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Surjective Multihomomorphisms between Cyclic Groups

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Abstract : A multifunction f from a group G into a group G' is called a *multihomomorphism* if

$$f(xy) = f(x)f(y) \ \big(= \{st \mid s \in f(x) \text{ and } t \in f(y)\}\big)$$

for all $x, y \in G$. Denote by $\mathrm{MHom}(G, G')$ the set of all multihomomorphisms from G into G'. We call $f \in \mathrm{MHom}(G, G')$ a *surjective multihomomorphism* if $f(G) = G'$ where $f(G) = \bigcup_{x \in G} f(x)$. The elements of $\mathrm{MHom}\big((\mathbb{Z}, +), (\mathbb{Z}, +)\big)$, $\mathrm{MHom}\big((\mathbb{Z}_n, +), (\mathbb{Z}, +)\big)$, $\mathrm{MHom}\big((\mathbb{Z}, +), (\mathbb{Z}_n, +)\big)$ and $\mathrm{MHom}\big((\mathbb{Z}_m, +), (\mathbb{Z}_n, +)\big)$ have been already characterized and counted. Our purpose is to characterize when these multihomomorphisms are surjective. The cardinalities of the subsets of surjective multihomomorphisms are also determined.

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1 Introduction

The cardinality of a set X is denoted by $|X|$. A *multifunction* from a nonempty set X into a nonempty set Y is a function from X into $P(Y)\backslash\{\emptyset\}$ where $P(Y)$ is the power set of Y. For $A \subseteq X$, we let $f(A) = \bigcup_{x \in A} f(x)$.

Semicontinuity of multifunctions between two topological spaces has been studied by Whyburn [6], Smithson [4] and Feichtinger [2]. Triphop, Harnchoowong and Kemprasit [5] have studied multifunctions in an algebraic sense. The definition of multihomomorphisms between groups was given naturally in [5] as follows : A multifunction f from a group G into a group G' is a *multihomomorphism* if

$$f(xy) = f(x)f(y) \ \big(= \{st \mid s \in f(x) \text{ and } t \in f(y)\}\big) \text{ for all } x, y \in G.$$

The set of all multihomomorphisms from G into G' is denoted by $\mathrm{MHom}(G, G')$. Then $\mathrm{MHom}(G, G')$ contains all homomorphisms from G into G'. We write $\mathrm{MHom}(G)$ for $\mathrm{MHom}(G, G)$.

In [5], the authors characterized the elements of $\mathrm{MHom}(G, G')$ and also determined $|\mathrm{MHom}(G, G')|$ where G and G' are cyclic groups. It is well-known that every infinite cyclic group is isomorphic to $(\mathbb{Z},+)$ and every finite cyclic group of order n is isomorphic to $(\mathbb{Z}_n,+)$ where $(\mathbb{Z},+)$ and $(\mathbb{Z}_n,+)$ are the additive group of integers and the additive group of integers modulo n, respectively. Recall that

$$\mathbb{Z}_n = \big\{[x]_n \mid x \in \mathbb{Z}\big\} = \big\{[0]_n, [1]_n, \ldots, [n-1]_n\big\}.$$

For $a_1, \ldots, a_k \in \mathbb{Z}$ not all zero, let $(a_1, \ldots, a_k)$ denote the g.c.d. of $a_1, \ldots, a_k$. We let $\mathbb{Z}^+ = \big\{x \in \mathbb{Z} \mid x > 0\big\}$, $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$, $\mathbb{Z}^- = \big\{x \in \mathbb{Z} \mid x < 0\big\}$ and $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$. It is clearly seen that for $a, b \in \mathbb{Z}$ not both zero, $a\mathbb{Z} + b\mathbb{Z} = (a,b)\mathbb{Z}$ and $a\mathbb{Z}_n + b\mathbb{Z}_n = (a,b)\mathbb{Z}_n$. We note here that if $k \mid n$ (k divides n), then $|k\mathbb{Z}_n| = \dfrac{n}{k}$ and $k\mathbb{Z}_n = \big\{[0]_n, [k]_n, \ldots, \big(\frac{n}{k} - 1\big)[k]_n\big\}$. Recall that the Euler ϕ-function is defined by $\phi(1) = 1$ and for $k \in \mathbb{Z}^+$ with $k > 1$, $\phi(k)$ is the number of positive integers less than k and relatively prime to k. Hence

$$\phi(k) = \Big|\big\{a \in \{0, 1, \ldots, k-1\} \mid (a,k) = 1\big\}\Big| \text{ for all } k \in \mathbb{Z}^+.$$

It is known that $\sum\limits_{k \mid n} \phi(k) = n$ ([3], page 191).

