Thai Journal of Mathematics
Volume 10 (2012) Number 3 : 551–563

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Coupled Coincidence and Coupled Common Fixed Point Theorems in Partially Ordered Metric Spaces[1]

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Abstract : The purpose of this paper is to prove some coupled coincidence point theorems for nonlinear contraction mappings having a mixed monotone property in a partially ordered metric spaces. We also give some examples to validate the main results in this paper. Our theorems are generalizations of the results of Luong and Thuan in [N.V. Luong, N.X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. 74 (2011) 983–992.], classical coupled fixed point theorems of Bhaskar and Lakshmikantham [T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379–1393.] and several results in fixed point theory.

Keywords : coupled fixed point; coupled coincidence point; a coupled point of coincidence; coupled common fixed point; mixed monotone property.
2010 Mathematics Subject Classification : 47H10; 54H25.

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[1]The first author would like to thank the Research Professional Development Project under the Science Achievement Scholarship of Thailand (SAST) and the second author would like to thank the National Research University Project of Thailand's Office of the Higher Education Commission for financial support under the project NRU-CSEC no. 54000267 for financial support during the preparation of this manuscript.
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1 Introduction and Preliminaries

The classical Banach's contraction principle [1] is a power tool in nonlinear analysis and has been extended and improved by many mathematicians (see [2–12] and others). Recently, the existence of fixed points for contraction mappings in partially ordered metric spaces has been studied by Ran and Reurings [13], Nieto and Lopez [14] and Agarwal et al. [15]. Extensions and applications of these works appear in [16, 17].

In 2006, Bhaskar and Lakshmikantham [18] introduced the concept of a coupled fixed point and the mixed monotone property. Furthermore, they proved some coupled fixed point theorems for mapping satisfies the mixed monotone property and give some applications in the existence and uniqueness of a solution for a periodic boundary value problem. A number of articles in this topic have been dedicated to the improvement and generalization see in [19–22] and reference therein.

Recall that, if $(X, \leq)$ is a partially ordered set and $F : X \to X$ is such that, for all $x, y \in X$, $x \leq y$ implies $F(x) \leq F(y)$, then a mapping F is said to be non-decreasing. Similarly, a non-increasing mapping is also defined.

Definition 1.1 (Bhaskar and Lakshmikantham [18])**.** Let $(X, \leq)$ be a partial ordered set and $F : X \times X \to X$ be a mapping. The mapping F is said to has the *mixed monotone property* if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X, x_1 \leq x_2 \quad \Longrightarrow \quad F(x_1, y) \leq F(x_2, y) \tag{1.1}$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \quad \Longrightarrow \quad F(x, y_1) \geq F(x, y_2). \tag{1.2}$$

Definition 1.2 (Bhaskar and Lakshmikantham [18])**.** Let X be a non-empty set. An element $(x, y) \in X \times X$ is call a *coupled fixed point* of the mapping $F : X \times X \to X$ if

$$x = F(x, y) \text{ and } y = F(y, x).$$

Theorem 1.3 (Bhaskar and Lakshmikantham [18])**.** *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \tag{1.3}$$

for all $x, y, u, v \in X$ for which $x \geq u$ and $y \leq v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0),$$

then F has a coupled fixed point.

Theorem 1.4 (Bhaskar and Lakshmikantham [18]). *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Suppose that X has the following property:*

1. *if $\{x_n\}$ is a nondecreasing sequence with $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,*
2. *if $\{y_n\}$ is a nonincreasing sequence with $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.*

Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)] \tag{1.4}$$

for all $x, y, u, v \in X$ for which $x \geq u$ and $y \leq v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0),$$

then F has a coupled fixed point.

In 2009, Lakshmikantham and Ćirić [23] introduced the notion of a coupled coincidence point and a coupled common fixed point and also improved the concept of mixed monotone property to mixed g-monotone property.

