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A Modified Eighth-Order Derivative-Free Root Solver

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Abstract : This paper proposes a modified technique for solving nonlinear equations. The technique is fully free from derivative calculation per full cycle and consumes only four pieces of function evaluations to reach the local convergence rate eight. This shows that our technique is optimal due to the conjecture of Kung and Traub. The contributed class is built by using weight function approach. In the sequel, theoretical results are given and finally numerical examples are employed to evaluate and illustrate the accuracy of the novel methods derived of the modified technique.

Keywords : Steffensen's method; optimality; derivative-free optimization; computational algorithms; multi-point iteration methods.
2010 Mathematics Subject Classification : 41A25; 65B99.

1 Introduction

The focus of this work is to find the simple roots for nonlinear equations of the general form $f(x) = 0$, which automatically makes our new contribution to be useful for finding the local minima/maxima of the sufficiently smooth function f, in the open domain D. Derivative-free algorithms are much more applicable in optimization problems due to the expensive cost of derivative evaluations.

To illustrate further, we should mention that the expensive optimization problems arise in science and engineering, because evaluation of the function f often

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requires a complex deterministic simulation based on solving the (nonlinear) equations (for example, nonlinear eigenvalue problems, ordinary or partial differential equations) that describe the underlying physical phenomena. The computational noise associated with these complex simulations means that obtaining derivatives is difficult and unreliable (especially for the case of n-dimensional space). Moreover, these simulations often rely on legacy or proprietary codes and hence must be treated as black-box functions, necessitating a derivative-free optimization algorithm.

The efficiency index of an iterative process for approximating simple roots is defined by the value $EI = p^{\frac{1}{m}}$, where p is the order of convergence and m is the number of (functional) evaluations (including the derivatives that are necessary to apply the algorithm that defines the iterative process.)

In 1974, the fundamental work in root-finding was published by Kung and Traub in [1]. In fact, they conjectured that a multi-point iteration without memory using m (functional) evaluation per iteration can reach the maximum convergence rate $2^{(m-1)}$. For further reading, one may consult [2, 3, 4, 5].

Our purpose here is to construct a robust wide optimal class which does not require computation of the first, second or higher order derivatives of the function f, per full cycle to proceed. Hence, this study is organized as follows. Section 2 gives our general class of three-step without memory iteration methods and proves that they reach the optimal order eight using only four pieces of information. This section is followed by Section 3, where a robust comparison with the existing new optimal three-step eighth-order derivative-free algorithms is presented to support that the derived methods from the class are efficient and reliable. Finally, Section 4 contains a short conclusion of the study.

2 Main Result

In recent years, many researchers have developed modifications of Newton's method or Newton-like methods in a number of ways to improve its order of convergence at the expense of additional evaluations of functions and/or derivatives mostly at the point iterated by the Newton's method (see, e.g., [6, 7, 8]).

However, the problem with the Newton's method is that it may fail to converge in some cases if the derivative of the function is small or even zero in the vicinity of the true root. Similar to the Newton's method, many modified Newton's methods have also such problems and many of those methods depend on the higher-order derivatives in the computing process, which make their practical application restricted. Thus, the research of developing higher-order methods, which are suitable in the problems where derivative evaluation is hard, is important for practical applications.

Recently, Zheng et al. [9] developed a family of eighth-order derivative-free optimal methods as follows

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]},\ w_n = x_n + \beta f(x_n), \beta \in \mathbb{R} - \{0\}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n,y_n]+f[y_n,w_n]-f[x_n,w_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,y_n,x_n](z_n-y_n)+f[z_n,y_n,x_n,w_n](z_n-y_n)(z_n-x_n)}, \end{cases} \tag{2.1}$$

where (using $c_j = \frac{f^{(j)}(\alpha)}{j!},\ j \geq 1$, and $e_n = x_n - \alpha$), the error equation of this without memory algorithm is

$$e_{n+1} = \frac{c_2^2(c_2^2 - c_1c_3)(c_2^3 - c_1c_2c_3 + c_1^2c_4)(1 + c_1\beta)^4}{c_1^7} e_n^8 + O(e_n^9). \tag{2.2}$$