An element $f \in \mathrm{MHom}(G, G')$ is called a *surjective multihomomorphism* from G into G' if $f(G) = G'$, that is, $\bigcup\limits_{x \in G} f(x) = G'$. For convenience, let

$$\mathrm{SMHom}(G, G') = \big\{f \in \mathrm{MHom}(G, G') \mid f \text{ is surjective}\big\},$$

that is,

$$\mathrm{SMHom}(G, G') = \big\{f \in \mathrm{MHom}(G, G') \mid f(G) = G'\big\},$$

and let $\mathrm{SMHom}(G) = \mathrm{SMHom}(G, G)$.

Our purpose is to characterize the surjective multihomomorphisms in MHom $(\mathbb{Z},+)$, $\mathrm{MHom}\big((\mathbb{Z}_n,+),(\mathbb{Z},+)\big)$, $\mathrm{MHom}\big((\mathbb{Z},+),(\mathbb{Z}_n,+)\big)$ and $\mathrm{MHom}\big((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)\big)$ and determine the cardinalities of $\mathrm{SMHom}(\mathbb{Z},+)$, $\mathrm{SMHom}\big((\mathbb{Z}_n,+),(\mathbb{Z},+)\big)$, $\mathrm{SMHom}\big((\mathbb{Z},+),(\mathbb{Z}_n,+)\big)$ and $\mathrm{SMHom}\big((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)\big)$.

In the remainder, let n and m be positive integers. For a subsemigroup H of $(\mathbb{Z},+)$ containing 0 and $a \in \mathbb{Z}$, define

$$F_{H,a}(x) = ax + H \quad \text{for all } x \in \mathbb{Z}$$

and

$$C_H([x]_n) = H \quad \text{for all } x \in \mathbb{Z}.$$

If $k, a \in \mathbb{Z}, k \neq 0$ and $\dfrac{k}{(k,n)} \Big| \, a$, let

$$D_{k,a}([x]_n) = ax + k\mathbb{Z} \quad \text{for all } x \in \mathbb{Z}.$$

Also, for $k, a \in \mathbb{Z}$, define

$$G_{k,a}(x) = [ax]_n + k\mathbb{Z}_n \quad \text{for all } x \in \mathbb{Z},$$

and for $k, a \in \mathbb{Z}$ with $\dfrac{(k,n)}{(k,m,n)} \mid a$, let

$$I_{k,a}([x]_m) = [ax]_n + k\mathbb{Z}_n \quad \text{for all } x \in \mathbb{Z}.$$

The following results will be referred and they were proved in [5].

Theorem 1.1 ([5]) *The following statements about $MHom(\mathbb{Z},+)$ are true.*

(a) $MHom(\mathbb{Z},+) = \{F_{H,a} \mid H \text{ is subsemigroup of } (\mathbb{Z},+) \text{ containing } 0 \text{ and } a \in \mathbb{Z}\}$.

(b) $|MHom(\mathbb{Z},+)| = \aleph_0$.

Theorem 1.2 ([5]) *The following statements about $MHom\big((\mathbb{Z}_n,+),(\mathbb{Z},+)\big)$ are true.*

(a) $$MHom\big((\mathbb{Z}_n,+),(\mathbb{Z},+)\big) = \{C_H \mid H \text{ is a subsemigroup of } (\mathbb{Z},+) \text{ containing } 0\} \cup \Big\{D_{k,a} \mid k,a \in \mathbb{Z}, k \neq 0 \text{ and } \frac{k}{(k,n)} \mid a\Big\}.$$

(b) $|MHom((\mathbb{Z}_n,+),(\mathbb{Z},+))| = \aleph_0$.

Brown and Curtis [1] noted that every subsemigroup of $(\mathbb{Z}_0^+,+)$ containing 0 is finitely generated. This fact is useful to obtain Theorem 1.1(b) and Theorem 1.2(b).

Theorem 1.3 ([5]) *The following statements about $MHom\big((\mathbb{Z},+),(\mathbb{Z}_n,+)\big)$ are true.*

(a) $MHom\big((\mathbb{Z},+),(\mathbb{Z}_n,+)\big) = \{G_{k,a} \mid k,a \in \mathbb{Z}\}$.