Definition 1.5 (Lakshmikantham and Ćirić [23]). Let X be a non-empty set. An element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$g(x) = F(x, y) \text{ and } g(y) = F(y, x).$$

Definition 1.6 (Lakshmikantham and Ćirić [23]). Let X be a non-empty set. An element $(x, y) \in X \times X$ is called a *coupled common fixed point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$x = g(x) = F(x, y) \text{ and } y = g(y) = F(y, x).$$

Definition 1.7 (Lakshmikantham and Ćirić [23]). Let $(X, \leq)$ be a partial ordered set and $F : X \times X \to X$, $g : X \to X$ be mappings. The mapping F is said to has the *mixed g-monotone property* if F is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \quad \Longrightarrow \quad F(x_1, y) \leq F(x_2, y) \tag{1.5}$$

and

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \quad \Longrightarrow \quad F(x, y_1) \geq F(x, y_2). \tag{1.6}$$

Recently, Luong and Thuan [24] proved some coupled fixed point theorems for mappings satisfy the mixed monotone property in partially ordered metric spaces under some control functions which are generalizations of the main results of Bhaskar and Lakshmikantham [18]. They applied this results to the existence and uniqueness for a solution of a nonlinear integral equation.

Let Φ denote all functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(φ_1) φ is continuous and non-decreasing;

(φ_2) $\varphi(a) = 0 \Longleftrightarrow a = 0$;

(φ_3) $\varphi(a + b) \leq \varphi(a) + \varphi(b)$ for all $a, b \in [0, \infty)$.

The following functions are in Φ:

(1) $\varphi_1(a) = ka$ where $k \in (0, \infty)$;

(2) $\varphi_2(a) = \dfrac{a}{a+1}$;

(3) $\varphi_3(a) = \ln(a + 1)$;

(4) $\varphi_3(a) = \min\{1, a\}$.

Let Ψ denote all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(ψ_1) $\lim_{r \to t} \psi(r) > 0$ for all $t > 0$;

(ψ_2) $\lim_{r \to 0^+} \psi(r) = 0$.

The following functions are in Ψ:

(1) $\psi_1(a) = ka$ where $k \in (0, \infty)$;

(2) $\psi_2(a) = \dfrac{1}{2}\ln(2a + 1)$;

(3) $\psi_3(a) = \begin{cases} 1, & a = 0, 1 \\ \dfrac{a}{a+1}, & a \in (0, 1) \\ \dfrac{a}{2}, & a > 1. \end{cases}$

Theorem 1.8 (Luong and Thuan [24]). *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d in X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ is the mapping such that F has the mixed monotone property. Suppose there exists a function $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi(d(F(x, y), F(u, v))) \leq \frac{1}{2}\varphi(d(x, u) + d(y, v)) - \psi\left(\frac{d(x, u) + d(y, v)}{2}\right) \qquad (1.7)$$

for all $x, y, u, v \in X$ for which $x \leq u$ and $y \geq v$. Suppose either
(a) F is continuous or
(b) X has the following property:

1. *if $\{x_n\}$ is a non-decreasing sequence with $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,*
2. *if $\{y_n\}$ is a non-increasing sequence with $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.*

If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0),$$

then F has a coupled fixed point.

Theorem 1.9 (Luong and Thuan [24]). *In addition to the hypotheses of Theorem 1.8, suppose that, for all $(x, y), (z, t) \in X \times X$, there exists $(u, v) \in X \times X$ which is comparable to (x, y) and (z, t). Then F has a unique coupled fixed point.*

In this paper, we are interesting in the improvement of the result due to Luong and Thuan [24]. We extend the coupled fixed point theorems of Luong and Thuan [24] to the coupled common fixed point theorems for mappings satisfy a new non-commuting condition. So our theorems are also generalization of classical coupled fixed point theorems of Bhaskar and Lakshmikantham [18].

The following lemma due to Haghi et al. [25] is useful tool for prove our main theorems:

Lemma 1.10 (Haghi, Rezapour and Shahzad [25]). *Let X be a nonempty set and $g : X \to X$ be a mapping. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.*

2 Main Results

We begin this section by prove the coupled coincidence point theorems which are essential tool in the partial order metric spaces to conclude the existence of coupled common fixed points for two mappings.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X. Let $F : X \times X \to X$ and $g : X \to X$ are the mappings such that F has the mixed g-monotone property, $F(X \times X) \subseteq g(X)$, $(g(X), d)$ is a complete metric space and g is continuous. Suppose there exists a function $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi(d(F(x,y), F(u,v))) \leq \frac{1}{2}\varphi(d(g(x), g(u)) + d(g(y), g(v))) - \psi\left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\right) \tag{2.1}$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(y) \geq g(v)$. Suppose either
(a) *F is continuous or*
(b) *X has the following property:*

1. *if $\{x_n\}$ is a non-decreasing sequence with $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,*
2. *if $\{y_n\}$ is a non-increasing sequence with $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.*

If there exists $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0), \quad g(y_0) \geq F(y_0, x_0),$$

then F and g have a coupled coincidence fixed point.

