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Three Solutions for a Class of $(p_1, ..., p_n)$-Biharmonic Systems via Variational Methods[1]

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Abstract : In this paper, we prove the existence of at least three weak solutions to a boundary value problem for a system of n coupled equations involving $(p_1, ..., p_n)$-biharmonic Laplacians. We use a variational approach based on a three critical points theorem due to Ricceri [B. Ricceri, A three critical points theorem revisited, Nonlinear Anal. 70 (2009) 3084–3089].

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1 Introduction

In this work, we study the existence of at least three weak solutions for the nonlinear elliptic equation of $(p_1, ..., p_n)$-biharmonic type under Navier boundary conditions:

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) = \lambda F_{u_i}(x, u_1, ..., u_n) + \mu G_{u_i}(x, u_1, ..., u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

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for $1 \leq i \leq n$, where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$, $p_i > \max\{1, \frac{N}{2}\}$ for $1 \leq i \leq n$, $\lambda, \mu > 0$, $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a function such that $F(\cdot, t_1, ..., t_n)$ is continuous in Ω for all $(t_1, ..., t_n) \in \mathbb{R}^n$, $F(x, \cdot, ..., \cdot)$ is a C^1 in $\mathbb{R}^n$ for every $x \in \Omega$ and $F(x, 0, ..., 0) = 0$ for all $x \in \Omega$, and $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function with respect to x in Ω for every $(t_1, ..., t_n) \in \mathbb{R}^n$, and is a C^1-function with respect to $(t_1, ..., t_n) \in \mathbb{R}^n$ for every x in Ω and satisfies the condition

$$\sup_{|(t_1,...,t_n)| \leq s} \sum_{i=1}^{n} |G_{t_i}(x, t_1, ..., t_n)| \leq h_s(x) \tag{1.2}$$

for all $s > 0$ and some $h_s \in L^1$ with $G(\cdot, 0, ..., 0) \in L^1$, and F_t and G_t denote the partial derivative of F and G with respect to t, respectively.

Here and in the next section, X will denote the Cartesian product of n Sobolev space $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega)$ for $i = 1, ..., n$, i.e., $X = W^{2,p_1}(\Omega) \cap W_0^{1,p_1}(\Omega) \times ... \times W^{2,p_n}(\Omega) \cap W_0^{1,p_n}(\Omega)$ endowed with the norm

$$||\ (u_1, ..., u_n)\ || = \sum_{i=1}^{n} ||u_i||_{p_i}$$

where

$$||u_i||_{p_i} = \left(\int_\Omega |\Delta u_i(x)|^{p_i} dx \right)^{1/p_i}$$

for $1 \leq i \leq n$.

In this paper, precisely we deal with the existence of an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q with the following property: for every $\lambda \in \Lambda$ and an arbitrary function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ measurable in Ω for all $(t_1, ..., t_n) \in \mathbb{R}^n$ and C^1 in R^n for every $x \in \Omega$ satisfying (1.2), there is a $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the system (1.1) admits at least three weak solutions in X whose norms are less than q. Our main result is Theorem 2.5, which provides intervals for the parameters such that if the parameters belong to those intervals, the corresponding system has at least three solutions satisfying some boundedness properties. Consequences of this result, examples and a detailed discussion on systems over 1-dimensional domains are given.

We say that $u = (u_1, ..., u_n)$ is a weak solution to (1.1) if $u = (u_1, ..., u_n) \in X$ and

$$\int_\Omega \sum_{i=1}^{n} |\Delta u_i(x)|^{p_i - 2} \Delta u_i(x) \Delta v_i(x) dx - \lambda \int_\Omega \sum_{i=1}^{n} F_{u_i}(x, u_1(x), ..., u_n(x)) v_i(x) dx$$

$$-\mu \int_\Omega \sum_{i=1}^{n} G_{u_i}(x, u_1(x), ..., u_n(x)) v_i(x) dx = 0$$

for every $(v_1, ..., v_n) \in X$.

There seems to be increasing interest in studying fourth-order boundary value problems, because the static form change of beam or the sport of rigid body can be described by a fourth-order equation, and specially a model to study travelling waves in suspension bridges can be furnished by the fourth-order equation of nonlinearity, so it is important to Physics. More general nonlinear fourth-order elliptic boundary value problems have been studied [1–12]. Very recently, Li and Tang, in an interesting paper [13] (also see [14]), employing Ricceri's three critical points theorem [15] investigated the system (1.1) in the case $n = 2$, establishing the the existence of an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q with the following property: for every $\lambda \in \Lambda$ and two Carathéodory functions $G_{u_1}, G_{u_2} : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\sup_{|s|\leq\zeta,|t|\leq\zeta} (|G_{u_1}(.,t,s)| + |G_{u_2}(.,t,s)|) \in L^1(\Omega)$$

for all $\zeta > 0$, there is a $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the system (1.1), in the case $n = 2$, admits at least three weak solutions in $W^{2,p_1}(\Omega) \cap W_0^{1,p_1}(\Omega) \times W^{2,p_2}(\Omega) \cap W_0^{1,p_2}(\Omega)$ whose norms are less than q.

Here, as in [13], our main tool is Ricceri's three critical points theorem; see Theorem 1 in the next section. We also recall that, again applying Ricceri's three critical points [15, 16] theorem, elliptic systems have been studied in [17–21]. The aim of the present paper is to extend the main result of [13] to the general case.

2 Main results

Our analysis is based on the following three critical points theorem (see also [22–24] for related results) to transfer the existence of three solutions of the system (1.1) into the existence of critical points of the Euler functional.