(b) *If* $k,l \in \mathbb{Z}^+$, $k \mid n, l \mid n$, $a \in \{0,1,\dots,k-1\}$, $b \in \{0,1,\dots,l-1\}$ *and* $G_{k,a} = G_{l,b}$, *then* $k = l$ *and* $a = b$.

(c) $MHom\big((\mathbb{Z},+),(\mathbb{Z}_n,+)\big) = \{G_{k,a} \mid k \in \mathbb{Z}^+, k \mid n \text{ and } a \in \{0,1,\dots,k-1\}\}$.

(d) $$\big|MHom\big((\mathbb{Z},+),(\mathbb{Z}_n,+)\big)\big| = \sum_{\substack{k\in\mathbb{Z}^+\\ k \mid n}} k.$$

Theorem 1.4 ([5]) *The following statements about $MHom\big((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)\big)$ are true.*

(a) $$MHom\big((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)\big) = \Big\{I_{k,a} \mid k,a \in \mathbb{Z} \text{ and } \frac{(k,n)}{(k,m,n)} \mid a\Big\}.$$

(b) *If* $k,l \in \mathbb{Z}^+$, $k \mid n, l \mid n$, $a \in \{0,1,\dots,k-1\}$, $b \in \{0,1,\dots,l-1\}$, $\dfrac{k}{(k,m)} \mid a$, $\dfrac{l}{(l,m)} \mid b$ *and* $I_{k,a} = I_{l,b}$, *then* $k = l$ *and* $a = b$.

(c) $MHom((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)) = \{I_{k,a} \mid k \in \mathbb{Z}^+, k \mid n, a \in \{0,1,\ldots,k-1\}$ *and* $\frac{k}{(k,m)} \mid a\}$.

(d) $\big|MHom\big((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)\big)\big| = \sum\limits_{\substack{k\in\mathbb{Z}^+\\ k\,|\,n}} (k,m)$.

2 Surjective Multihomomorphisms

We shall characterize surjective multihomomorphisms between cyclic groups by using the notations introduced above.

Theorem 2.1 *Let H be a subsemigroup of $(\mathbb{Z},+)$ containing 0 and $a \in \mathbb{Z}$. Then $F_{H,a} \in SMHom(\mathbb{Z},+)$ if and only if the following two statements are satisfied.*

(i) a is relatively prime to some $h \in H$.

(ii) $a = 0$ implies $H = \mathbb{Z}$.

Proof. We have $F_{H,a}(\mathbb{Z}) = a\mathbb{Z} + H$ by the definition of $F_{H,a}$. Assume that $F_{H,a}$ is surjective. Then $F_{H,a}(\mathbb{Z}) = a\mathbb{Z} + H = \mathbb{Z}$, so $as + h = 1$ for some $s \in \mathbb{Z}$ and $h \in H$. This implies that $(a,h) = 1$. If $a = 0$, then $H = a\mathbb{Z} + H = \mathbb{Z}$.

Conversely, assume that (i) and (ii) hold. From (i), $as + ht = 1$ for some $s,t \in \mathbb{Z}$. If $a = 0$, then by (ii), $H = \mathbb{Z}$, so $F_{H,a}(\mathbb{Z}) = H = \mathbb{Z}$. Next, assume that $a \neq 0$.

Case 1: $t \geq 0$. Then $ht \in H$ since H is a subsemigroup of $(\mathbb{Z},+)$ and $0 \in H$, so $1 = as + ht \in a\mathbb{Z} + H$. Hence for all $x \in \mathbb{Z}_0^+, x = asx + htx \in a\mathbb{Z} + H$. Let $y \in \mathbb{Z}^-$. Since $a \neq 0$, $y + ar \in \mathbb{Z}^+$ for some $r \in \mathbb{Z}$. From the above proof, $y + ar \in a\mathbb{Z} + H$ which implies that $y \in a\mathbb{Z} + H - ar = a\mathbb{Z} + H$.

Case 2: $t < 0$. Then $-ht = h(-t) \in H$, so $-1 = -as + (-ht) \in a\mathbb{Z} + H$. Consequently, for all $x \in \mathbb{Z}_0^+, -x = -asx + (-htx) \in a\mathbb{Z} + H$. This shows that $x \in a\mathbb{Z} + H$ for all $x \in \mathbb{Z}_0^-$. If $y \in \mathbb{Z}^+$, then $y + ar \in \mathbb{Z}^-$ for some $r \in \mathbb{Z}$ since $a \neq 0$. This implies that $y + ar \in a\mathbb{Z} + H$. Thus $y \in a\mathbb{Z} + H - ar = a\mathbb{Z} + H$.