Unfortunately, they missed to study the convergence order of their family by using backward finite difference (FD) approximation in the first step instead of forward FD. Herein, we study a modified version of (2.1) by using backward FD and also weight functions at the end of the second and third steps in order to extent the family to a wide general class of methods. Thus, we suggest the following three-step iteration

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]},\ w_n = x_n - \beta f(x_n), \beta \in \mathbb{R} - \{0\}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n,y_n]+f[y_n,w_n]-f[x_n,w_n]} G(t), \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,y_n,x_n](z_n-y_n)+f[z_n,y_n,x_n,w_n](z_n-y_n)(z_n-x_n)} H(\tau)P(\omega), \end{cases} \tag{2.3}$$

where $t = \frac{f(y_n)}{f(x_n)}$, $\tau = \frac{f(z_n)}{f(y_n)}$, $\omega = \frac{f(y_n)}{f(w_n)}$, and its error equation is wider than (2.2).

Theorem 2.1. *Let the function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ has a simple root α in the open interval D. Furthermore the first, second, third and fourth derivatives of the function $f(x)$ belongs in the open interval D. Then the class of methods defined by (2.3) is of optimal order eight and satisfies the follow-up error equation*

$$\begin{aligned} e_{n+1} = \frac{-1}{48c_1^7}&(c_2(-1 + c_1\beta)^2(-2c_1c_3 + c_2^2(2 + (-1 + c_1\beta)G''(0)))(-24c_1^2c_2c_4(-1 \\ &+ c_1\beta)^2 + 12c_1^2c_3^2(-1 + c_1\beta)^2H''(0) - 12c_1c_2^2c_3(-1 + c_1\beta)^2(-2 + (2 \\ &+ (-1 + c_1\beta)G''(0))H''(0)) + c_2^4(3(-1 + c_1\beta)^2(2 + (-1 + c_1\beta)G''(0)) \\ &\times (-4 + (2 + (-1 + c_1\beta)G''(0))H''(0)) + P^{(4)}(0))))e_n^8 + O(e_n^9), \end{aligned} \tag{2.4}$$

where $G(0) = H(0) = P(0) = 1,\ G'(0) = H'(0) = P'(0) = P''(0) = P^{(3)}(0) = 0,$ $|G''(0)| < \infty,\ |H''(0)| < \infty, |P^{(4)}(0)| < \infty.$

Proof. To find the asymptotic error constant of (2.3) wherein

$$c_j = \frac{f^{(j)}(\alpha)}{j!},\ j \geq 1,$$

and $e_n = x_n - \alpha$, we expand any terms of (2.3) around the simple root α in the nth iterate. Thus, we write

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9). \quad (2.5)$$

Accordingly, we attain by Taylor's series expanding around the simple root

$$y_n - \alpha = c_2 \left(\frac{1}{c_1} - \beta \right) e_n^2 + \cdots + O(e_n^9). \quad (2.6)$$

In the same vein, we have for the second step (with $G(0) = 1,\ G'(0) = 0$)

$$z_n - \alpha = \frac{c_2(-1 + c_1\beta)^2(-2c_1c_3 + c_2^2(2 + (-1 + c_1\beta)G''(0)))}{2c_1^3} e_n^4 + \cdots + O(e_n^9). \quad (2.7)$$