Theorem 2.1 ([15]). *Let X be a reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, $\Phi : X \longrightarrow \mathbb{R}$ a sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose derivative admits a continuous inverse on X^* and $J : X \longrightarrow \mathbb{R}$ a C^1 functional with compact derivative.*
Assume that

$$\lim_{||x||\to+\infty} (\Phi(x) + \lambda J(x)) = +\infty$$

for all $\lambda \in I$, and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\in I} \inf_{x\in X} (\Phi(x) + \lambda(J(x) + \rho)) < \inf_{x\in X} \sup_{\lambda\in I} (\Phi(x) + \lambda(J(x) + \rho)).$$

Then, there exist a non-empty open set interval $A \subseteq I$ and a positive real number q with the following property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \longrightarrow \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) + \lambda J'(u) + \mu\Psi'(u) = 0$$

has at least three solutions in X whose norms are less than q.

For using later, we also recall the following result, Proposition 3.1 of [22] with Ψ replaced by $-J$, precisely to show the minimax inequality in Theorem 2.1.

Proposition 2.2 ([22]). *Let X be a non-empty set and Φ and J two real function on X. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = J(x_0) = 0, \ \Phi(x_1) > r,$$

$$\sup_{x\in\Phi^{-1}(]-\infty,r])} J(x) < r\frac{J(x_1)}{\Phi(x_1)}.$$

Then, for each ρ satisfying

$$\sup_{x\in\Phi^{-1}(]-\infty,r])} J(x) < \rho < r\frac{J(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x) + \lambda(\rho - J(x))) < \inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x) + \lambda(\rho - J(x))).$$

Put

$$k = \max\left\{\sup_{u_i\in W^{2,p_1}(\Omega)\cap W_0^{1,p_1}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|u_i(x)|^{p_i}}{||u_i||_{p_i}^{p_i}}; \ \text{ for } 1 \leq i \leq n\right\}. \quad (2.1)$$

Since $p_i > \max\{1, \frac{N}{2}\}$ for $1 \leq i \leq n$, and the embedding $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega) \hookrightarrow C^0(\Omega)$ for $1 \leq i \leq n$ is compact, one has $k < +\infty$.

Now, fix $x^0 \in \Omega$ and pick r_1, r_2 with $o < r_1 < r_2$ such that

$$S(x^0, r_1) \subset S(x^0, r_2) \subseteq \Omega$$

where $S(x^0, r_i)$ denote the ball with center at x^0 and radius of r_i for $i = 1, ..., n$. Put

$$\sigma_i = \sigma_i(N, p_i, r_1, r_2) = \frac{12(N+2)^2(r_1+r_2)}{(r_2-r_1)^3}\left(\frac{k\pi^{\frac{N}{2}}(r_2^N - r_1^N)}{\Gamma(1+\frac{N}{2})}\right)^{1/p_i} \ \text{ for } 1 \leq i \leq n \quad (2.2)$$

and

$$\theta_i = \theta_i(N, p_i, r_1, r_2) = \begin{cases} \frac{3N}{(r_2-r_1)(r_1+r_2)}\left(\frac{k\pi^{\frac{N}{2}}((r_1+r_2)^N-(2r_1)^N)}{2^N\Gamma(1+\frac{N}{2})}\right)^{1/p_i} & \text{if } N < \frac{4r_1}{r_2-r_1}, \\ \frac{12r_1}{(r_2-r_1)^2(r_1+r_2)}\left(\frac{k\pi^{\frac{N}{2}}((r_1+r_2)^N-(2r_1)^N)}{2^N\Gamma(1+\frac{N}{2})}\right)^{1/p_i} & \text{if } N \geq \frac{4r_1}{r_2-r_1} \end{cases} \quad (2.3)$$

where $\Gamma(.)$ is the Gamma function.

For all $\gamma > 0$ we denote by $K(\gamma)$ the set

$$\left\{(t_1, ..., t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \le \gamma\right\}. \tag{2.4}$$

Our main results fully depend on the following technical lemma:

Lemma 2.3. *Assume that there exist two positive constants c and d with $\sum_{i=1}^{n} \frac{(d\theta_i)^{p_i}}{p_i} > \frac{c}{\prod_{i=1}^{n} p_i}$ such that*

(A_1) $F(x, t_1, ..., t_n) \ge 0$ *for each* $(x, t_1, ..., t_n) \in (\Omega \setminus S(x^0, r_1)) \times [0, d] \times \cdots \times [0, d]$;

(A_2)

$$\sum_{i=1}^{n} \frac{(d\sigma_i)^{p_i}}{p_i} \int_\Omega \sup_{(t_1,...,t_n) \in K\left(\frac{c}{\prod_{i=1}^{n} p_i}\right)} F(x, t_1, ..., t_n)dx < \frac{c}{\prod_{i=1}^{n} p_i} \int_{S(x^0, r_1)} F(x, d, ..., d)dx$$

where σ_i and θ_i are given by (2.2) and (2.3), respectively, and $K(\frac{c}{\prod_{i=1}^{n} p_i}) = \{(t_1, ..., t_n) | \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \le \frac{c}{\prod_{i=1}^{n} p_i}\}$ (see (2.4)).

Then, there exist $r > 0$ and $w = (w_1, ..., w_n) \in X$ such that $\sum_{i=1}^{n} \frac{||w_i||_{p_i}^{p_i}}{p_i} > r$ and

$$\int_\Omega \sup_{(t_1,...,t_n) \in K(kr)} F(x, t_1, ..., t_n)dx < \left(r \prod_{i=1}^{n} p_i\right) \frac{\int_\Omega F(x, w_1(x), ..., w_n(x))dx}{\sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} p_j ||w_i||_{p_i}^{p_i}}$$

where $K(kr) = \{(t_1, ..., t_n) | \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \le kr\}$.

