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Common Fixed Point Theorems of Ćirić Type Weak Contractions in Cone Metric Spaces

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Abstract : We prove the existence of common fixed points for a pair of weakly compatible selfmaps satisfying Ćirić type weak contractions in cone metric spaces where the underlying cone is neither regular nor normal. Our theorems extend the results of Choudhury and Metiya [B.S. Choudhury, N. Metiya, Fixed points of weak contractions in cone metric spaces, Nonlinear Anal. 72 (2010) 1589–1593] to non-normal cones. Several examples are provided in support of our results.

Keywords : cone metric space; point of coincidence; common fixed point; asymptotically regular maps; property (E. A).
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1 Introduction

In 2007, Huang and Zhang [1] introduced cone metric spaces by using ordered Banach space instead of the set of real numbers as a codomain, and established Banach contraction principle and some other common fixed point theorems in cone metric spaces where the underlying cone is normal. Later, many authors [2–4] proved common fixed point theorems in cone metric spaces.

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In 2011, Jankovic et al. [5] has shown that all fixed point results in cone metric spaces obtained recently, in which the underlying cone is assumed to be normal, can be reduced to the corresponding results in metric spaces. They have also shown that when the underlying cone is non-normal and solid this is not possible. Also, in recent papers by Du [6] and Amini-Harindi and Fakhar [7] it has been shown that fixed point results in the setting of cone metric spaces in which linear contractive conditions appear can be reduced to the respective results in the metric setting via scalarization method.

In this paper, we use non-linear contractive conditions in cone metric spaces and establish the existence of point of coincidence and common fixed point theorems for a pair of weakly compatible selfmaps without assuming the normality of the cone.

Throughout this paper, let R^+ denote $[0, \infty)$ and Z^+, the set of all positive integers.

Let E be a real Banach space and P a subset of E. Then P is called a *cone* if

(i) P is closed, non-empty and $P \neq \{0\}$;

(ii) $a,\ b \in R,\ a,\ b \geq 0$ and $x,\ y \in P$ implies $ax + by \in P$; and

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subseteq E$, we define a partial order '$\leq$' with respect to P by "$x \leq y$ if and only if $y - x \in P$". We write "$x < y$" to denote "$x \leq y$ but $x \neq y$" and "$x \ll y$" means "$y - x \in intP$", where $intP$ denotes the interior of P.

A cone P is called *normal* if there exists a number $K > 0$ such that for all $x,\ y \in E,\ 0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number K satisfying the above inequality is called the *normal constant* of P. There are no normal cones with normal constant $K < 1$, see [8].

A cone P is said to be *regular* if every increasing sequence which is bounded from above is convergent, *i.e.,* if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$, for some $y \in E$, then there exists x in E such that $\| x_n - x \| \to 0$ as $n \to \infty$.

Equivalently, P is regular if and only if every decreasing sequence which is bounded from below is convergent.

Every regular cone is normal but it's converse need not be true [8].

A cone $P \subseteq E$ is said to be *solid* if $intP \neq \emptyset$ [3]. There are ordered Banach spaces with cone P which is not normal but solid.

Example 1.1 ([8])**.** *Let* $E = C'_R[0,1]$ *with* $\|f\| = \|f\|_\infty + \|f'\|_\infty$ *and* $P = \{f \in E/f \geq 0\}$. *Then* P *is a non-normal cone with* $intP \neq \emptyset$.

Definition 1.2. Let X be a non-empty set. If a mapping $d : X \times X \to E$ satisfies the following conditions

(i) $0 \leq d(x,y)$ for all $x,\ y \in X$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x,\ y \in X$, and

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,\ y,\ z \in X$,

then d is called a *cone metric* on X, and $(X,\ d)$ is called a *cone metric space*.

Definition 1.3. Let $(X,\ d)$ be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then we say that $\{x_n\}$ is a

(i) convergent sequence in X, if for every $c \in E$ with $0 \ll c$ there is an $N \in Z^+$ such that $d(x_n, x) \ll c$ for all $n > N$. We denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) Cauchy sequence in X, if for every $c \in E$ with $0 \ll c$ there is an $N \in Z^+$ such that $d(x_n, x_m) \ll c$ for all $m,\ n > N$.

A cone metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent in X.

Remark 1.4. *Let E be an ordered Banach space with cone P. Then*

(1) if $u \leq v$ and $v \ll w$ then $u \ll w$;

(2) if $u \ll v$ and $v \ll w$ then $u \ll w$;

(3) if $0 \leq u \ll c$ for each $c \in intP$, then $u = 0$;

(4) $c \in intP$ if and only if $[-c, c]$ is a neighborhood of 0;

(5) if P is a solid cone and if a sequence $\{x_n\}$ is convergent in a cone metric space (X, d), then the limit of $\{x_n\}$ is unique.

Remark 1.5. *If $u \leq v + c$ for each $c \in intP$ then $u \leq v$.*

Proof. Let $c \in intP$. Then $\frac{1}{n}c \in intP$ for each $n \in Z^+$. Hence by our assumption $u \leq v + \frac{1}{n}c$ for each $n \in Z^+$ *i.e.*, $v + \frac{1}{n}c - u \in P$. Now, on letting n tends to ∞, we get $v - u \in P$. Hence $u \leq v$. □

In 2010, Choudhury and Metiya [9] established the following fixed point theorem for a weakly contractive map in cone metric spaces with regular cone.

Theorem 1.6. *Let (X, d) be a complete cone metric space with regular cone P such that $d(x, y) \in intP$ for $x, y \in X$ with $x \neq y$. Let $T : X \to X$ be a mapping satisfying the inequality*

$$d(x, y) \leq d(x, y) - \varphi(d(x, y)) \text{ for each } x, y \in X$$

where $\varphi : intP \cup \{0\} \to intP \cup \{0\}$ is continuous and monotone increasing function with

(i) $\varphi(t) = 0$ if and only if $t = 0$;

(ii) $\varphi(t) \ll t$, for $t \in intP$;

(iii) either $\varphi(t) \leq d(x, y)$ or $d(x, y) \leq \varphi(t)$, for $t \in intP \cup \{0\}$ and $x, y \in X$.