Proof. We use Lemma 1.10, we have there exists $E \subseteq X$ with $g(E) = g(X)$ and $g : E \to X$ is one-to-one mapping. Next, we define a mapping $H : g(E) \times g(E) \to g(E)$ by $H(g(x), g(y)) = F(x, y)$. It follows from g is one-to-one on E that the mapping H is well-defined. From (2.1), we have

$$\varphi(d(H(g(x), g(y)), H(g(u), g(v)))) \leq \frac{1}{2}\varphi(d(g(x), g(u)) + d(g(y), g(v))) - \psi\left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\right) \quad (2.2)$$

for all $g(x), g(y), g(u), g(v) \in g(E)$ with $g(x) \leq g(y)$ and $g(y) \geq g(v)$.

As F has the mixed g-monotone property that H has the mixed monotone property. Since F is continuous, H is also continuous.

Now, we can apply Theorem 1.8 with mapping H. So there exists a coupled fixed point $m, n \in g(X)$ such that

$$m = H(m, n) \text{ and } n = H(n, m).$$

Since $m, n \in g(X)$, we have $m = g(m_1)$ and $n = g(n_1)$ for some $m_1, n_1 \in X$. Thus

$$g(m_1) = H(g(m_1), g(n_1)) \text{ and } g(n_1) = H(g(n_1), g(m_1))$$

and then

$$g(m_1) = F(m_1, n_1) \text{ and } g(n_1) = F(n_1, m_1).$$

Therefore, F and g have a coupled coincidence point. This completes the proof. □

Corollary 2.2 ([24, Theorem 2.1]). *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d in X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ is the mapping such that F has the mixed monotone property. Suppose there exists a function $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi(d(F(x, y), F(u, v))) \leq \frac{1}{2}\varphi(d(x, u) + d(y, v)) - \psi\left(\frac{d(x, u) + d(y, v)}{2}\right) \quad (2.3)$$

for all $x, y, u, v \in X$ for which $x \leq u$ and $y \geq v$. Suppose either

(a) F is continuous or

(b) X has the following property:

1. *if $\{x_n\}$ is a non-decreasing sequence with $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,*
2. *if $\{y_n\}$ is a non-increasing sequence with $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.*

If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0),$$

then F has a coupled fixed point.

Proof. In Theorem 2.1, taking $g = I_X$, where I_X is the identity mapping on X. □

Corollary 2.3 ([24, Corollary 2.2])**.** *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d in X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ is the mapping such that F has the mixed monotone property. Suppose there exists a function $\psi \in \Psi$ such that*

$$d(F(x,y), F(u,v)) \leq \frac{d(x,u) + d(y,v)}{2} - \psi\left(\frac{d(x,u) + d(y,v)}{2}\right) \tag{2.4}$$

for all $x, y, u, v \in X$ for which $x \leq u$ and $y \geq v$. Suppose either

(a) F is continuous or

(b) X has the following property:

1. *if $\{x_n\}$ is a non-decreasing sequence with $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,*
2. *if $\{y_n\}$ is a non-increasing sequence with $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.*

If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0),$$

then F has a coupled fixed point.

Proof. In Theorem 2.1, taking $g = I_X$, where I_X is the identity mapping on X and taking $\varphi(a) = a$. □

Corollary 2.4 ([18, Theorem 2.1 and 2.2])**.** *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d in X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ is the mapping such that F has the mixed monotone property. Suppose there exists a constant number $k \in [0, 1)$ such that*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)] \tag{2.5}$$

for all $x, y, u, v \in X$ *for which* $x \leq u$ *and* $y \geq v$. *Suppose either*
(a) F *is continuous or*
(b) X *has the following property:*

1. *if* $\{x_n\}$ *is a non-decreasing sequence with* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* $n \in \mathbb{N}$,

2. *if* $\{y_n\}$ *is a non-increasing sequence with* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all* $n \in \mathbb{N}$.