Proof. We put $w(x) = (w_1(x), ..., w_n(x))$ such that for $1 \le i \le n$,

$$w_i(x) = \begin{cases} 0 & \text{if} \quad x \in \Omega \setminus S(x^0, r_2) \\ \frac{d\left(3(l^4 - r_2^4) - 4(r_1 + r_2)(l^3 - r_2^3) + 6r_1 r_2 (l^2 - r_2^2)\right)}{(r_2 - r_1)^3 (r_1 + r_2)} & \text{if} \quad x \in S(x^0, r_2) \setminus S(x^0, r_1) \\ d & \text{if} \quad x \in S(x^0, r_1) \end{cases}$$

where $l = \text{dist}(x, x^0) = \sqrt{\sum_{i=1}^{N} (x_i - x_i^0)^2}$ and $r = \frac{c}{k \prod_{i=1}^{n} p_i}$. We have

$$\frac{\partial w_i(x)}{\partial x_i}$$

$$= \begin{cases} 0 & \text{if} \quad x \in \Omega \setminus S(x^0, r_2) \cup S(x^0, r_1) \\ \frac{12d\left(l^2(x_i - x_i^0) - (r_1 + r_2)l(x_i - x_i^0) + r_1 r_2 (x_i - x_i^0)\right)}{(r_2 - r_1)^3 (r_1 + r_2)} & \text{if} \quad x \in S(x^0, r_2) \setminus S(x^0, r_1), \end{cases}$$

$$\frac{\partial^2 w_i(x)}{\partial^2 x_i}$$
$$= \begin{cases} 0 & \text{if} \quad x \in \Omega \setminus S(x^0, r_2) \cup S(x^0, r_1) \\ \frac{12d\big(r_1r_2+(2l-r_1-r_2)(x_i-x_i^0)^2/l-(r_2+r_1-l)l\big)}{(r_2-r_1)^3(r_1+r_2)} & \text{if} \quad x \in S(x^0, r_2) \setminus S(x^0, r_1) \end{cases}$$

and

$$\sum_{i=1}^{N} \frac{\partial^2 w_i(x)}{\partial^2 x_i}$$
$$= \begin{cases} 0 & \text{if} \quad x \in \Omega \setminus S(x^0, r_2) \cup S(x^0, r_1) \\ \frac{12d\big((N+2)l^2-(N+1)(r_1+r_2)l+Nr_1r_2)\big)}{(r_2-r_1)^3(r_1+r_2)} & \text{if} \quad x \in S(x^0, r_2) \setminus S(x^0, r_1). \end{cases}$$

It is easy to see that $w = (w_1, ..., w_n) \in X$ and, in particular, one has

$$||w_i||_{p_i}^{p_i} = \frac{(12d)^{p_i} 2\pi^{\frac{N}{2}}}{(r_2 - r_1)^{3p_i}(r_1 + r_2)^{p_i}\Gamma(\frac{N}{2})} \times \int_{r_1}^{r_2} |(N+2)\xi^2 - (N+1)(r_1 + r_2)\xi + Nr_1r_2|^{p_i} \xi^{N-1} d\xi \qquad (2.5)$$

for $1 \le i \le n$. Hence, from (2.2), (2.3) and (2.5) we get

$$\frac{(d\theta_i)^{p_i}}{k} < ||w_i||_{p_i}^{p_i} < \frac{(d\sigma_i)^{p_i}}{k} \qquad (2.6)$$

for $1 \le i \le n$. However, taking into account that $\sum_{i=1}^{n} \frac{(d\theta_i)^{p_i}}{p_i} > \frac{c}{\prod_{i=1}^{n} p_i}$, from (2.6) one has

$$\sum_{i=1}^{n} \frac{||w_i||_{p_i}^{p_i}}{p_i} > r.$$

Since $0 \le w_i(x) \le d$ for each $x \in \Omega$ for $1 \le i \le n$, the condition (A_1) ensures hat

$$\int_{\Omega \setminus S(x^0, r_2)} F(x, w_1(x), ..., w_n(x))dx + \int_{S(x^0, r_2) \setminus S(x^0, r_1)} F(x, w_1(x), ..., w_n(x))dx \ge 0.$$

Moreover, owing to our assumptions, we have

$$\begin{aligned} \int_{\Omega} \sup_{(t_1,...,t_n) \in K(kr)} F(x, t_1, ..., t_n)dx &< \frac{c \int_{S(x^0, r_1)} F(x, d, ..., d)dx}{\left(\sum_{i=1}^{n} \frac{(d\sigma_i)^{p_i}}{p_i}\right)\left(\prod_{i=1}^{n} p_i\right)} \\ &\le \frac{c}{k} \frac{\int_{\Omega} F(x, w_1(x), ..., w_n(x))dx}{\sum_{i=1}^{n} \prod_{j=1, j \ne i}^{n} p_j ||w_i||_{p_i}^{p_i}} \\ &= \left(r \prod_{i=1}^{n} p_i\right) \frac{\int_{\Omega} F(x, w_1(x), ..., w_n(x))dx}{\sum_{i=1}^{n} \prod_{j=1, j \ne i}^{n} p_j ||w_i||_{p_i}^{p_i}} \end{aligned}$$

where $K(kr) = \{(t_1, ..., t_n) | \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \le kr\}$. So, the proof is complete. □

We need the following proposition in the proof of Theorem 2.5.