Then T has a unique fixed point in X.

Definition 1.7. Let X be any nonempty set. Let f, g be selfmaps of X. Then the pair (f, g) is said to be weakly compatible if $fgx = gfx$ whenever $fx = gx$, $x \in X$.

In 2011, Arandjelovic et al. [10] defined a comparison function in cone metric spaces and established the following fixed point theorem.

Theorem 1.8. *Let (X, d) be a complete cone metric space with solid cone P. Let (f, g) be a pair of weakly compatible self mappings on X such that*

$$d(fx, fy) \leq \varphi(u), \ \ \text{for } x, y \in X$$

where $u \in \{d(gx, gy), d(fx, gx), d(fy, gy)\}$ and $\varphi : P \to P$ is a function such that

(i) $k_1 \leq k_2$ *implies* $\varphi(k_1) \leq \varphi(k_2)$*;*

(ii) $\varphi(0) = 0$ *and* $0 < \varphi(k) < k$ *for* $k \in P \setminus \{0\}$*;*

(iii) $k \in intP$ *implies* $k - \varphi(k) \in intP$*; and*

(iv) if $k \in P \setminus \{0\}$ *and* $c \in intP$*, then there exists* $n_0 \in Z^+$ *such that* $\varphi^n(k) \ll c$ *for each* $n \geq n_0$*.*

Suppose that $f(X) \subseteq g(X)$ and that either $f(X)$ or $g(X)$ is a complete subspace of X. Then the mappings f and g have a unique common fixed point in X.

Definition 1.9. Let (X, d) be a cone metric space with cone P. Suppose that f and g are selfmaps of X such that $f(X) \subseteq g(X)$. Let $x_0 \in X$. Then we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = fx_n = gx_{n+1}$, $n = 0, 1, 2, \ldots$.

We say that the pair (f, g) is *asymptotically regular at x_0* if for each $c \in intP$ there exists $n_0 \in Z^+$ such that $d(y_n, y_{n+1}) \ll c \ \forall \ n \geq n_0$.

If the pair (f, g) is asymptotically regular at each point of X then we say that (f, g) is asymptotically regular on X. If $g = I_X$, the identity map on X, then clearly f is asymptotically regular on X.

Example 1.10. *Let $E = C'_R[0, 1]$ with supremum norm and $P = \{x \in E/x \geq 0\}$. Let $X = [0, 1]$ and $d : X \times X \to E$ be defined by $d(x, y) = |x - y|\varphi$, $\varphi(t) = e^t$, $t \geq 0$. We define $f, g : X \to X$ by $f(x) = \frac{x^2}{2}$ and $g(x) = \frac{x}{2}$. Let $x_0 \in X$. Since $f(X) \subseteq g(X)$, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n = fx_n = gx_{n+1}$, $n = 0, 1, 2, \ldots$. where $y_n = \left(\frac{x_o^2}{2}\right)^{2^n}$, $n = 0, 1, 2, \ldots$. Then $d(y_n, y_{n+1}) = |\left(\frac{x_o^2}{2}\right)^{2^n} - \left(\frac{x_o^2}{2}\right)^{2^{n+1}}| \to 0$ as n tends to ∞, because $x_0 \leq 1$. Hence (f, g) is asymptotically regular on X.*

Definition 1.11 ([11])**.** Let (X, d) be a cone metric space with cone P. Then the pair of selfmaps (f, g) of X is said to satisfy *property (E. A)* if there exists a sequence $\{x_n\}$ in X and a point z in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$.

Remark 1.12. *Asymptotic regularity and property (E. A) are independent of each other.*

Example 1.13. *Let $X = [0, \infty)$. E, P and d be as in Example 1.10. We define $f, g : X \to X$ by $f(x) = \frac{x}{2}$ and $g(x) = \frac{x}{3}$. Then the pair (f, g) satisfies property (E. A) with the sequence $\{x_n\}$ defined by $x_n = \frac{1}{n}$, $n = 1, 2, 3, \ldots$. But the pair (f, g) is not asymptotically regular on X. For, let $x_0 = \frac{1}{2}$. Since $f(X) \subseteq g(X)$, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n = fx_n = gx_{n+1} = \frac{1}{4}\left(\frac{3}{2}\right)^n$, $n = 0, 1, 2, \ldots$. Here we observe that $d(y_n, y_{n+1}) = \frac{1}{8}\left(\frac{3}{2}\right)^n e^t \to \infty$ as $n \to \infty$.*

Example 1.14. *Let E and P be as in Example 1.10. Let $X = \{x_1, x_2, x_3, \ldots\}$ where $x_n = \sum\limits_{i=1}^{n} \frac{1}{i}$. We define $d : X \times X \to E$ by $d(x, y) = |x - y|\varphi$, $\varphi(t) = e^t$, $t \geq 0$. We define $f, g : X \to X$ by $f(x_n) = x_{n+1}$ and $g(x_n) = x_n$, $n = 1, 2, 3, \ldots$. Then $f(X) \subseteq g(X)$ and so we construct a sequence $\{y_n\}$ in X such that $y_n = fx_n = gx_{n+1}$, $n = 0, 1, 2, \ldots$. Now, $d(y_n, y_{n+1}) = \frac{1}{n+1}\varphi \to 0$ as $n \to \infty$. Hence (f, g) is asymptotically regular on X. Here we observe that the pair (f, g) does not satisfy property (E. A), because for any sequence $\{x_n\}$ in X the sequences $\{fx_n\}$ and $\{gx_n\}$ diverge to ∞.*

Ćirić et al. [12] defined Ćirić type weak contractions on metric spaces and proved a common fixed point theorem for maps satisfying Ćirić type weak contraction.