Now we have $f(z_n) = (c_2(-1+c_1\beta)^2(-2c_1c_3+c_2^2(2+(-1+c_1\beta)G''(0)))e_n^4)/(2c_1^2) + (1/(6c_1^3))(-1 + c_1\beta)(6c_1^2c_3^2(-2 + c_1\beta)(-1 + c_1\beta) + 6c_1^2c_2c_4(-2 + c_1\beta)(-1 + c_1\beta) + 3c_1c_2^2c_3(-4(4 + c_1\beta(-5 + 2c_1\beta)) - 3(-2 + c_1\beta)(-1 + c_1\beta)^2G''(0)) + c_2^4(12(2 + c_1\beta(-2 + c_1\beta)) + 3(-1 + c_1\beta)(8 + c_1\beta(-8 + 3c_1\beta))G''(0) - (-1 + c_1\beta)^3G^3(0)))e_n^5 + 1/(24c_1^4)(-24c_1^3c_3c_4(-1 + c_1\beta)^2(7 + c_1\beta(-7 + 2c_1\beta)) + 12c_1^2c_2^2c_4(2(-1 + c_1\beta)(-12 + c_1\beta(19 + c_1\beta(-13 + 4c_1\beta))) + 3(-1 + c_1\beta)^3(3 + c_1\beta(-3 + c_1\beta))G''(0)) + 12c_1^2c_2(-1 + c_1\beta)(-2c_1c_5(-1 + c_1\beta)(3 + c_1\beta(-3 + c_1\beta)) + c_3^2(-2 + c_1\beta)(2(9 + 5c_1\beta(-2 + c_1\beta)) + 3(-2 + c_1\beta)(-1 + c_1\beta)^2G''(0))) + 8c_1c_2^3c_3(3(-30 + 29G''(0) + c_1\beta(68 - 100G''(0) + c_1\beta(-66 - 9c_1\beta(-4 + c_1\beta) + (138 + c_1\beta(-98 + c_1\beta(37 - 6c_1\beta)))G''(0)))) + 2(-2 + c_1\beta)(-1 + c_1\beta)^4G^{(3)}(0)) + c_2^5(24(10 + c_1\beta(-2 + c_1\beta)(10 + c_1\beta(-4 + 3c_1\beta))) + 12(-1 + c_1\beta)(40 + c_1\beta(-84 + c_1\beta(72 + c_1\beta(-31 + 6c_1\beta))))G''(0) - 4(-1 + c_1\beta)^3(11 + c_1\beta(-11 + 4c_1\beta))G^{(3)}(0) + (-1 + c_1\beta)^5G^{(4)}(0)))e_n^6 + O(e_n^7)$. For $f[z_n, y_n]$, we attain

$$\begin{aligned} f[z_n, y_n] = c_1 &+ (c_2^2(1 - c_1\beta)e_n^2)/c_1 + (c_2(c_1c_3(-2 + c_1\beta)(-1 + c_1\beta) - c_2^2(2 \\ &+ c_1\beta(-2 + c_1\beta)))e_n^3)/c_1^2 + (1/(2c_1^3))c_2(-2c_1^2c_4(-1 + c_1\beta)(3 \\ &+ c_1\beta(-3 + c_1\beta)) + 2c_1c_2c_3(-7 + c_1\beta(10 + c_1\beta(-7 + 2c_1\beta))) \\ &+ c_2^3(-2(-5 + c_1\beta(7 + c_1\beta(-4 + c_1\beta))) + (-1 + c_1\beta)^3G''(0)))e_n^4 \\ &+ O(e_n^5). \end{aligned}$$

Using this new formula and (2.7), we can obtain

$$\begin{aligned} &f[z_n, y_n] + f[z_n, y_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n) \\ &\qquad = c_1 + \frac{c_2(-1 + c_1\beta)^2(-2c_1c_2c_3 + c_1^2c_4 + c_2^3(2 + (-1 + c_1\beta)G''(0)))}{c_1^3} e_n^4 \\ &\qquad\quad + \cdots + O(e_n^9). \qquad (2.8) \end{aligned}$$

Considering (2.8) in the last step of (2.3) and $H(0) = P(0) = 1, H'(0) = P'(0) = P''(0) = P^{(3)}(0) = 0$, we have the error equation (2.4). This shows that our contributed class (2.3) achieves the optimal order eight by using only four pieces of information. The proof is complete. □

In terms of computational point of view, each member from the proposed general class (2.3) includes four function evaluations and is totally free from derivative per full iteration, this shows that our class achieves the optimality order conjectured by Kung-Traub. Accordingly, the new class has the optimal efficiency index $8^{\frac{1}{4}} \approx 1.682$, which is much better than $2^{\frac{1}{2}} \approx 1.414$ and $4^{\frac{1}{3}} \approx 1.587$ of optimal one- and two-step iterations without memory. Some forms of the weight functions in the Theorem 2.1, which make the order optimal are listed in Table 1.