Proposition 2.4. *Let $T : X \to X^*$ be the operator defined by*

$$T(u_1, ..., u_n)(h_1, ..., h_n) = \int_\Omega \sum_{i=1}^{n} |\Delta u_i(x)|^{p_i-2} \Delta u_i(x) \Delta h_i(x) dx$$

for every $(u_1, ..., u_n), (h_1, ..., h_n) \in X$. Then T admits a continuous inverse on X^*.

Proof. Taking into account (2.2) of [25] for $p \geq 2$ there exists a positive constant c_p such that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq c_p|x - y|^p$$

where $\langle\cdot,\ \cdot\rangle$ denotes the usual inner product in $\mathbb{R}^N$, for every $x,\ y \in \mathbb{R}^N$. Thus, it is easy to see that

$$(T(u_1, ..., u_n)-T(v_1, ..., v_n))(u_1-v_1, ..., u_n-u_n) \geq \min\{c_{p_1}, ..., c_{p_n}\} \sum_{i=1}^{n} ||u_i-v_i||_{p_i}^{p_i}$$

for every $(u_1, ..., u_n), (v_1, ..., v_n) \in X$, which means that T is uniformly monotone. Therefore, since T is coercive and hemicontinuous in X (for more details, see [13, Lemma 2]), by applying Theorem 26.A. of [26], we have that T admits a continuous inverse on X^*. □

Now, we state our main result.

Theorem 2.5. *Let $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a function such that $F(\cdot, t_1, ..., t_n)$ is continuous in Ω for all $(t_1, ..., t_n) \in \mathbb{R}^n$, $F(x, \cdot, ..., \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in \Omega$ and $F(x, 0, ..., 0) = 0$ for every $x \in \Omega$. Assume that there exist two positive constants c and d such that $\sum_{i=1}^{n} \frac{(d\theta_i)^{p_i}}{p_i} > \frac{c}{\prod_{i=1}^{n} p_i}$, and Assumptions (A_1) and (A_2) in Lemma 2.3 hold. Furthermore, assume that there exist n positive constants s_i for $1 \leq i \leq n$ with $s_i < p_i$ and a positive function $a \in L^1$ such that*

- (A_3) *$F(x, t_1, ..., t_n) \leq a(x)(1+\sum_{i=1}^{n} |t_i|^{s_i})$ for every $x \in \Omega$ and for all $t_i \in \mathbb{R},\ 1 \leq i \leq n$.*

Then, there exist a non-empty open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q with the following property: for every $\lambda \in \Lambda$ and an arbitrary function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ measurable in Ω for all $(t_1, ..., t_n) \in \mathbb{R}^n$ and C^1 in R^n for every $x \in \Omega$ satisfying (1.2), there is a $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the system (1.1) admits at least three weak solutions in X whose norms are less than q.

Proof. In order to apply Theorem 2.1 to our problem, we introduce the functionals $\Phi,\ J : X \to \mathbb{R}$ for each $u = (u_1, ..., u_n) \in X$, as follows

$$\Phi(u) = \sum_{i=1}^{n} \frac{||u_i||_{p_i}^{p_i}}{p_i}$$

and

$$J(u) = -\int_{\Omega} F(x, u_1(x), ..., u_n(x))dx.$$

Since $p, q > N$, X is compactly embedded in $C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$ and it is well known that Φ and J are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1, ..., u_n) \in X$ are the functionals $\Phi'(u),\ J'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_{\Omega} \sum_{i=1}^{n} |\nabla u_i(x)|^{p_i - 2} \nabla u_i(x) \nabla v_i(x) dx$$

and

$$J'(u)(v) = -\int_{\Omega} \sum_{i=1}^{n} F_{u_i}(x, u_1(x), ..., u_n(x)) v_i(x) dx$$

for every $v = (v_1, ..., v_n) \in X$, respectively.

Note Φ is bounded on each bounded subset of X. Furthermore, Proposition 2.4 gives that Φ' admits a continuous inverse on X^*, and since Φ' is monotone, we obtain that Φ is sequentially weakly lower semi continuous (see [26, Proposition 25.20]). We claim that $J' : X \to X^*$ is a compact operator. To this end, it is enough to show that J' is strongly continuous on X. For this, for fixed $(u_1, ..., u_n) \in X$ let $(u_{1m}, ..., u_{nm}) \to (u_1, ..., u_n)$ weakly in X as $m \to +\infty$, then we have $(u_{1m}, ..., u_{nm})$ converges uniformly to $(u_1, ..., u_n)$ on Ω as $m \to +\infty$ (see [25]). Since $F(x, ., ..., .)$ is C^1 in $\mathbb{R}^n$ for every $x \in \Omega$, so it is continuous in $\mathbb{R}^n$ for every $x \in \Omega$, and we get that $F(x, u_{1m}, ..., u_{nm}) \to F(x, u_1, ..., u_n)$ strongly as $m \to +\infty$ which follows $J'(u_{1m}, ..., u_{nm}) \to J'(u_1, ..., u_n)$ strongly as $m \to +\infty$. Thus we proved that J' is strongly continuous on X, which implies that J' is a compact operator by Proposition 26.2 of [26]. Hence the claim is true. Thanks to assumption $(A3)$, for each $\lambda > 0$ one has that

$$\lim_{||u|| \to +\infty} (\Phi(u) + \lambda J(u)) = +\infty,$$

and so one of the assumptions of Theorem 2.1 holds. We claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} J(u) < r \frac{J(w)}{\Phi(w)}.$$