Definition 1.15. Let (X, d) be a metric space and f, T be selfmaps of X. Then T is said to be a *Ćirić type f-weak contraction* if there exists a mapping $\varphi : R^+ \to R^+$ satisfying

(i) $\varphi(t) > 0$ for all $t > 0$,

(ii) $\lim_{s \to t^+} \varphi(s) > 0$ for all $t > 0$,

(iii) $t - \varphi(t)$ is non-decreasing,

(iv) $\lim_{t \to \infty} \varphi(t) = \infty$,

such that

$$d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y)) \ \forall\ x, y \in X$$

where $M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}$.

Theorem 1.16 ([12])**.** *Let K be a subset of a metric space (X, d) and let f and T be self mappings of K. Assume that $clT(K) \subseteq f(K)$, $clT(K)$ is complete and T is a Ćirić type f-weak contraction. Then T and f have a unique coincidence point in K. If, in addition, the pair (f, T) is weakly compatible then T and f have a unique common fixed point in K. Here $clT(K)$ denotes the closure of $T(K)$.*

In this paper, we prove common fixed point theorems for Ćirić type weak contractions in cone metric spaces where the underlying cone is neither regular nor normal, using asymptotic regularity (Theorem 2.1) and property (E. A) (Theorem 2.9). These results extend Theorem 1.6 to non-normal cones. Also, we prove a common fixed point theorem (Theorem 2.11) which improves Theorem 1.8. Supporting examples are provided to the results established in this paper.

2 Main Results

Theorem 2.1. *Let (X, d) be a cone metric space with solid cone P. Suppose that f and g are selfmaps of X such that $f(X) \subseteq g(X)$ and the pair (f, g) is asymptotically regular at some point $x_0 \in X$. Suppose that there exists a mapping $\varphi : P \to P$ satisfying φ is continuous, $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t \in P \setminus \{0\}$ such that*

$$d(fx, fy) \leq P(x, y) - \varphi(P(x, y)) \text{ for all } x, y \in X, \tag{2.1.1}$$

where $P(x, y) \in \{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}$. If either $f(X)$ or $g(X)$ is a complete subspace of X then f and g have a unique point of coincidence. Moreover, if the pair (f, g) is weakly compatible then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Since $f(X) \subseteq g(X)$ there exists $x_1 \in X$ such that $fx_0 = gx_1 = y_0$ (say). Having defined y_n, we define

$$y_{n+1} = fx_{n+1} = gx_{n+2}, \ n = 0, 1, 2, \ldots.$$

Since (f, g) is asymptotically regular at x_0, for each $c \in intP$ there exists $n_0 \in Z^+$ such that $d(y_n, y_{n+1}) \ll c \ \forall \ n \geq n_0$. We now show that $\{y_n\}$ is a Cauchy sequence in X. Fix a positive integer n such that $n \geq n_0$. We first show that

$$d(y_n, y_{n+p}) \ll c \ \text{ for each } c \in intP \text{ and } p = 1, 2, 3, \ldots. \tag{2.1.2}$$

By induction. Clearly (2.1.2) holds with $p = 1$. We suppose that (2.1.2) is true for some k. Hence, we have $d(y_n, y_{n+k}) \ll c$ for each $c \in intP$. Now consider,

$$\begin{aligned} d(y_n, y_{n+k+1}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+k+1}) \\ &= d(y_n, y_{n+1}) + d(fx_{n+1}, fx_{n+k+1}) \\ &\leq d(y_n, y_{n+1}) + P(x_{n+1}, x_{n+k+1}) - \varphi(P(x_{n+1}, x_{n+k+1})) \end{aligned} \tag{2.1.3}$$

where

$$\begin{aligned} P(x_{n+1}, x_{n+k+1}) \in \{&d(gx_{n+1}, gx_{n+k+1}), d(fx_{n+1}, gx_{n+1}), d(fx_{n+k+1}, gx_{n+k+1}), \\ &d(fx_{n+1}, gx_{n+k+1}), d(fx_{n+k+1}, gx_{n+1})\} \\ = \{&d(y_n, y_{n+k}), d(y_{n+1}, y_n), d(y_{n+k+1}, y_{n+k}), d(y_{n+1}, y_{n+k}), \\ &d(y_n, y_{n+k+1})\}. \end{aligned}$$

For infinitely many n, we get the following five cases:

Case (i): $P(x_{n+1}, x_{n+k+1}) = d(y_n, y_{n+k})$. Then (2.1.3) implies

$$\begin{aligned} d(y_n, y_{n+k+1}) &\leq d(y_n, y_{n+1}) + d(y_n, y_{n+k}) - \varphi(d(y_n, y_{n+k})) \\ &\leq d(y_n, y_{n+1}) + d(y_n, y_{n+k}) \\ &\ll \frac{c}{2} + \frac{c}{2} \\ &= c. \end{aligned}$$

Hence, $d(y_n, y_{n+k+1}) \ll c$ for each $c \in intP$.

Case (ii): $P(x_{n+1}, x_{n+k+1}) = d(y_n, y_{n+1})$. Now (2.1.3) implies

$$\begin{aligned} d(y_n, y_{n+k+1}) &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_n) - \varphi(d(y_{n+1}, y_n)) \\ &\le d(y_n, y_{n+1}) + d(y_n, y_{n+1}) \\ &\ll \frac{c}{2} + \frac{c}{2} \\ &= c. \end{aligned}$$

Therefore, $d(y_n, y_{n+k+1}) \ll c$ for each $c \in intP$.

Case (iii): $P(x_{n+1}, x_{n+k+1}) = d(y_{n+k+1}, y_{n+k})$. Then (2.1.3) implies

$$\begin{aligned} d(y_n, y_{n+k+1}) &\le d(y_n, y_{n+1}) + d(y_{n+k+1}, y_{n+k}) - \varphi(d(y_{n+k+1}, y_{n+k})) \\ &\le d(y_n, y_{n+1}) + d(y_{n+k+1}, y_{n+k}) \\ &\ll \frac{c}{2} + \frac{c}{2} \\ &= c. \end{aligned}$$

Therefore, $d(y_n, y_{n+k+1}) \ll c$ for each $c \in intP$.