Clearly, based on Theorem 2.1, we can produce the following iterations

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]},\ w_n = x_n - f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,y_n]+f[y_n,w_n]-f[x_n,w_n]}\{1 - \frac{1}{-1+f[x_n,w_n]}(\frac{f(y_n)}{f(x_n)})^2\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,y_n,x_n](z_n-y_n)+f[z_n,y_n,x_n,w_n](z_n-y_n)(z_n-x_n)} \\ \qquad \times\{(1+(\frac{f(z_n)}{f(y_n)})^3)(1+(\frac{f(y_n)}{f(w_n)})^5)\}, \end{cases} \tag{2.9}$$

wherein the error equation reads

$$e_{n+1} = \frac{c_2^2 c_3(c_2c_3 - c_1c_4)(-1+c_1)^4}{c_1^5} e_n^8 + O(e_n^9). \tag{2.10}$$

As can be seen this error equation is much simpler than (2.2) which shows the simplicity and accuracy of our derived method from the class (2.3). We can also have the following iteration from (2.3)

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]},\ w_n = x_n - f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,y_n]+f[y_n,w_n]-f[x_n,w_n]}\{1 + (\frac{f(y_n)}{f(x_n)})^3\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,y_n,x_n](z_n-y_n)+f[z_n,y_n,x_n,w_n](z_n-y_n)(z_n-x_n)} \\ \qquad \times\{(1+(\frac{f(z_n)}{f(y_n)})^3)(1+(\frac{f(y_n)}{f(w_n)})^5)\}, \end{cases} \tag{2.11}$$

wherein

$$e_{n+1} = \frac{c_2^2(c_2^2 - c_1c_3)(c_2^3 - c_1c_2c_3 + c_1^2c_4)(-1+c_1)^4}{c_1^7} e_n^8 + O(e_n^9). \tag{2.12}$$

Table 1. Some typical forms of the real valued weight functions in (2.3).

Weight function	$G(t)$	$H(\tau)$	$P(\omega)$
Forms	$1+t^2+\gamma_1 t^3$	$1+\tau^2+\gamma_1\tau^3$	$1+\omega^4+\gamma_1\omega^5$

$t = \frac{f(y_n)}{f(x_n)},\ \tau = \frac{f(z_n)}{f(y_n)},\ \omega = \frac{f(y_n)}{f(w_n)},\ \gamma_1 \in \mathbb{R}.$

According to (2.4), we conclude that iterative adjustment of a parameter $\beta \in \mathbb{R} - \{0\}$ leads to the increase of convergence speed of the three-step methods derived from the class (2.3). Numerical results could present evidently that the increase of convergence speed of our class (2.3) could be attained by choosing very small positive values of $\beta \in \mathbb{R} - \{0\}$.

One of the most refined methods can now be generated using (2.9) and a small value for β, i.e. we can have

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]},\ w_n = x_n - 0.01f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,y_n]+f[y_n,w_n]-f[x_n,w_n]}\{1 - \frac{1}{-1+0.01f[x_n,w_n]}(\frac{f(y_n)}{f(x_n)})^2\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,y_n,x_n](z_n-y_n)+f[z_n,y_n,x_n,w_n](z_n-y_n)(z_n-x_n)} \\ \qquad \times \left(1 + \left(\frac{f(z_n)}{f(y_n)}\right)^3\right)\left(1 + \left(\frac{f(y_n)}{f(w_n)}\right)^5\right), \end{cases} \tag{2.13}$$

wherein the error equation now reads

$$e_{n+1} = \frac{c_2^2 c_3(c_2c_3 - c_1c_4)(-1 + 0.01c_1)^4}{c_1^5} e_n^8 + O(e_n^9). \tag{2.14}$$

Table 2. The examples considered in this study.