Moreover, since from (2.1), for $1 \le i \le n$,

$$\sup_{x \in \Omega} |u_i(x)|^{p_i} \le k ||u_i||_{p_i}^{p_i}$$

for each $u = (u_1, ..., u_n) \in X$, then we have

$$\sup_{x \in \Omega} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \le k \sum_{i=1}^{n} \frac{||u_i||_{p_i}^{p_i}}{p_i}$$

for each $u = (u_1, ..., u_n) \in X$, and so for each $r > 0$

$$\begin{aligned}\Phi^{-1}(]-\infty, r]) &= \{u = (u_1, u_2, ..., u_n) \in X;\ \Phi(u) \leq r\} \\ &= \left\{u = (u_1, u_2, ..., u_n) \in X;\ \sum_{i=1}^{n} \frac{||u_i||_{p_i}^{p_i}}{p_i} \leq r\right\} \\ &\subseteq \left\{u = (u_1, u_2, ..., u_n) \in X;\ \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq kr \ \text{ for each } x \in \Omega\right\},\end{aligned}$$

which follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])} J(u) &= \sup_{\sum_{i=1}^{n} \frac{||u_i||_{p_i}^{p_i}}{p_i} \leq r} \int_\Omega F(x, u_1(x), ..., u_n(x))dx \\ &\leq \int_\Omega \sup_{(t_1,...,t_n)\in K(kr)} F(x, t_1, ..., t_n)dx,\end{aligned}$$

where $K(kr) = \{(t_1, ..., t_n)|\ \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \leq kr\}$.

Now, thanks to Lemma 2.3, there exist $r > 0$ and $w \in X$ such that

$$\begin{aligned}\int_\Omega \sup_{(t_1,...,t_n)\in K(kr)} F(x, t_1, ..., t_n)dx &< \left(r \prod_{i=1}^{n} p_i\right) \frac{\int_\Omega F(x, w_1(x), ..., w_n(x))dx}{\sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} p_j ||w_i||_{p_i}^{p_i}} \\ &= r\frac{\int_\Omega F(x, w_1(x), ..., w_n(x))dx}{\sum_{i=1}^{n} \frac{||w_i||_{p_i}^{p_i}}{p_i}} \\ &= r\frac{J(w)}{\Phi(w)}.\end{aligned}$$

So

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} J(u) < r\frac{J(w)}{\Phi(w)}.$$

Fix ρ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} J(u) < \rho < r\frac{J(w)}{\Phi(w)},$$

from Proposition 2.2, with $x_0 = 0$ and $x_1 = w$ we obtain

$$\sup_{\lambda\geq 0} \inf_{u\in X} (\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u\in X} \sup_{\lambda\geq 0} (\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

For any fixed function $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ as in the statement of the theorem, set

$$\Psi(u) = -\int_\Omega G(x, u_1(x), ..., u_n(x))dx.$$

It is well known that Ψ is a continuously differentiable functional whose differential $\Psi'(u) \in X^*$, at $u = (u_1, ..., u_n) \in X$ is given by

$$\Psi'(u)(v) = -\int_\Omega \sum_{i=1}^{n} G_{u_i}(x, u_1(x), ..., u_n(x))v_i(x)dx \text{ for every } v = (v_1, ..., v_n) \in X,$$

such that $\Psi' : X \to X^*$ is a compact operator. Now, all the assumptions of Theorem 2.1, are satisfied. Hence, applying Theorem 2.1, and taking into account that the critical points of the functional $\Phi + \lambda J + \mu\Psi$ are exactly the weak solutions of the system (1.1), we have the conclusion. □

Remark 2.6. *If $n = 2$, Theorem 2.5 gives back the same result of Theorem 1 obtained in [13].*

Example 2.7. *Let $\Omega = \{(x, y) \in \mathbb{R}^2; x^2+y^2 \leq 9\}$, $p_1 = p_2 = 3$ and $F : \Omega\times\mathbb{R}^2 \to \mathbb{R}$ be the function defined as*

$$F(x, y, t_1, t_2) = \begin{cases} 0 & \textit{for all } \ t_i < 0, i = 1, 2; \\ (x^2 + y^2)t_2^{100}e^{-t_2} & \textit{for}\, t_1 < 0, t_2 \geq 0; \\ (x^2 + y^2)t_1^{100}e^{-t_1} & \textit{for}\, t_1 \geq 0, t_2 < 0; \\ (x^2 + y^2)\sum_{i=1}^{2} t_i^{100}e^{-t_i} & \textit{for}\, t_i \geq 0, i = 1, 2 \end{cases}$$

for each $(x, y, t_1, t_2) \in \Omega \times \mathbb{R}^2$. In fact, by choosing $r_1 = 1$ and $r_2 = 2$, taking into account that $k = \frac{40}{\pi}$, we have $\sigma_1 = \sigma_2 = 1152\sqrt{30}$ and $\theta_1 = \theta_2 = 10\sqrt{2}$. Clearly, by choosing $x^0 = (0, 0)$, $c = 3$ and $d = 100$ we observe that the assumptions $(A1)$ and $(A3)$ in Theorem 2.5 are satisfied. For $(A2)$,