Case (iv): $P(x_{n+1}, x_{n+k+1}) = d(y_{n+1}, y_{n+k})$. Therefore from (2.1.3), we get

$$\begin{aligned} d(y_n, y_{n+k+1}) &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+k}) - \varphi(d(y_{n+1}, y_{n+k})) \\ &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+k}) \\ &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_n) + d(y_n, y_{n+k}) \\ &\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} \\ &= c. \end{aligned}$$

Hence, $d(y_n, y_{n+k+1}) \ll c$ for each $c \in intP$.

Case (v): $P(x_{n+1}, x_{n+k+1}) = d(y_n, y_{n+k+1})$. Now (2.1.3) implies

$$d(y_n, y_{n+k+1}) \le d(y_n, y_{n+1}) + d(y_n, y_{n+k+1}) - \varphi(d(y_n, y_{n+k+1})).$$

This implies $\varphi(d(y_n, y_{n+k+1})) \le d(y_n, y_{n+1}) \ll c$. Hence $0 \le \varphi(d(y_n, y_{n+k+1})) \ll c$ for each $c \in intP$. Therefore, from Remark 1.4 (3), we get $\varphi(d(y_n, y_{n+k+1})) = 0$ which implies that $d(y_n, y_{n+k+1}) = 0$. Hence $d(y_n, y_{n+k+1}) \ll c$ for each $c \in intP$.

From all the above five cases, we conclude that $d(y_n, y_{n+k+1}) \ll c$ for each $c \in intP$. Hence (2.1.2) holds for each $p = 1, 2, 3, \ldots$. Therefore $\{y_n\}$ is a Cauchy sequence in X.

Suppose that $g(X)$ is a complete subspace of X. Hence there exists $z \in g(X)$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_{n+1} = z.$$

Since $z \in g(X)$, there exists $u \in X$ such that $z = gu$.

We show that

$$fu = z \tag{2.1.4}$$

Now,

$$\begin{aligned} d(fu,z) &\le d(fu,fx_n) + d(fx_n,z) \\ &\le P(u,x_n) - \varphi(P(u,x_n) + d(fx_n,z) \end{aligned} \tag{2.1.5}$$

where

$$\begin{aligned} P(u,x_n) \in \{&d(gu,gx_n), d(fu,gu), d(fx_n,gx_n), d(fu,gx_n), d(fx_n,gu)\} \\ = \{&d(z,gx_n), d(fu,z), d(fx_n,gx_n), d(fu,gx_n), d(z,fx_n)\}. \end{aligned}$$

For infinitely many n we get the following cases:

Case 1: $P(u,x_n) = d(z,gx_n)$. Then from (2.1.5), we get

$$\begin{aligned} d(fu,z) &\le d(z,gx_n) - \varphi(d(z,gx_n)) + d(fx_n,z) \\ &\le d(z,gx_n) + d(fx_n,z) \\ &\ll \frac{c}{2} + \frac{c}{2} \text{ for each } n \ge n_0 = n_0(c) \\ &= c. \end{aligned}$$

Hence $0 \le d(fu,z) \ll c$ for each $c \in intP$. Therefore $d(fu,z) = 0$.

Case 2: $P(u,x_n) = d(fu,z)$. Then from (2.1.5), we get

$$d(fu,z) \le d(fu,z) - \varphi(d(fu,z)) + d(fx_n,z).$$

This implies $\varphi(d(fu,z)) \le d(fx_n,z) \ll c$ for each $n \ge n_0 = n_0(c)$ Hence $0 \le d(fu,z) \ll c$ for each $c \in intP$. Therefore $\varphi(d(fu,z)) = 0$ so that $d(fu,z) = 0$.

Case 3: $P(u,x_n) = d(fx_n,gx_n)$. Then from (2.1.5), we get

$$\begin{aligned} d(fu,z) &\le d(fx_n,gx_n) - \varphi(d(fx_n,gx_n)) + d(fx_n,z) \\ &\le d(fx_n,gx_n) + d(fx_n,z) \\ &\le d(fx_n,z) + d(z,gx_n) + d(fx_n,z) \\ &\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} \text{ for each } n \ge n_0 = n_0(c) \\ &= c. \end{aligned}$$

Hence $0 \le d(fu,z) \ll c$ for each $c \in intP$. Therefore $d(fu,z) = 0$.

Case 4: $P(u,x_n) = d(fu,gx_n)$. Then from (2.1.5), we get

$$\begin{aligned} d(fu,z) &\le d(fu,gx_n) - \varphi(d(fu,gx_n)) + d(fx_n,z) \\ &\le d(fu,z) + d(z,gx_n) - \varphi(d(fu,gx_n)) + d(fx_n,z). \end{aligned}$$

This implies,

$$\begin{aligned}\varphi(d(fu, gx_n)) &\leq d(z, gx_n) + d(fx_n, z)\\ &\ll \frac{c}{2} + \frac{c}{2} \text{ for each } n \geq n_0 = n_0(c)\\ &= c.\end{aligned}$$

Therefore for each $c \in intP$ there exists $n_0 = n_0(c) \in Z^+$ such that $\varphi(d(fu, gx_n)) \ll c$ for all $n \geq n_0$, *i.e.,* $\lim_{n\to\infty} \varphi(d(fu, gx_n)) = 0$. Since φ is continuous, we get $\lim_{n\to\infty} d(fu, gx_n) = 0$. Hence, $\lim_{n\to\infty} gx_n = fu$. Therefore, $fu = z$.

Case 5: $P(u, x_n) = d(z, fx_n)$. Then from (2.1.5), we get

$$\begin{aligned}d(fu, z) &\leq d(z, fx_n) - \varphi(d(z, fx_n)) + d(fx_n, z)\\ &\leq d(z, fx_n) + d(fx_n, z)\\ &\ll \frac{c}{2} + \frac{c}{2} \text{ for each } n \geq n_0 = n_0(c)\\ &= c.\end{aligned}$$

Hence $0 \leq d(fu, z) \ll c$ for each $c \in intP$. Therefore $d(fu, z) = 0$.

