Thai Journal of Mathematics
Volume 10 (2012) Number 3 : 535–539
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On $\triangle_v^m$-Cesáro Summable Double Sequences

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Abstract : In this article we prove that for $0 < p < \infty$ if a double sequence is strongly $\triangle_{v_p^2}^m$-Cesáro summable to L , then it is $\triangle_v^m$-statistically convergent to L. If a bounded double sequence is $\triangle_v^m$-statistically convergent to L, then it is strongly $\triangle_{v_p^2}^m$-Cesaro summable to L.

Keywords : difference sequnce; double sequence; statistical convergence; strong Cesáro summablility.
2010 Mathematics Subject Classification : 40A05; 46A45.

1 Introduction

The difference sequence space $X(\triangle)$ was introduced by Kizmaz [1] as follows

$$X(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in X\} \text{ for } X = l_\infty, \ c \text{ and } c_0,$$

where $\triangle x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Difference sequence spaces have been studied by Colak and Et [2], Et [3], Et and Esi [4], Khan [5–7] and many others. The definition of v-invariance of a sequence space X was given by Colak [8] as follows:

Definition 1.1. Let $v = \{v_k\}$ be any sequence. A sequence space X is *v-invariant* if $X_v = X$, where $X_v = \{x = (x_k) : (v_k x_k) \in X\}$.

The definition of v-invariance of the sequence space $\triangle^m(X)$ was given by Isik [9] as follows:

Definition 1.2. The sequence space $\triangle^m(X)$ is *v-invariant* if

$$\triangle_v^m(X) = \triangle^m(X),$$

where $\triangle_v^m(X) = \{x = (x_k) : (\triangle_v^m x_k) \in X\},\ m \in \mathbb{N}$,

$$\triangle_v^m x_k = \sum_{i=0}^{m} (-1)^i \begin{bmatrix} m \\ i \end{bmatrix} x_{k+i} v_{k+i}$$

and X is any sequence space (see Isik [9]).

Note that if X is linear space, then $\triangle_v^m(X)$ is also a linear space.

Lemma 1.3. *Let X and Y be sequence spaces, $v = \{v_k\}$, $m, n \in \mathbb{N}$.*

1. *If $X \subset Y$, then $\triangle_v^m(X) \subset \triangle_v^m(Y)$ and the inclusion is strict.*
2. *If $n < m$, then $\triangle_v^n(X) \subset \triangle_v^m(X)$ and the inclusion is strict.*

Proof. See [9]. □

Let C_1 be the Cesaro matrix of order 1, that is $(C_1)_{nk} = 1/n$ for $0 \le k \le n$ and 0 for $k > n$ $(n = 0, 1, \ldots)$. For $0 < p < \infty$ Maddox [10] defined the sets

$$w_p^0 = (c_0)_{[C_1]^p} = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right) = 0 \right\},$$
$$w_p = \left\{ x = (x_k) \in \omega : x - le \in w_p^0 \ \text{ for some complex number } \ell \right\},$$
$$w_p^\infty = (l_\infty)_{[C_1]^p}.$$

In the case $X = w_p$, Isik have introduced $\triangle_v^m(w_p)$ in [9] as:

$$\triangle_v^m(w_p) = \left\{ x = (x_k) : \frac{1}{n} \sum_{k=1}^{n} |\triangle_v^m x_k - L|^p \to 0, n \to \infty, p > 0, \text{for some } L \right\}.$$

If $x \in \triangle_v^m(w_p)$, then we say that x is strongly $\triangle_{v_p}^m$-Cesáro summable to L.

In this paper, we denote double sequence by (x_{jk}). Double sequences have been studied by Hardy [11], Moricz [12], Moricz and Rhoades [13], Tripathy [14] and others.

2 Main Results

In this paper, we define the concepts of $\triangle_v^m$-statistically convergent and strongly Cesáro summable for double sequences as follows:

Definition 2.1. A sequence (x_{jk}) is said to be $\triangle_v^m$-*statistically convergent* if there is a complex number L such that

$$\lim_{r,n} \frac{1}{rn} \left|\{j \leq r, k \leq n : |\triangle_v^m x_{jk} - L| \geq \varepsilon\}\right| = 0,$$

for every $\varepsilon > 0$.

Definition 2.2.

$$\triangle_v^m(w_p^2) = \left\{x = (x_{jk}) : \frac{1}{rn}\sum_{j=1}^{r}\sum_{k=1}^{n} |\triangle_v^m x_{jk} - L|^p \to 0, r, n \to \infty, p > 0, \text{for some } L\right\}$$

where w_p^2 denote the space of all p-Cesáro summable double sequences. If $x \in \triangle_v^m(w_p^2)$, then we say that a double sequence $x = (x_{jk})$ is *strongly* $\triangle_{v_p^2}^m$*-Cesáro summable to* L.