$$\begin{aligned} &\sum_{i=1}^{2} \frac{(d\sigma_i)^{p_i}}{p_i} \int_\Omega \sup_{(t_1,t_2)\in K\left(\frac{c}{\Pi_{i=1}^2 p_i}\right)} F(x, y, t_1, t_2)dxdy \\ &= \frac{2}{3}(100 \times 1152\sqrt{30})^3 \int_\Omega \sup_{(t_1,t_2)\in K(\frac{1}{3})} F(x, y, t_1, t_2)dxdy \\ &\leq \frac{2}{3}(100 \times 1152\sqrt{30})^3 \int_\Omega \sup_{(t_1,t_2)\in K(\frac{1}{3})} (x^2 + y^2)\sum_{i=1}^{2} t_i^{100}e^{-t_i}dxdy \\ &= \frac{2}{3}(100 \times 1152\sqrt{30})^3 \max_{(t_1,t_2)\in K(\frac{1}{3})} \sum_{i=1}^{2} t_i^{100}e^{-t_i} \int_{x^2+y^2\leq 9} (x^2 + y^2)dxdy \\ &\leq \frac{2}{3}(100 \times 1152\sqrt{30})^3 \left(2 \max_{|t|\leq 1} t^{100}e^{-t}\right) \int_{x^2+y^2\leq 9} (x^2 + y^2)dxdy \\ &\leq 54\pi(100 \times 1152\sqrt{30})^3 e \leq \frac{\pi}{3}(100)^{100}e^{-100} \\ &= \frac{2}{3}(100)^{100}e^{-100} \int_{x^2+y^2\leq 1} (x^2 + y^2)dxdy \end{aligned}$$

$$= \frac{c}{\prod_{i=1}^{2} p_i} \int_{S(x^0,r_1)} F(x,y,d,d)dxdy.$$

So, Theorem 2.5 is applicable to the system

$$\begin{cases} \Delta(|\Delta u_1|\Delta u_1) = \lambda(x^2+y^2)(u_1^+)^{99}e^{-u_1^+}(100-u_1^+) + \mu G_{u_1}(x,y,u_1,u_2) & in\ \Omega, \\ \Delta(|\Delta u_2|\Delta u_2) = \lambda(x^2+y^2)(u_2^+)^{99}e^{-u_2^+}(100-u_2^+) + \mu G_{u_2}(x,y,u_1,u_2) & in\ \Omega, \\ u_1 = \Delta u_1 = u_2 = \Delta u_2 = 0 & on\ \partial\Omega \end{cases}$$

where $u_i^+ = \max\{u_i, 0\}$ *and* $G : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ *is an arbitrary function which is measurable in* Ω *and* C^1 *in* $\mathbb{R}^n$ *satisfying*

$$\sup_{|(t_1,t_2)| \le s} (|G_{t_1}(x,t_1,t_2)| + |G_{t_2}(x,t_1,t_2)|) \le h_s(x)$$

for all $s > 0$ *and some* $h_s \in L^1(\Omega)$ *with* $G(.,0,0) \in L^1(\Omega)$.

Put

$$\tau_i = \tau_i(N,p_i,r_1,r_2) = \frac{12(N+2)^2(r_1+r_2)}{(r_2-r_1)^3}\left(\frac{k(r_2^N - r_1^N)}{r_1^N}\right)^{1/p_i} \quad \text{for } 1 \le i \le n. \tag{2.7}$$

Here is a remarkable consequence of Theorem 2.5.

Theorem 2.8. *Let* $F : \mathbb{R}^n \to \mathbb{R}$ *be a* C^1*-function such that* $F(0,...,0) = 0$. *Assume that there exist* $n+3$ *positive constants* c, d, η *and* s_i *for* $1 \le i \le n$ *with* $\sum_{i=1}^n \frac{(d\theta_i)^{p_i}}{p_i} > \frac{c}{\prod_{i=1}^n p_i}$ *and* $s_i < p_i$ *such that*

(A_1') $F(t_1,...,t_n) \ge 0$ *for each* $(t_1,...,t_n) \in [0,d] \times \cdots \times [0,d]$;

(A_2') $m(\Omega)\sum_{i=1}^n \frac{(d\tau_i)^{p_i}}{p_i} \max_{(t_1,...,t_n)\in K(\frac{c}{\prod_{i=1}^n p_i})} F(t_1,...,t_n) < \frac{c}{\prod_{i=1}^n p_i} F(d,...,d)$;
where τ_i *is given by* (2.7) *and* $K(\frac{c}{\prod_{i=1}^n p_i}) = \{(t_1,...,t_n)| \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \le \frac{c}{\prod_{i=1}^n p_i}\}$,

(A_3') $F(t_1,...,t_n) \le \eta(1 + \sum_{i=1}^n |t_i|^{s_i})$ *for all* $t_i \in \mathbb{R}$, $1 \le i \le n$.

Then, there exist a non-empty open interval $\Lambda \subseteq [0,+\infty[$ *and a positive real number* q *with the following property: for every* $\lambda \in \Lambda$ *and an arbitrary function* $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *measurable in* Ω *for all* $(t_1,...,t_n) \in \mathbb{R}^n$ *and* C^1 *in* $\mathbb{R}^n$ *for every* $x \in \Omega$ *satisfying* (1.2), *there is a* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$ *the systems*

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) = \lambda F_{u_i}(u_1,...,u_n) + \mu G_{u_i}(x,u_1,...,u_n) & in\ \Omega, \\ u_i = \Delta u_i = 0 & on\ \partial\Omega \end{cases} \tag{2.8}$$

for $1 \le i \le n$, *admits at least three weak solutions in* X *whose norms are less than* q.