Hence from all the above cases we get $fu = z$ so that (2.1.4) holds. Therefore, $fu = gu = z$. That is z is a point of coincidence of f and g. We now show that this z is unique. Now suppose that there exist $u', z' \in X$ such that $fu' = gu' = z'$. Then

$$d(z, z') = d(fu, fu') \leq P(u, u') - \varphi(P(u, u')),$$

where

$$\begin{aligned}P(u, u') &= \{d(gu, gu'), d(fu, gu), d(fu', gu'), d(fu, gu'), d(fu', gu)\}\\ &= \{d(z, z'), 0, 0, d(z, z'), d(z, z')\}\\ &= \{0, d(z, z')\}.\end{aligned}$$

If $P(u, u') = 0$ then $d(z, z') = 0$ so that $z = z'$. If $P(u, u') = d(z, z')$ then $d(z, z') \leq d(z, z') - \varphi(d(z, z'))$. Hence $\varphi(d(z, z')) = 0$ which implies that $d(z, z') = 0$. Therefore z is the unique point of coincidence of f and g.

If (f, g) is weakly compatible then $fz = fgu = gfu = gz = w$ (say). That is w is a point of coincidence of f and g. But since z is a unique point of coincidence of f and g, we have $w = z$. Hence $fz = gz = z$. Therefore z is a unique common fixed point of f and g.

Now, if $f(X)$ is complete, since $f(X) \subseteq g(X)$ there exists $z \in g(X)$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z$, and the proof in this case is similar as above. □

The following results follow as corollaries to Theorem 2.1.

Corollary 2.2. *Let (X,d) be a cone metric space with solid cone P. Suppose that f and g are selfmaps of X such that $f(X)\subseteq g(X)$ and the pair (f,g) is asymptotically regular at some point $x_0\in X$. Suppose that there exists a mapping $\varphi:P\to P$ satisfying φ is continuous, $\varphi(0)=0$ and $\varphi(t)>0$ for $t\in P\setminus\{0\}$ such that*

$$d(fx,fy)\leq d(gx,gy)-\varphi(d(gx,gy)) \text{ for each } x,y\in X.$$

If either $f(X)$ or $g(X)$ is a complete subspace of X then f and g have a unique point of coincidence. Moreover, if the pair (f,g) is weakly compatible then f and g have a unique common fixed point in X.

Corollary 2.3. *Let (X,d) be a complete cone metric space with solid cone P. Suppose that f is a selfmap of X which is asymptotically regular at some point $x_0\in X$. Suppose that there exists a mapping $\varphi:P\to P$ satisfying φ is continuous, $\varphi(0)=0$ and $\varphi(t)>0$ for $t\in P\setminus\{0\}$ such that*

$$d(fx,fy)\leq P(x,y)-\varphi(P(x,y)) \text{ for all } x,y\in X, \tag{2.3.1}$$

where $P(x,y)\in\{d(x,y),d(x,fx),d(y,fy),d(x,fy),d(y,fx)\}$. Then f has a unique fixed point in X.

Proof. The conclusion of this corollary follows by taking $g=I_X$, the identity map on X, in Theorem 2.1. □

Corollary 2.4. *Let (X,d) be a complete cone metric space with solid cone P. Suppose that f is a selfmap of X which is asymptotically regular at some point $x_0\in X$. Suppose that there exists a mapping $\varphi:P\to P$ satisfying φ is continuous, $\varphi(0)=0$ and $\varphi(t)>0$ for $t\in P\setminus\{0\}$ such that*

$$d(fx,fy)\leq d(x,y)-\varphi(d(x,y)) \text{ for all } x,y\in X, \tag{2.4.1}$$

Then f has a unique fixed point in X.

Proof. By taking $u=d(x,y)$, in Corollary 2.3, the conclusion of this corollary follows. □

Remark 2.5. *Corollary* 2.4 *extends Theorem* 1.6 *to non-normal cones. Also, properties* (ii), (iii) *and monotone increasing property of φ in Theorem* 1.6 *can be relaxed.*

Example 2.6. *Let $E=C'_R[0,1]$ with $\|x\|=\|x\|_\infty+\|x'\|_\infty$ and $P=\{x\in E/x\geq 0\}$. Then P is a non-normal solid cone. Let $X=[0,1]$. We define $d:X\times X\to E$ by $d(x,y)=|x-y|\psi$, $\psi(t)=e^t$, $t\geq 0$. We define $f:X\to X$ by*

$$fx=\begin{cases}\frac{x^2}{4}, & if \quad x\in[0,\ \frac{1}{2}]\\ \frac{x}{2}, & if \quad x\in(\frac{1}{2},\ 1].\end{cases}$$

Then f is asymptotically regular at each point on X. We define $\varphi : P \to P$ by $\varphi(x(t)) = \frac{x(t)}{4}$, $x(t) \geq 0$. We now show that f satisfies (2.3.1).

Case (i): $x,\ y \in [0, \frac{1}{2}]$. *In this case,* $d(fx, fy) = \frac{1}{4}|x^2 - y^2|\psi$ *and* $d(x, y) = |x - y|\psi$. *Now*

$$\begin{aligned} d(fx, fy) &= \frac{1}{4}|x^2 - y^2|\psi = \frac{1}{4}|x + y||x - y|\psi \\ &< \frac{3}{4}|x - y|\psi \\ &= |x - y|\psi - \varphi(|x - y|\psi) \\ &= d(x, y) - \varphi(d(x, y)). \end{aligned}$$

Case (ii): $x \in [0, \frac{1}{2}],\ y \in (\frac{1}{2}, 1]$. *In this case* $d(fx, fy) = |\frac{x^2}{4} - \frac{y}{2}|\psi$ *and* $d(y, fx) = (y - \frac{x^2}{4})\psi$. *Now*

$$\begin{aligned} d(fx, fy) &= |\frac{x^2}{4} - \frac{y}{2}|\psi = \frac{1}{2}(y - \frac{x^2}{2})\psi \leq \frac{1}{2}(y - \frac{x^2}{4})\psi \\ &< \frac{3}{4}(y - \frac{x^2}{2})\psi \\ &= d(y, fx) - \varphi(d(y, fx)) \end{aligned}$$