From Case 1 and Case 2, $a\mathbb{Z} + H = \mathbb{Z}$. Hence $F_{H,a}$ is surjective. □

Corollary 2.2 $|SMHom(\mathbb{Z},+)| = \aleph_0$.

Proof. Note that $F_{H,1}(0) = H$ for every subsemigroup H of $(\mathbb{Z},+)$ containing 0. This implies $F_{k\mathbb{Z},1} \neq F_{l\mathbb{Z},1}$ for all distinct $k,l \in \mathbb{Z}^+$. By Theorem 2.1, $F_{k\mathbb{Z},1} \in$ SMHom$(\mathbb{Z},+)$ for every $k \in \mathbb{Z}$. Then

$$\begin{aligned} |\mathrm{SMHom}(\mathbb{Z},+)| &\geq \big|\{F_{k\mathbb{Z},1} \mid k \in \mathbb{Z}^+\}\big| \\ &= |\mathbb{Z}^+| = \aleph_0. \end{aligned}$$

But $|\mathrm{MHom}(\mathbb{Z},+)| = \aleph_0$ by Theorem 1.1(b), so we have $|\mathrm{SMHom}(\mathbb{Z},+)| = \aleph_0$. □

Theorem 2.3 *Let $k, a \in \mathbb{Z}$ be such that $k \neq 0$ and $\dfrac{k}{(k,n)} \mid a$. Then $D_{k,a} \in$ SMHom$((\mathbb{Z}_n, +), (\mathbb{Z}, +))$ if and only if k and a are relatively prime.*

Proof. We have that $D_{k,a}(\mathbb{Z}_n) = a\mathbb{Z} + k\mathbb{Z} = (k,a)\mathbb{Z}$, and $(k,a)\mathbb{Z} = \mathbb{Z}$ if and only if $(k,a) = 1$. Hence $D_{k,a} \in \text{SMHom}((\mathbb{Z}_n, +), (\mathbb{Z}, +))$ if and only if $(k,a) = 1$. □

Corollary 2.4 $\big|\mathit{SMHom}\big((\mathbb{Z}_n, +), (\mathbb{Z}, +)\big)\big| = n.$

Proof. First, we note that for a subsemigroup H of $(\mathbb{Z}, +)$ containing 0, $C_H \in \text{SMHom}\big((\mathbb{Z}_n, +), (\mathbb{Z}, +)\big)$ if and only if $H = \mathbb{Z}$. Also, we have that $C_{\mathbb{Z}} = D_{1,0}$ and $(1,0) = 1$. Hence by Theorem 2.3, we have

$$\text{SMHom}\big((\mathbb{Z}_n, +), (\mathbb{Z}, +)\big) = \Big\{D_{k,a} \mid k, a \in \mathbb{Z}, k \neq 0, \frac{k}{(k,n)} \Big| \ a \text{ and } (k,a) = 1\Big\}.$$

Let

$$K = \big\{D_{k,a} \mid k \in \mathbb{Z}^+, k \mid n, a \in \{0, 1, \ldots, k-1\} \text{ and } (k,a) = 1\big\}.$$

To show that $\text{SMHom}\big((\mathbb{Z}_n, +), (\mathbb{Z}, +)\big) = K$, it is clear that $K \subseteq \text{SMHom}\big((\mathbb{Z}_n, +), (\mathbb{Z}, +)\big)$. For the reverse inclusion, let $k, a \in \mathbb{Z}, k \neq 0$, $\dfrac{k}{(k,n)} \Big| \ a$ and $(k,a) = 1$. Let $s, b \in \mathbb{Z}$ be such that $a = s|k| + b$ and $b \in \{0, 1, \ldots, |k| - 1\}$. Then

$$ax - bx = s|k|x \in |k|\mathbb{Z} = k\mathbb{Z} \quad \text{for all } x \in \mathbb{Z}$$

which implies that

$$ax + k\mathbb{Z} = bx + |k|\mathbb{Z} \quad \text{for all } x \in \mathbb{Z}.$$

By assumption, $k \mid a(k,n)$ and $(k,a) = 1$. Then $k \mid (k,n)$, so $k \mid n$. Hence $|k| \mid n$. Since $(|k|, a) = (k,a) = 1$ and $|k| \mid a - b$, it follows that $(|k|, b) = 1$. Consequently, $D_{k,a} = D_{|k|,b} \in K$.