Proof. Set $F(x,t_1,...,t_n) = F(t_1,...,t_n)$ for all $x \in \Omega$ and $t_i \in \mathbb{R}$ for $1 \le i \le n$. Clearly, from $(A'1)$ and $(A'3)$ we arrive at $(A1)$ and $(A3)$, respectively. In particular, since $m(S(x^0,r_1)) = r_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)}$, Assumption (A_2') follows that Assumption (A_2) is fulfilled. So, we have the conclusion by using Theorem 2.5. □

Here, a particular case of Theorem 2.8, in which the function F has separated variables is presented.

Corollary 2.9. *Let $f_i : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $f_i(0) = 0$ for $1 \le i \le n$. Assume that there exist $n+3$ positive constants c, d, μ and s_i for $1 \le i \le n$ with $\sum_{i=1}^n \frac{(d\theta_i)^{p_i}}{p_i} > \frac{c}{\prod_{i=1}^n p_i}$, $s_i < p_i$ such that*

(A_1'') *$f_i(t_i) \ge 0$ for $t_i \in [0, d]$ for $1 \le i \le n$;*

(A_2'') *$m(\Omega) \sum_{i=1}^n \frac{(d\tau_i)^{p_i}}{p_i} \max_{(t_1,...,t_n) \in K(\frac{c}{\prod_{i=1}^n p_i})} \prod_{i=1}^n f_i(t_i) < \frac{c \prod_{i=1}^n f_i(d)}{\prod_{i=1}^n p_i}$ where τ_i is given by (2.7) and $K(\frac{c}{\prod_{i=1}^n p_i}) = \{(t_1,...,t_n) | \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \le \frac{c}{\prod_{i=1}^n p_i}\}$;*

(A_3'') *$\prod_{i=1}^n f_i(t_i) \le \mu(1 + \sum_{i=1}^n |t_i|^{s_i})$ for all $t_i \in \mathbb{R}$, $1 \le i \le n$.*

Then, there exist a non-empty open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q with the following property: for every $\lambda \in \Lambda$ and an arbitrary function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ measurable in Ω for all $(t_1, ..., t_n) \in \mathbb{R}^n$ and C^1 in $\mathbb{R}^n$ for almost every $x \in \Omega$ satisfying (1.2), there is a $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the systems

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) = \lambda f_i'(u_i)(\prod_{j=1, j\ne i}^n f_j(u_j)) + \mu G_{u_i}(x, u_1, ..., u_n) & in\, \Omega, \\ u_i = \Delta u_i = 0 & on\, \partial\Omega \end{cases} \tag{2.9}$$

admits at least three solutions in X whose norms are less than q.

Proof. The conclusion follows immediately from Theorem 2.8 by setting

$$F(u_1, ..., u_n) = \prod_{i=1}^n f_i(u_i)$$

for each $(u_1, ..., u_n) \in \mathbb{R}^n$. □

3 Existence Results in the Case $N = 1$

Consider the following nonlinear elliptic equation of $(p_1, ..., p_n)$-biharmonic type under Navier boundary conditions:

$$\begin{cases} (|u_i''|^{p_i-2}u_i'')'' = \lambda F_{u_i}(x, u_1, ..., u_n) + \mu G_{u_i}(x, u_1, ..., u_n), \\ u_i(0) = u_i(1) = u_i''(0) = u_i''(1) = 0 \end{cases} \tag{3.1}$$

for $1 \le i \le n$, where $p_i > 1$ for $1 \le i \le n$, $\lambda, \mu > 0$, $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that $F(\cdot, t_1, ..., t_n)$ is continuous in $[0,1]$ for all $(t_1, ..., t_n) \in \mathbb{R}^n$, $F(x, \cdot, ..., \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in [0,1]$ and $F(x, 0, ..., 0) = 0$ for all $x \in [0,1]$, and $G : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function with respect to x in $[0,1]$ for every $(t_1, ..., t_n) \in \mathbb{R}^n$, and is a C^1-function with respect to $(t_1, ..., t_n) \in \mathbb{R}^n$ for every x in $[0,1]$ and satisfies the condition

$$\sup_{|(t_1,...,t_n)| \le s} \sum_{i=1}^n |G_{t_i}(x, t_1, ..., t_n)| \le h_s(x) \tag{3.2}$$

for all $s > 0$ and some $h_s \in L^1([0,1]$ with $G(.,0,...,0) \in L^1([0,1])$, and F_t and G_t denote the partial derivative of F and G with respect to t, respectively.

Put

$$k = \max\{\frac{1}{2^{p_i}} {p_i}^{-1},\ i = 1,...,n\}. \tag{3.3}$$

Lemma 3.1. *Assume that there exist two positive constants c and d with $\sum_{i=1}^n \frac{(32d)^{p_i}}{2kp_i} > \frac{c}{\prod_{i=1}^n p_i}$ such that*

(B_1) *$F(x, t_1, ..., t_n) \geq 0$ for each $(x, t_1, ..., t_n) \in [0, \frac{1}{4}] \cup]\frac{3}{4}, 1] \times [0, d] \times \cdots \times [0, d]$,*

(B_2)

$$\sum_{i=1}^n \frac{(32d)^{p_i}}{2kp_i} \int_0^1 \sup_{(x,t_1,...,t_n) \in K\left(\frac{c}{\prod_{i=1}^n p_i}\right)} F(x, t_1, ..., t_n) dx < \frac{c}{\prod_{i=1}^n p_i} \int_{\frac{1}{4}}^{\frac{3}{4}} F(x, d, ..., d) dx,$$

where k is given as in (3.3) and $K(\frac{c}{\prod_{i=1}^n p_i}) = \{(t_1, ..., t_n) | \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \leq \frac{c}{\prod_{i=1}^n p_i}\}$(see (2.4)).