Case (iii): $x,\ y \in (\frac{1}{2}, 1]$. *In this case* $d(fx, fy) = \frac{1}{2}|x - y|\psi$ *and* $d(x, y) = |x - y|\psi$. *Now*

$$\begin{aligned} d(fx, fy) &= \frac{1}{2}|x - y|\psi < \frac{3}{4}|x - y|\psi = |x - y|\psi - \varphi(|x - y|\psi) \\ &= d(x, y) - \varphi(d(x, y)). \end{aligned}$$

Hence from all the above cases, it is clear that (2.3.1) *holds and 0 is the unique fixed point of f. Further, we observe that when $x = \frac{1}{2}$ and $y = \frac{3}{4}$, $d(fx, fy) = \frac{5}{16}$ and $d(x, y) = \frac{1}{4}$ so that* (1.6.1) *does not hold for any φ. Hence Theorem* 1.6 *is not applicable.*

One more example in this direction is the following.

Example 2.7. *Let E, P, X and d be as in Example* 2.6. *We define $f, g : X \to X$ and $\varphi : P \to P$ by*

$$fx = \begin{cases} \frac{x}{4}, & if \quad x \in [0, 1) \\ \frac{1}{5}, & if \quad x = 1. \end{cases} \qquad and \quad \varphi(x(t)) = \frac{x(t)^2}{\psi(t) + x(t)},\ x(t) \geq 0.$$

Then f is asymptotically regular at each point on X. We now verify the inequality (2.3.1). *Since* (2.3.1) *is symmetric in x, y we consider only the following cases.*

Case (i): $x,\ y \in [0,1)$. *In this case* $d(fx,fy) = \frac{1}{4}|x-y|\psi$ *and* $d(x,y) = |x-y|\psi$ *so that*

$$d(fx,fy) = \frac{1}{4}|x-y|\psi < |x-y|\psi - \frac{(|x-y|\psi)^2}{\psi + |x-y|\psi} = d(x,y) - \varphi(d(x,y)).$$

Case (ii): $x \in [0,1),\ y = 1$. *In this case* $d(fx,fy) = |\frac{x}{4} - \frac{1}{5}|\psi$ *and* $d(y,fy) = \frac{4}{5}$. *Therefore,*

$$d(fx,fy) = |\frac{x}{4} - \frac{1}{5}|\psi \leq \frac{4}{5} - \frac{(\frac{4}{5})^2}{\psi + \frac{4}{5}\psi} = d(y,fy) - \varphi(d(y,fy)).$$

Case (iii): $x = 1,\ y = 1$. *In this case* $d(fx,fy) = 0$ *so that* (2.3.1) *holds.*

Hence all the hypotheses of Corollary 2.3 *hold and 0 is the unique fixed point of* f. *Further, with* $x = \frac{24}{25}$ *and* $y = 1$, *we get* $d(fx,fy) = \frac{1}{25}$ *and* $d(x,y) = \frac{1}{25}$ *so that* (1.6.1) *does not hold for any* φ. *Hence Theorem* 1.6 *is not applicable.*

The following is an example in support of Theorem 2.1.

Example 2.8. *Let* E, P *and* d *be as in Example* 2.6. *Let* $X = (0,1]$. *We define* $f, g : X \to X$ *and* $\varphi : P \to P$ *by*

$$fx = \begin{cases} \frac{2}{5}, & if \quad x \in (0,\ \frac{2}{3}) \\ 1 - \frac{x}{2}, & if \quad x \in [\frac{2}{3}, 1) \\ \frac{23}{30}, & if \quad x = 1. \end{cases} \quad and \quad gx = \begin{cases} \frac{23}{30}, & if \quad x \in (0,\ \frac{2}{3}) \\ \frac{4}{3} - x, & if \quad x \in [\frac{2}{3}, 1]. \end{cases}$$

and $\varphi(x(t)) = \frac{x(t)^2}{4\psi(t)},\ x(t) \geq 0$. *Here* $f(X) = \{\frac{2}{5}, \frac{23}{30}\} \cup (\frac{1}{2}, \frac{2}{3}]$ *and* $g(X) = \{\frac{23}{30}\} \cup [\frac{1}{3}, \frac{2}{3}]$ *so that* $f(X) \subset g(X)$ *and* $g(X)$ *is complete subspace of* X. *We now verify the inequality* (2.1.1). *Since* (2.1.1) *is symmetric in* x, y *we consider only the following cases.*

Case (i): $x,\ y \in (0, \frac{2}{3})$. *In this case* $d(fx,fy) = 0$ *so that* (2.1.1) *holds obviously.*

Case (ii): $x \in (0, \frac{2}{3}),\ y \in [\frac{2}{3}, 1)$. *In this case* $d(fx,fy) = (\frac{3}{5} - \frac{y}{2})\psi$ *and* $d(fx,gx) = \frac{11}{30}\psi$. *Now*

$$d(fx,fy) = (\frac{3}{5} - \frac{y}{2})\psi < \frac{11}{30}\psi - \frac{(\frac{11}{30}\psi)^2}{4\psi} = d(fx,gx) - \varphi(d(fx,gx)).$$

Case (iii): $x \in (0, \frac{2}{3}),\ y = 1$. *In this case* $d(fx,fy) = \frac{11}{30}\psi$ *and* $d(gx,gy) = \frac{13}{30}\psi$. *Now*

$$d(fx,fy) = \frac{11}{30}\psi \leq \frac{13}{30}\psi - \frac{(\frac{13}{30}\psi)^2}{4\psi} = d(gx,gy) - \varphi(d(gx,gy)).$$

Case (iv): $x,\ y \in [\frac{2}{3}, 1)$. *In this case* $d(fx, fy) = \frac{1}{2}|x-y|\psi$ *and* $d(gx, gy) = |x-y|\psi$ *so that*

$$d(fx, fy) = \frac{1}{2}|x-y|\psi < |x-y|\psi - \frac{(|x-y|\psi)^2}{4\psi} = d(gx, gy) - \varphi(d(gx, gy)).$$

Case (v): $x \in [\frac{2}{3}, 1),\ y = 1$. *In this case* $d(fx, fy) = (\frac{x}{2} - \frac{7}{30})\psi$ *and* $d(fy, gy) = \frac{13}{30}\psi$. *Now*

$$d(fx, fy) = (\frac{x}{2} - \frac{7}{30})\psi < \frac{13}{30}\psi - \frac{(\frac{13}{30}\psi)^2}{4\psi} = d(fy, gy) - \varphi(d(fy, gy)).$$

Case (vi): $x = 1,\ y = 1$. *In this case* $d(fx, fy) = 0$ *so that* (2.1.1) *holds obviously.*

Hence all the hypotheses of Theorem 2.1 *hold and* $\frac{2}{3}$ *is the unique common fixed point of* f *and* g.