If $k, l \in \mathbb{Z}^+$, $k \mid n$, $l \mid n$, $a \in \{0, 1, \ldots, k-1\}$ and $b \in \{0, 1, \ldots, l-1\}$ are such that $D_{k,a} = D_{l,b}$, then $k\mathbb{Z} = D_{k,a}([0]_n) = D_{l,b}([0]_n) = l\mathbb{Z}$ and $a + k\mathbb{Z} = D_{k,a}([1]_n) = D_{l,b}([1]_n) = b + l\mathbb{Z}$. It follows that $k = l$, $|a - b| \in \{0, 1, \ldots, k-1\}$ and $0 \leq |a - b| \in k\mathbb{Z}$. Thus $a - b = 0$, so $a = b$. Hence

$$\begin{aligned}
&\big|\text{SMHom}\big((\mathbb{Z}_n, +), (\mathbb{Z}, +)\big)\big| \\
&\qquad = \big|\{D_{k,a} \mid k \in \mathbb{Z}^+, k \mid n, a \in \{0, 1, \ldots, k-1\} \text{ and } (k,a) = 1\}\big| \\
&\qquad = \big|\{\langle k, a\rangle \mid k \in \mathbb{Z}^+, k \mid n, a \in \{0, 1, \ldots, k-1\} \text{ and } (k,a) = 1\}\big| \\
&\qquad = \sum_{k \mid n} \phi(k) = n
\end{aligned}$$

where $\langle k, a\rangle$ denotes the ordered pair of k and a. □

Theorem 2.5 *For $k, a \in \mathbb{Z}$, $G_{k,a} \in$ SMHom$((\mathbb{Z}, +), (\mathbb{Z}_n, +))$ if and only if (k,n) and a are relatively prime.*

Proof. By the definition of $G_{k,a}$, $G_{k,a}(\mathbb{Z}) = a\mathbb{Z}_n + k\mathbb{Z}_n$.

First, assume that $G_{k,a} \in \text{SMHom}((\mathbb{Z},+),(\mathbb{Z}_n,+))$. Then $a\mathbb{Z}_n + k\mathbb{Z}_n = \mathbb{Z}_n$. Then $[1]_n = [as+kt]_n$ for some $s,t \in \mathbb{Z}$. Hence $1 = as+kt+nl$ for some $l \in \mathbb{Z}$. It follows that $as + (k,n)\big(\frac{k}{(k,n)}t + \frac{n}{(k,n)}l\big) = 1$. This implies that $((k,n),a) = 1$.

For the converse, assume that (k,n) and a are relatively prime. Then $1 = as + (k,n)t$ for some $s,t \in \mathbb{Z}$, and thus for every $x \in \mathbb{Z}$, $[x]_n = [asx + (k,n)tx]_n \in a\mathbb{Z}_n + (k,n)\mathbb{Z}_n$. Hence $\mathbb{Z}_n = a\mathbb{Z}_n + (k,n)\mathbb{Z}_n = a\mathbb{Z}_n + k\mathbb{Z}_n$. Therefore $G_{k,a} \in \text{SMHom}((\mathbb{Z},+),(\mathbb{Z}_n,+))$, as desired. □

Corollary 2.6 $\big|SMHom\big((\mathbb{Z},+),(\mathbb{Z}_n,+)\big)\big| = n.$

Proof. By Theorem 1.3(c),

$$\text{MHom}\big((\mathbb{Z},+),(\mathbb{Z}_n,+)\big) = \big\{G_{k,a} \mid k \in \mathbb{Z}^+, k \,|\, n \text{ and } a \in \{0,1,\ldots,k-1\}\big\}.$$

From this fact and Theorem 2.5, we have

$$\text{SMHom}\big((\mathbb{Z},+),(\mathbb{Z}_n,+)\big) = \big\{G_{k,a} \mid k \in \mathbb{Z}^+, k \,|\, n, a \in \{0,1,\ldots,k-1\}, \text{and } (k,a) = 1\big\}.$$

Hence by Theorem 1.3(b),

$$\begin{aligned}\big|\text{SMHom}\big((\mathbb{Z},+),(\mathbb{Z}_n,+)\big)\big| &= \big|\big\{\langle k,a\rangle \mid k \in \mathbb{Z}^+, k \,|\, n, a \in \{0,1,\ldots,k-1\} \\ &\qquad \text{and } (k,a) = 1\big\}\big| \\ &= \sum_{k \,|\, n} \phi(k) = n.\end{aligned}$$

