notin E(s)$. Then, by (ii), we get that $(y, x) \in E(s)$ and there exists $S \subseteq E(s)$ such that $(h(x), h(y)) \in E$. Therefore, h is a homomorphism from $G(t)$ into G. In the same way, we can prove that if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Hence by Proposition 3.3, we get $s \approx t \in Id\mathcal{K}_5$. □

Theorem 3.10. *Let $s \approx t$ be a non-trivial equation, $L(s) = L(t)$, $V(s) = V(t)$. Then, $s \approx t$ is an identity in the graph variety $\mathcal{K}_6$ if and only if*

(i) for any $x \in V(s)$, there exists $y \in H^*_x(s)$ such that $(y, y) \in E(s)$ if and only if there exists $y' \in H^*_x(t)$ such that $(y', y') \in E(t)$;

(ii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and, there exist $y' \in H^*_y(s)$, $x' \in H^*_x(s)$ such that $(y', y') \in E(s)$ and $(x', x') \in E(s)$ if and only if $(x, y) \in E(t)$ or $(y, x) \in E(t)$ and, there exist $y'' \in H^*_y(t)$, $x'' \in H^*_x(t)$ such that $(y'', y'') \in E(t)$ and $(x'', x'') \in E(t)$.

Proof. The proof is similar to the proof of Theorem 3.1. □

Theorem 3.11. *Let $s \approx t$ be a non-trivial equation, $L(s) = L(t)$, $V(s) = V(t)$. Then, $s \approx t$ is an identity in the graph variety $\mathcal{K}_7$ if and only if*

(i) for any $x \in V(s)$, there exists $y \in I^*_x(s)$ such that $(y, y) \in E(s)$ if and only if there exists $y' \in I^*_x(t)$ such that $(y', y') \in E(t)$;

(ii) for any $x, y \in V(s)$ with $x \neq y$, $(x, y) \in E(s)$ or $(y, x) \in E(s)$ and, there exists $S \subseteq E(s)$ such that if $G \in \mathcal{K}_7$ and h is a homomorphism from $G(s)$ into G, then $(h(x), h(y)) \in E$ if and only if $(x, y) \in E(t)$ or $(y, x) \in E(t)$ and, there exists $S' \subseteq E(t)$ such that if $G' \in \mathcal{K}_7$ and h' is a homomorphism from $G(t)$ into G', then $(h'(x), h'(y)) \in E'$.

Proof. The proof is similar to the proof of Theorem 3.5. □

Theorem 3.12. *Let $s \approx t$ be a non-trivial equation, $L(s) = L(t)$, $V(s) = V(t)$. Then, $s \approx t \in Id\mathcal{K}_8$ if and only if the following conditions are satisfied:*

(i) for any $x \in V(s)$, there exists $y \in A^*_x(s) \cup B^*_x(s)$ such that $(y, y) \in E(s)$ if and only if there exists $y' \in A^*_x(t) \cup B^*_x(t)$ such that $(y', y') \in E(t)$;

(ii) *for any* $x, y \in V(s)$ *with* $x \neq y$, $(x, y) \in E(s)$ *or* $(y, x) \in E(s)$ *and, there exists* $y' \in A^*_y(s) \cup B^*_y(s)$ *such that* $(y', y') \in E(s)$ *and there exists* $x' \in A^*_x(s) \cup B^*_x(s)$ *such that* $(x', x') \in E(s)$ *if and only if* $(x, y) \in E(t)$ *or* $(y, x) \in E(t)$ *and, there exists* $y'' \in A^*_y(t) \cup B^*_y(t)$ *such that* $(y'', y'') \in E(t)$ *and there exists* $x'' \in A^*_x(t) \cup B^*_x(t)$ *such that* $(x'', x'') \in E(t)$.

Proof. The proof is similar to the proof of Theorem 3.1. □

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