We now relax the containment relation $f(X) \subseteq g(X)$ of range spaces and asymptotic regularity of the pair (f, g) in Theorem 2.1 and by imposing property (E. A) and prove the following theorem.

Theorem 2.9. *Let* (X, d) *be a cone metric space with solid cone* P. *Suppose that* f *and* g *are selfmaps of* X *satisfying* (2.1.1) *and suppose that the pair* (f, g) *satisfies property* (*E. A*). *If* $g(X)$ *is closed subspace of* X *then* f *and* g *have a unique point of coincidence. Moreover, if the pair* (f, g) *is weakly compatible then* f *and* g *have a unique common fixed point in* X.

Proof. Since the pair (f, g) satisfies property (E. A), there exists a sequence $\{x_n\}$ in X and a point z in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$. Since $g(X)$ is a closed subspace of X there exists u in X such that $gu = z$. We show that $fu = z$. From here onwards we proceed on the similar lines from (2.1.4) as in the proof of Theorem 2.1, and the conclusion follows. ☐

The following is an example in support of Theorem 2.9.

Example 2.10. *Let* E, P, X *and* d *be as in Example* 2.8. *We define* $f, g : X \to X$ *and* $\varphi : P \to P$ *by*

$$fx = \begin{cases} \frac{2}{3}, & if \ \ x \in (0,\ \frac{3}{4}),\ x = 1 \\ 1 - \frac{x}{3}, & if \ \ x \in [\frac{3}{4}, 1). \end{cases} \quad and \ \ gx = \begin{cases} 1, & if \ \ x \in (0,\ \frac{3}{4}) \\ x, & if \ \ x \in [\frac{3}{4}, 1]. \end{cases}$$

and $\varphi(x(t)) = \frac{x(t)^2}{2\psi(t)},\ x(t) \geq 0$. *Here* $f(X) = (\frac{1}{3}, \frac{3}{4})$ *and* $g(X) = \{\frac{5}{6}\} \cup [\frac{3}{4}, 1]$. *so that neither* $f(X) \subseteq g(X)$ *nor* $g(X) \subseteq f(X)$. *Also* $g(X)$ *is closed subspace of* X. *The pair* (f, g) *satisfies property (E. A) with the sequence* $\{x_n\}$ *defined by* $x_n = \frac{3}{4} + \frac{1}{n},\ n = 1, 2, 3, \ldots$. *We now verify the inequality* (2.1.1). *Since* (2.1.1) *is symmetric in* x, y *we consider only the following cases.*

Case (i): $x, y \in (0, \frac{3}{4})$. *In this case* $d(fx, fy) = 0$ *so that* (2.1.1) *holds obviously.*

Case (ii): $x \in (0, \frac{3}{4}),\ y \in [\frac{3}{4}, 1)$. *In this case* $d(fx, fy) = \frac{1}{3}|1-y|\psi$ *and* $d(gx, gy) = |1-y|\psi$ *so that*

$$d(fx, fy) = \frac{1}{3}|1-y|\psi < |1-y|\psi - \frac{(|1-y|\psi)^2}{2\psi} = d(gx, gy) - \varphi(d(gx, gy)).$$

Case (iii): $x \in (0, \frac{3}{4}),\ y = 1$. *In this case* $d(fx, fy) = 0$ *so that* (2.1.1) *holds obviously.*

Case (iv): $x, y \in [\frac{3}{4}, 1)$. *In this case* $d(fx, fy) = \frac{1}{3}|x-y|\psi$ *and* $d(gx, gy) = |x-y|\psi$ *so that*

$$d(fx, fy) = \frac{1}{3}|x-y|\psi < |x-y|\psi - \frac{(|x-y|\psi)^2}{2\psi} = d(gx, gy) - \varphi(d(gx, gy)).$$

Case (v): $x \in [\frac{3}{4}, 1),\ y = 1$. *In this case* $d(fx, fy) = \frac{1}{3}|1-x|\psi$ *and* $d(gx, gy) = |1-x|\psi$ *so that*

$$d(fx, fy) = \frac{1}{3}|1-x|\psi < |1-x|\psi - \frac{(|1-x|\psi)^2}{2\psi} = d(gx, gy) - \varphi(d(gx, gy)).$$

Case (vi): $x = 1,\ y = 1$. *In this case* $d(fx, fy) = 0$ *so that* (2.1.1) *holds obviously.*

Hence all the hypotheses of Theorem 2.9 *hold and* $\frac{3}{4}$ *is the unique common fixed point of* f *and* g.

Theorem 2.11. *Let* (X, d) *be a cone metric space with solid cone* P. *Suppose that* f *and* g *are selfmaps of* X *and suppose that the pair* (f, g) *satisfies property (E. A). Suppose that there exists a mapping* $\varphi : P \to P$ *satisfying* $\varphi(0) = 0$ *and* $0 < \varphi(t) < t$ *for* $t \in P \setminus \{0\}$ *such that*

$$d(fx, fy) \leq \varphi(P(x, y)) \ \text{for all } x, y \in X, \tag{2.11.1}$$

where $P(x, y) \in \{d(gx, gy), d(fx, gx), d(fy, gy), \frac{1}{2}[d(fx, gy) + d(fy, gx)]\}$. *If either* $f(X)$ *or* $g(X)$ *is a complete subspace of* X *then* f *and* g *have a unique point of coincidence. Moreover, if the pair* (f, g) *is weakly compatible then* f *and* g *have a unique common fixed point in* X.