If there exists $x_0, y_0 \in X$ *such that*

$$x_0 \leq F(x_0, y_0), \quad y_0 \geq F(y_0, x_0),$$

then F *has a coupled fixed point.*

Proof. In Theorem 2.1, taking $g = I_X$, where I_X is the identity mapping on X, $\varphi(a) = a$ and $\psi(a) = (1 - k)a$. □

Next, we give the notion of a coupled point of coincidence between mappings $F : X \times X \to X$ and $g : X \to X$.

Definition 2.5 (Abbas et al. [26])**.** Let X be a non-empty set. If $F : X \times X \to X$ and $g : X \to X$ have a coupled coincidence point $(x, y) \in X \times X$, then we called a point $(a, b) := (g(x), g(y))$ that a *coupled point of coincidence* of F and g.

Example 2.6. *Let* $X = \mathbb{R}^+ \cup \{0\}$. *The mapping* $F : X \times X \to X$ *defined by*

$$F(x, y) = x^2 + y^2 + 1$$

for all $x, y \in X$ *and the mapping* $g : X \to X$ *defined by*

$$g(x) = \begin{cases} x^2 + 1, & x \leq 1, \\ x^2 + 2, & x > 1. \end{cases}$$

It obvious that a coupled coincidence point of F *and* g *is only point* $(0, 0)$ *and a coupled point of coincidence is* $(g(0), g(0)) = (1, 1)$.

Example 2.7. *Let* $X = [\sqrt{3}, \infty)$. *The mapping* $F : X \times X \to X$ *defined by*

$$F(x, y) = x^2 + y^2 - 2$$

for all $x, y \in X$ *and the mapping* $g : X \to X$ *defined by*

$$g(x) = \begin{cases} x^2, & x \in \{3, 5, 7, ...\}, \\ x^2 + 14, & \text{otherwise}. \end{cases}$$

It obvious that a coupled coincidence point of F *and* g *is only point* $(4, 4)$ *and a coupled point of coincidence is* $(g(4), g(4)) = (30, 30)$.

Example 2.8. *Let* $X = \mathbb{R}$*. The mapping* $F : X \times X \to X$ *defined by*

$$F(x, y) = 1$$

for all $x, y \in X$ *and the mapping* $g : X \to X$ *defined by*

$$g(x) = x^2 - 3$$

for all $x \in X$*. It obvious that a coupled coincidence point of* F *and* g *are* $(2, 2), (2, -2), (-2, 2), (-2, -2)$ *and a coupled point of coincidence is* $(1, 1)$*.*

Theorem 2.9. *In addition to the hypotheses of Theorem 2.1, for all* $(g(x), g(y))$*,* $(g(z), g(t)) \in g(X) \times g(X)$*, there exists* $(g(u), g(v)) \in g(X) \times g(X)$ *that is comparable to* $(g(x), g(y))$ *and* $(g(z), g(t))$*. Then* F *and* g *have a coupled coincidence point. Moreover,* F *and* g *have a unique coupled point of coincidence.*

Proof. Similar in the proof of Theorem 2.1, we can prove this result by use Lemma 1.10 and Theorem 1.9. □

Next, we give the concept of weakly compatible (w-compatible) between the binary mapping $F : X \times X \to X$ and the unitary mapping $g : X \to X$. This concept was introduced by Abbas et al. [26] We also establish some coupled common fixed point theorems.

Definition 2.10 (Abbas et al. [26])**.** Let X be a non-empty set, $F : X \times X \to X$ and $g : X \to X$. The mappings F and g are said to be *weakly compatible* if

$$g(F(x, y)) = F(g(x), g(y))$$

whenever $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, F and g commute at all coupled coincidence points.

Example 2.11. *Let* $X = \mathbb{R}^+ \cup \{0\}$*. The mapping* $F : X \times X \to X$ *defined by*

$$F(x, y) = 2(x + y)$$

for all $x, y \in X$ *and the mapping* $g : X \to X$ *defined by*

$$g(x) = \begin{cases} x, & x < 1, \\ 2x - 1, & x \geq 1. \end{cases}$$

It is easy to see that a coupled coincidence point of F *and* g *is only* $(0, 0)$*. Since*

$$g(F(0, 0)) = g(0) = 0 = F(0, 0) = F(g(0), g(0)),$$

we get F *and* g *are weakly compatible.*

Next, we establish the coupled common fixed point theorem for weakly compatible mappings.