Theorem 2.3. *Let* $0 < p < \infty$. *If a double sequence is strongly* $\triangle_{v_p^2}^m$*-Cesáro summable to L, then it is* $\triangle_v^m$*-statistically convergent to L. If a bounded double squennce is* $\triangle_v^m$*-statistically convergent to L, then it strongly* $\triangle_{v_p^2}^m$*-Cesáro summable to L.*

Proof. Suppose that $(x_{jk}) \in \triangle_v^m(w_p^2)$. Then for any $\varepsilon > 0$ we have

$$\frac{1}{rn}\sum_{j=1}^{r}\sum_{k=1}^{n} |\triangle_v^m x_{jk} - L|^p \geq \frac{1}{rn}|\{(j,k) : j \leq r; k \leq n \;\text{ and }\; |\triangle_v^m x_{jk} - L|^p \geq \varepsilon\}|\varepsilon^p.$$

Taking as $r, n \to \infty$ we have (x_{jk}) is $\triangle_v^m$-statistically convergent to L.

Conversely suppose that (x_{jk}) be bounded and $\triangle_v^m$-statistically convergent to L and let $H = ||B||_\infty + |L|$, where $B = (x_{jk})$ be a double sequence. Let $\varepsilon > 0$, then there exist m_0, n_0 such that

$$\frac{1}{rn}\left|\left\{(j,k) : j \leq r; k \leq n \;\text{ and }\; |\triangle_v^m x_{jk} - L| \geq \left(\frac{\varepsilon}{2}\right)^{1/p}\right\}\right| < \frac{\varepsilon}{2H^p}$$

for all $r > m_0$ and $n > n_0$. Let

$$L_{rn} = \left\{(j,k) : j \leq r; k \leq n \;\text{ and }\; |\triangle_v^m x_{jk} - L|^p \geq \frac{\varepsilon}{2}\right\}.$$

Now for all $r > m_0$ and $n > n_0$ we have

$$\begin{aligned}\frac{1}{rn}\sum_{j=1}^{r}\sum_{k=1}^{n}|\triangle_v^m x_{jk} - L|^p &= \frac{1}{rn}\left\{\sum\sum_{(j,k)\in L_{rn}} |\triangle_v^m x_{jk} - L|^p\right\} \\ &\quad + \frac{1}{rn}\left\{\sum_{\substack{(j,k)\in L_{rn} \\ j\le r, k\le n}} \sum |\triangle_v^m x_{jk} - L|^p\right\} \\ &< \frac{1}{rn}\left\{rn\frac{\varepsilon}{2H^p}H^p + rn\frac{\varepsilon}{2}\right\} \\ &< \varepsilon.\end{aligned}$$

Hence (x_{jk}) is strongly $\triangle_{v_p^2}^m$-Cesáro summable to L. □

Corollary 2.4. *Let $0 < p < q < \infty$. Then*

$$\triangle_v^m(w_q^2) \subseteq \triangle_v^m(w_p^2)$$

and

$$\triangle_v^m(w_p^2) \cap \triangle_v^m l_\infty^2 = \triangle_v^m(w_q^2) \cap \triangle_v^m l_\infty^2,$$

where l_∞^2 denote the space of bounded double sequences.

Corollary 2.5. *If a bounded double sequence is $\triangle_v^m$-statistically convergent to L, then it is $\triangle_{v_p^2}^m$-Cesáro summable to L.*

Acknowledgement : The author would like to record their gratitude to the reviewer for his careful reading and making some useful corrections which improved the presentation of the paper.

References

[1] H. Kizmaz, On certain sequence spaces, Canad. Math. Bull. 24 (2) (1981) 169–176.

[2] R. Colak, M. Et, On some generalized difference sequence spacesand related matrix transformation, Hokkoido Mathematical Journal 26 (3) (1997) 483–492.

[3] M. Et, On some topological properties of generalized difference sequence spaces, Internat. J. Math. Math. Sci. 24 (11) (2000) 785–791.

[4] M. Et, A. Esi, On Kothe Toeplitz duals of generalized difference sequence spaces, Bull. Malaysian Math. Sci. Soc. 24 (11) (2000) 25–32.

[5] V.A. Khan, Some inclusion relations between the difference sequence spaces defined by sequence of moduli, Indian Mathematical Society 73 (2006) 77–81.

[6] V.A. Khan, Q.M.D. Lohani, Difference sequence spaces defined by a sequence of moduli, Southeast Asian Bull. Math. 30 (2006) 1061–1067.

[7] V.A. Khan, Q.M.D. Lohani, New Lacunary strong convergence diffference sequence spaces defined by sequence of moduli, Kyungpook Mathematical Journal 46 (2006) 591–595.

[8] R. Colak, On invariant sequence spaces, Erciyes Univ. Journal of Sci. 5 (1989), 881–887.

[9] M. Isik, On statistica convergence of generalized difference sequences, Soochow J. Math. 30 (2) (2004) 197–205.

[10] I.J. Maddox, Elements of Functional Analysis, Cambridge University Press, 1970.

[11] G.H. Hardy, On the convergence of certain multiple series, Proc. Camb. Phil. Soc. 19 (1917) 86–95.

[12] F. Moricz, Extension of the spaces c and c_0 from single to double sequences, Acta. Math. Hung. 57 (19991) 129–136.

[13] F. Moricz, B.E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, Math. Proc. Camb. Phil. Soc. 104 (1988) 283–294.

[14] B.C. Tripathy, Statistically convergent double sequences, Tamkang J. Math. 34 (3) (2003) 231–237.

(Received 25 July 2010)
(Accepted 7 February 2012)