Proof. Since the pair (f, g) satisfies property (E. A), there exists a sequence $\{x_n\}$ in X and a point z in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$. Since $g(X)$ is a closed subspace of X there exists u in X such that $gu = z$. We show that $fu = z$. Consider

$$\begin{aligned} d(fu, z) &\leq d(fu, fx_n) + d(fx_n, z) \\ &\leq \varphi(P(u, x_n) + d(fx_n, z) \end{aligned} \tag{2.11.2}$$

where

$$P(u, x_n) \in \left\{ d(gu, gx_n), d(fu, gu), d(fx_n, gx_n), \frac{1}{2}[d(fu, gx_n) + d(fx_n, gu)] \right\}$$
$$= \left\{ d(z, gx_n), d(fu, z), d(fx_n, gx_n), \frac{1}{2}[d(fu, gx_n) + d(z, fx_n)] \right\}.$$

For infinitely many n we get the following cases:

Case 1: $P(u, x_n) = d(z, gx_n)$. Then from (2.11.2), we get

$$\begin{aligned} d(fu, z) &\leq \varphi(d(z, gx_n)) + d(fx_n, z) \\ &< d(z, gx_n) + d(fx_n, z) \\ &\ll \frac{c}{2} + \frac{c}{2} \text{ for each } n \geq n_0 = n_0(c) \\ &= c. \end{aligned}$$

Hence $0 \leq d(fu, z) \ll c$ for each $c \in intP$. Therefore $d(fu, z) = 0$.

Case 2: $P(u, x_n) = d(fu, z)$. Now from (2.11.2), we get

$$\begin{aligned} d(fu, z) &\leq \varphi(d(fu, z)) + d(fx_n, z) \\ &\ll \varphi(d(fu, z)) + c \text{ for each } n \geq n_0 = n_0(c). \end{aligned}$$

Hence $0 \leq d(fu, z) \ll \varphi(d(fu, z)) + c$ for each $c \in intP$. Therefore $d(fu, z) \leq \varphi(d(fu, z)) < d(fu, z)$, a contradiction.

Case 3: $P(u, x_n) = d(fx_n, gx_n)$. Then from (2.11.2), we get

$$\begin{aligned} d(fu, z) &\leq \varphi(d(fx_n, gx_n)) + d(fx_n, z) \\ &< d(fx_n, gx_n) + d(fx_n, z) \\ &\leq d(fx_n, z) + d(z, gx_n) + d(fx_n, z) \\ &\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} \text{ for each } n \geq n_0 = n_0(c) \\ &= c. \end{aligned}$$

Hence $0 \leq d(fu, z) \ll c$ for each $c \in intP$. Therefore $d(fu, z) = 0$.

Case 4: $P(u, x_n) = \frac{1}{2}[d(fu, gx_n) + d(z, fx_n)]$. Now from (2.11.2), we get

$$\begin{aligned} d(fu, z) &\leq \varphi(\frac{1}{2}[d(fu, gx_n) + d(z, fx_n)]) + d(fx_n, z) \\ &< \frac{1}{2}[d(fu, gx_n) + d(z, fx_n)] + d(fx_n, z) \\ &\leq \frac{1}{2}[d(fu, z) + d(z, gx_n) + d(z, fx_n)] + d(fx_n, z). \end{aligned}$$

This implies,

$$\begin{aligned} d(fu,z) &\leq d(z,gx_n) + 3d(fx_n,z) \\ &\ll \frac{c}{2} + 3\frac{c}{6} \text{ for each } n \geq n_0 = n_0(c) \\ &= c. \end{aligned}$$

Hence $0 \leq d(fu,z) \ll c$ for each $c \in intP$. Therefore $d(fu,z) = 0$.

Hence from all the above cases we get $fu = z$. Therefore, $fu = gu = z$. That is z is a point of coincidence of f and g. We now show that this z is unique. Now suppose that there exist $u', z' \in X$ such that $fu' = gu' = z'$. Suppose $z \neq z'$. Then

$$d(z,z') = d(fu,fu') \leq \varphi(P(u,u')),$$

where

$$\begin{aligned} P(u,u') &= \left\{ d(gu,gu'), d(fu,gu), d(fu',gu'), \frac{1}{2}[d(fu,gu') + d(fu',gu)] \right\} \\ &= \{d(z,z'), 0, 0, d(z,z')\} \\ &= \{0, d(z,z')\}. \end{aligned}$$

If $P(u,u') = 0$ then $d(z,z') = 0$ so that $z = z'$. If $P(u,u') = d(z,z')$ then $d(z,z') \leq \varphi(d(z,z')) < d(z,z')$, a contradiction. Therefore z is the unique point of coincidence of f and g.

If (f,g) is weakly compatible then $fz = fgu = gfu = gz = w$ (*say*). That is w is a point of coincidence of f and g. But since z is a unique point of coincidence of f and g, we have $w = z$. Hence $fz = gz = z$. Therefore z is a unique common fixed point of f and g. □

Property (E. A) of the pair (f,g) of Theorem 2.11, relaxes the containment $f(X) \subseteq g(X)$ of range spaces and properties (i), (iii) and (iv) of φ in Theorem 1.8.

Example 2.12. *Let E, P, X, d and f, g be as in Example* 2.10. *We define $\varphi : P \to P$ by $\varphi(x(t)) = \frac{x(t)(2\psi(t)+x(t))}{2(\psi(t)+x(t))}$, $x(t) \geq 0$. Clearly $\varphi(0) = 0$ and $0 < \varphi(x(t)) < x(t)$ for each $x(t) \in P \setminus \{0\}$. With this φ, f and g satisfy all the hypotheses of Theorem* 2.11 *and $\frac{3}{4}$ is the unique common fixed point of f and g.*

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