Test Functions	Zeros
$f_1(x) = (\sin x)^2 + x$	$\alpha_1 = 0$
$f_2(x) = (1 + x)\cos(\frac{\pi x}{2}) - \sqrt{1 - x^2}$	$\alpha_2 \approx -0.728584046444826\cdots$
$f_3(x) = (\sin x)^2 - x^2 + 1$	$\alpha_3 \approx 1.404491648215341\cdots$
$f_4(x) = e^{-x} + \sin(x) - 2$	$\alpha_4 \approx -1.0541271240912128\cdots$
$f_5(x) = xe^{-x} - 0.1$	$\alpha_5 \approx 0.111832559158963\cdots$
$f_6(x) = \sqrt{x^4 + 8}\sin(\frac{\pi}{x^2+2}) + \frac{x^3}{x^4+1} - \sqrt{6} + \frac{8}{17}$	$\alpha_6 \approx -1.1492126746090871\cdots$
$f_7(x) = \sqrt{x^2 + 2x + 5} - 2\sin(x) - x^2 + 3$	$\alpha_7 \approx 2.331967655883964\cdots$
$f_8(x) = \arcsin(x^2 - 1) - \frac{x}{2} + 1$	$\alpha_8 \approx 0.594810968398369\cdots$
$f_9(x) = (\sin(x) - \frac{2}{2})(x + 1)$	$\alpha_9 \approx 0.785398163397448\cdots$
$f_{10}(x) = x - \sin(\cos(x)) + 1$	$\alpha_{10} \approx -0.1660390510510295\cdots$

3 Numerical Experiments

We check the effectiveness of the novel derivative-free class of iterative methods (2.3) in this section. We have compared (2.9) and (2.11) with the the eighth-order technique of Soleymani [10], which is defined by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]},\ w_n = x_n + f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,y_n]+f[y_n,w_n]-f[x_n,w_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)\{1+(\frac{f(y_n)}{f(x_n)})^4-(1+f[x_n,w_n])(\frac{f(y_n)}{f(w_n)})^3-(\frac{f(z_n)}{f(y_n)})^2+\frac{f(z_n)}{f(w_n)}+(\frac{f(z_n)}{f(x_n)})^2\}}{f[x_n,z_n]+f[z_n,y_n]-f[x_n,y_n]}, \end{cases} \tag{3.1}$$

and the optimal eighth-order family of Zheng et al. (2.1) with $\beta = 1$, using the examples listed in Table 2. The results of comparisons are given in Table 3 in terms of the number significant digits for each test function after the specified number of iterations. For numerical comparisons, the stopping criterion is $|f(x_n)| < 1.E - 1000$.

Table 3. Results of convergence for different derivative-free methods.