□

Theorem 2.7 *Let $k,a \in \mathbb{Z}$ be such that $\frac{(k,n)}{(k,m,n)} \mid a$. Then $I_{k,a} \in SMHom\big((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)\big)$ if and only if (k,n) and a are relatively prime.*

Proof. We have from the definition of $I_{k,a}$ that $I_{k,a}(\mathbb{Z}_m) = a\mathbb{Z}_n + k\mathbb{Z}_n$. The remainder of the proof is exactly the same as that of Theorem 2.5. □

Corollary 2.8 $\big|SMHom\big((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)\big)\big| = (m,n).$

Proof. By Theorem 1.4(c),

$$\text{MHom}\big((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)\big) = \Big\{I_{k,a} \mid k \in \mathbb{Z}^+, k \,|\, n, a \in \{0,1,\ldots,k-1\} \text{ and } \frac{k}{(k,m)} \Big| \, a\Big\}.$$

This fact and Theorem 2.7 yield

$$\begin{aligned}\text{SMHom}\big((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)\big) = &\Big\{I_{k,a} \mid k \in \mathbb{Z}^+, k \,|\, n, a \in \{0,1,\ldots,k-1\}, \\ &\frac{k}{(k,m)} \Big| \, a \text{ and } (k,a) = 1\Big\}.\end{aligned}$$

Then it follows from Theorem 1.4(b) that

$$\left|\mathrm{SMHom}\big((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)\big)\right| = \Big|\Big\{\langle k,a\rangle \mid k\in\mathbb{Z}^+, k\mid n, a\in\{0,1,\ldots,k-1\}, \frac{k}{(k,m)}\Big|\, a \text{ and } (k,a)=1\Big\}\Big|.$$

Claim that for $k\in\mathbb{Z}^+$ and $a\in\{0,1,\ldots,k-1\}$,

$$k\mid n,\ \frac{k}{(k,m)}\Big|\, a \text{ and } (k,a)=1 \ \Leftrightarrow\ k|(m,n) \text{ and } (k,a)=1.$$

If $k\mid n$, $\frac{k}{(k,m)}\Big|\, a$ and $(k,a)=1$, then $k\mid n$, $k\mid a(k,m)$ and $(k,a)=1$. This implies that $k\mid n$ and $k\mid (k,m)$, so $k\mid n$ and $k\mid m$. Hence $k\mid (m,n)$. The converse is evident. Consequently,

$$\begin{aligned}&\left|\mathrm{SMHom}\big((\mathbb{Z}_m,+),(\mathbb{Z}_n,+)\big)\right|\\ &\quad= \left|\{\langle k,a\rangle \mid k\in\mathbb{Z}^+,\ k\mid (m,n), a\in\{0,1,\ldots,k-1\} \text{ and } (k,a)=1\}\right|\\ &\quad= \sum_{k\mid (m,n)}\phi(k) = (m,n).\end{aligned}$$

□

Example 2.9. It follows from Corollary 2.4 and Corollary 2.6 that

$$\left|\mathrm{SMHom}\big((\mathbb{Z}_{12},+),(\mathbb{Z},+)\big)\right| = 12 = \left|\mathrm{SMHom}\big((\mathbb{Z},+),(\mathbb{Z}_{12},+)\big)\right|.$$

Then by Theorem 1.2(b) and Theorem 1.3(d), we have respectively that

$$\Big|\Big\{f\in\mathrm{MHom}\big((\mathbb{Z}_{12},+),(\mathbb{Z},+)\big) \mid f \text{ is not surjective}\Big\}\Big| = \aleph_0$$

and

$$\Big|\Big\{f\in\mathrm{MHom}\big((\mathbb{Z},+),(\mathbb{Z}_{12},+)\big) \mid f \text{ is not surjective}\Big\}\Big| = \sum_{k\mid 12} k - 12 = 28-12=16.$$

We have from Corollary 2.8 that

$$\left|\mathrm{SMHom}\big((\mathbb{Z}_{12},+),(\mathbb{Z}_{15},+)\big)\right| = (12,15) = 3.$$

Hence by Theorem 1.4(d),

$$\Big|\Big\{f\in\mathrm{MHom}\big((\mathbb{Z}_{12},+),(\mathbb{Z}_{15},+)\big) \mid f \text{ is not surjective}\Big\}\Big| = \sum_{k\mid 15}(k,12)-3 = 8-3=5.$$

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