Theorem 2.12. *In addition to the hypotheses of Theorem 2.1, suppose that F and g are weakly compatible and, for all $(g(x), g(y)), (g(z), g(t)) \in g(X) \times g(X)$, there exists $(g(u), g(v)) \in g(X) \times g(X)$ that is comparable to $(g(x), g(y))$ and $(g(z), g(t))$. Then F and g have a unique coupled common fixed point.*

Proof. From Theorem 2.9, we get F and g have a coupled coincidence point (x, y) that is

$$g(x) = F(x, y) \text{ and } g(y) = F(y, x). \tag{2.6}$$

Moreover, by Theorem 2.9, we also $(g(x), g(y))$ is a unique coupled point of coincidence. Using condition of weakly compatible of F and g, we get

$$g(g(x)) = g(F(x, y)) = F(g(x), g(y)) \tag{2.7}$$

and

$$g(g(y)) = g(F(y, x)) = F(g(y), g(x)). \tag{2.8}$$

We denote $g(x) = m$ and $g(y) = n$. From (2.7) and (2.8), we have

$$g(m) = F(m, n) \text{ and } g(n) = F(n, m). \tag{2.9}$$

Thus (m, n) is a coupled coincidence point of F and g and $(g(m), g(n))$ is a coupled point of coincidence of F and g. Since $(g(x), g(y))$ is a unique coupled point of coincidence, we get $(g(m), g(n)) = (g(x), g(y))$, which implies that

$$g(m) = g(x) \text{ and } g(n) = g(y). \tag{2.10}$$

Hence

$$g(m) = m \text{ and } g(n) = n. \tag{2.11}$$

From (2.9) and (2.11), we have

$$m = g(m) = F(m, n) \text{ and } n = g(n) = F(n, m). \tag{2.12}$$

Therefore, a coupled common fixed point of F and g is (m, n).

Finally, we prove the uniqueness of a coupled common fixed point (m, n). We may assume that (m_1, n_1) is another coupled common fixed point of F and g and then $(g(m_1), g(n_1))$ is also a coupled point of coincidence of F and g. Hence $(g(m_1), g(n_1)) = (g(m), g(n))$ and so $g(m_1) = g(m)$ and $g(n_1) = g(n)$. Thus $m_1 = g(m_1) = g(m) = m$ and $n_1 = g(n_1) = g(n) = n$. Above statement implies (m, n) is a unique coupled common fixed point of F and g. This completes the proof. □

We finish this section by give an example which satisfy the requirements of Theorem 2.1 as follows:

Example 2.13. *Let $X = \mathbb{R}$ and defined a partially order $\leq$ by $a \leq b \Longleftrightarrow b - a \in \mathbb{R}^+ \cup \{0\}$. Define a mapping $d : X \times X \to [0, \infty)$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Let $F : X \times X \to X$ and $g : X \to X$ be the mappings defined by*

$$F(x, y) = 2 \text{ and } g(x) = x^2 + 1$$

for all $x \in X$. Define a mapping $\varphi, \psi : [0, \infty) \to [0, \infty)$ by

$$\varphi(a) = ka \text{ and } \psi(a) = (1 - k)a$$

where $k \in [0, 1)$. By simple calculation, we see that F and g satisfy (2.1) and F has the mixed g-monotone property. Moreover, g and F are continuous and there exists point $0, 2 \in X$ such that

$$g(0) = 1 \leq 2 = F(0, 2) \quad \text{and} \quad g(2) = 5 \geq 2 = F(2, 0).$$

So all the conditions of Theorem 2.1 are satisfied. Therefore, we conclude that F and g have a coupled coincidence point in X. This coupled coincidence point are $(1, 1), (1, -1), (-1, 1), (-1, -1)$.

Remark 2.14. *Although main results of Luong and Thuan in [24] are essential tool in the partially ordered metric spaces to show the existence of coupled fixed points of binary mapping $F : X \times X \to X$. However, some problems in nonlinear analysis can not apply to only one binary mapping. Therefore, it is very necessary use the main results of this paper to contribute in conclude that existence of coupled coincidence points and coupled common fixed points in the partially ordered metric spaces. Moreover, not only our theorems hold in partially order metric spaces, but also, by using a similar the proof, it is a consequence of many results in other spaces such as a partially ordered cone metric space due to Huang and Zhang [27].*

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(Received 19 July 2011)
(Accepted 21 May 2012)