f & Guess		(2.1)	(3.1)	(2.11)	(2.9)
$f_1, 0.2$	$\lvert f(x_1)\rvert$	0.4e-5	0.3e-5	0.6e-8	0.6e-8
	$\lvert f(x_2)\rvert$	0.1e-41	0.1e-41	0.1e-89	0.9e-90
	$\lvert f(x_3)\rvert$	0.8e-334	0.1e-333	0.3e-988	0.7e-990
$f_2, -0.4$	$\lvert f(x_1)\rvert$	0.7e-3	0.1e-3	0.1e-3	0.1e-3
	$\lvert f(x_2)\rvert$	0.4e-23	0.1e-29	0.5e-32	0.3e-32
	$\lvert f(x_3)\rvert$	0.4e-185	0.2e-237	0.1e-259	0.6e-261
$f_3, 1.6$	$\lvert f(x_1)\rvert$	0.9e-5	0.1e-4	0.7e-7	0.1e-6
	$\lvert f(x_2)\rvert$	0.3e-43	0.1e-40	0.2e-58	0.5e-56
	$\lvert f(x_3)\rvert$	0.4e-350	0.2e-328	0.8e-471	0.6e-451
$f_4, -1.2$	$\lvert f(x_1)\rvert$	0.3e-6	0.7e-6	0.1e-7	0.4e-7
	$\lvert f(x_2)\rvert$	0.1e-54	0.1e-51	0.7e-64	0.3e-59
	$\lvert f(x_3)\rvert$	0.4e-441	0.5e-417	0.2e-514	0.1e-476
$f_5, 0.15$	$\lvert f(x_1)\rvert$	0.2e-10	0.2e-9	0.1e-13	0.2e-13
	$\lvert f(x_2)\rvert$	0.1e-83	0.4e-74	0.1e-112	0.2e-111
	$\lvert f(x_3)\rvert$	0.2e-668	0.6e-592	0.1e-905	0.1e-894
$f_6, -1.3$	$\lvert f(x_1)\rvert$	0.2e-3	0.4e-0	0.5e-4	0.8e-4
	$\lvert f(x_2)\rvert$	0.9e-25	0.3e-4	0.6e-33	0.4e-31
	$\lvert f(x_3)\rvert$	0.3e-196	0.3e-31	0.7e-264	0.1e-248
$f_7, 2$	$\lvert f(x_1)\rvert$	0.2e-7	0.8e-7	0.4e-8	0.4e-8
	$\lvert f(x_2)\rvert$	0.3e-68	0.1e-62	0.6e-72	0.6e-72
	$\lvert f(x_3)\rvert$	0.8e-555	0.6e-508	0.7e-583	0.1e-582
$f_8, 0.3$	$\lvert f(x_1)\rvert$	0.1e-6	0.3e-6	0.8e-14	0.8e-14
	$\lvert f(x_2)\rvert$	0.3e-57	0.2e-53	0.2e-120	0.2e-120
	$\lvert f(x_3)\rvert$	0.1e-461	0.1e-430	0.3e-972	0.3e-972
$f_9, 0.6$	$\lvert f(x_1)\rvert$	0.1e-6	0.9e-6	0.7e-12	0.7e-12
	$\lvert f(x_2)\rvert$	0.3e-59	0.1e-49	0.4e-104	0.4e-104
	$\lvert f(x_3)\rvert$	0.7e-479	0.5e-400	0.6e-842	0.8e-842
$f_{10}, 0.3$	$\lvert f(x_1)\rvert$	0.3e-5	0.2e-6	0.4e-10	0.4e-10
	$\lvert f(x_2)\rvert$	0.2e-44	0.2e-54	0.5e-89	0.5e-89
	$\lvert f(x_3)\rvert$	0.9e-358	0.3e-438	0.3e-720	0.3e-720

It can be observed from Table 3 that almost in most cases our derived methods from the suggested class of derivative-free without memory iterations (2.3) is superior in solving nonlinear equations.

In this study, numerical computations have been carried out using variable precision arithmetic in MATLAB 7.6. In general, computational accuracy strongly depends on the structures of the iterative methods, the sought zeros and the test functions as well as good initial approximations. For further derivative-free root-solvers, refer to [11, 12].

4 Concluding Remarks

This research article has suggested a general class of three steps without memory iterations for solving nonlinear scalar equations using the similar structure as (2.1), but with different approximation in the first step and also using weight functions in the second and third steps in order to attain a wide class with a general error equation in which very efficient methods with simple error equation could be attained. The analytical study of the class was furnished and shows that our class is consistent with the optimality conjecture of Kung and Traub.

Each member from our class has $8^{\frac{1}{4}} \approx 1.682$ as its efficiency index, while this index was totally supported through the numerical examples in Section 3 and the results in Table 3. Hence, our derived methods from the proposed class can be considered as novel and robust alternatives in the literature for solving nonlinear equations while no derivative evaluation per full cycle is needed. The new methods are so fruitful for optimization-oriented problems.

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