Then, there exist $r > 0$ and $w = (w_1, ..., w_n) \in X$ such that $\sum_{i=1}^n \frac{\int_0^1 |w_i''(x)|^{p_i} dx}{p_i} > r$ and

$$\int_0^1 \sup_{(x,t_1,...,t_n) \in K(kr)} F(x, t_1, ..., t_n) dx < \left(r \prod_{i=1}^n p_i \right) \frac{\int_0^1 F(x, w_1(x), ..., w_n(x)) dx}{\sum_{i=1}^n \prod_{j=1, j\neq i}^n p_j \int_0^1 |w_i''(x)|^{p_i} dx}$$

where $K(kr) = \{(t_1, ..., t_n) | \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \leq kr\}$.

Proof. We put $w(x) = (w_1(x), ..., w_n(x))$ such that for $1 \leq i \leq n$,

$$w_i(x) = \begin{cases} d - 16d(\frac{1}{4} - |x - \frac{1}{2}|)^2 & \text{if} \quad x \in [0, \frac{1}{4}] \cup]\frac{3}{4}, 1] \\ d & \text{if} \quad x \in]\frac{1}{4}, \frac{3}{4}] \end{cases}$$

and $r = \frac{c}{k \prod_{i=1}^n p_i}$. It is easy to see that $w = (w_1, ..., w_n) \in X$ and, in particular, one has for $1 \leq i \leq n$,

$$\int_0^1 |w_i''(x)|^{p_i} dx = \frac{(32d)^{p_i}}{2},$$

then, taking into account that $\sum_{i=1}^n \frac{(32d)^{p_i}}{2kp_i} > \frac{c}{\prod_{i=1}^n p_i}$, one has

$$\sum_{i=1}^n \frac{\int_0^1 |w_i''(x)|^{p_i} dx}{p_i} > r.$$

Since $0 \leq w_i(x) \leq d$ for each $x \in (0,1)$ for $1 \leq i \leq n$, the condition (B_1) ensures hat

$$\int_0^{\frac{1}{4}} F(x, w_1(x), ..., w_n(x)) dx + \int_{\frac{3}{4}}^1 F(x, w_1(x), ..., w_n(x)) dx \geq 0.$$

Moreover, owing to our assumptions, we have

$$\begin{aligned}\int_0^1 \sup_{(x,t_1,...,t_n)\in K(kr)} F(x,t_1,...,t_n)dx &< \frac{c\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,d,...,d)dx}{\sum_{i=1}^n \frac{(32d)^{p_i}}{2kp_i}(\prod_{i=1}^n p_i)}\\ &\le \frac{c}{k}\frac{\int_0^1 F(x,w_1(x),...,w_n(x))dx}{\sum_{i=1}^n\prod_{j=1,j\neq i}^n p_j\int_0^1|w_i''(x)|^{p_i}dx}\\ &= \left(r\prod_{i=1}^n p_i\right)\frac{\int_0^1 F(x,w_1(x),...,w_n(x))dx}{\sum_{i=1}^n\prod_{j=1,j\neq i}^n p_j\int_0^1|w_i''(x)|^{p_i}dx}\end{aligned}$$

where $K(kr)=\{(t_1,...,t_n)|\sum_{i=1}^n\frac{|t_i|^{p_i}}{p_i}\le kr\}$. So, the proof is complete. □

Now we state the main result of this section.

Theorem 3.2. *Let $F:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ be a function such that $F(.,t_1,...,t_n)$ is continuous in $[0,1]$ for all $(t_1,...,t_n)\in\mathbb{R}^n$, $F(x,.,...,.)$ is C^1 in R^n for every $x\in[0,1]$ and $F(x,0,...,0)=0$ for every $x\in[0,1]$. Assume that there exist two positive constants c and d such that $\sum_{i=1}^n\frac{(32d)^{p_i}}{2kp_i}>\frac{c}{\prod_{i=1}^n p_i}$, and Assumptions (B_1) and (B_2) in Lemma 3.1 hold. Furthermore, assume that there exist n positive constants s_i for $1\le i\le n$ with $s_i<p_i$ and a positive function $\alpha\in L^1$ such that*

- (B_3) *$F(x,t_1,...,t_n)\le\alpha(x)(1+\sum_{i=1}^n|t_i|^{s_i})$ for almost every $x\in[0,1]$ and for all $t_i\in\mathbb{R},\ 1\le i\le n$.*

Then, there exist a non-empty open interval $\Lambda\subseteq[0,+\infty[$ and a positive real number q with the following property: for every $\lambda\in\Lambda$ and an arbitrary function $G:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ measurable in $[0,1]$ for all $(t_1,...,t_n)\in\mathbb{R}^n$ and C^1 in $\mathbb{R}^n$ for every $x\in[0,1]$ satisfying (3.2), there is a $\delta>0$ such that, for each $\mu\in[0,\delta]$ the system (3.1) admits at least three weak solutions in $W^{2,p_1}([0,1])\cap W_0^{1,p_1}([0,1])\times\cdots\times W^{2,p_n}([0,1])\cap W_0^{1,p_n}([0,1])$ whose norms are less than q.

Proof. Take $X=W^{2,p_1}([0,1])\cap W_0^{1,p_1}([0,1])\times\cdots\times W^{2,p_n}([0,1])\cap W_0^{1,p_n}([0,1])$ equipped with the norm

$$||\ (u_1,...,u_n)\ ||=\sum_{i=1}^n\left(\int_0^1|u_i''(x)|^{p_i}dx\right)^{1/p_i}.$$

Set

$$\Phi(u)=\sum_{i=1}^n\frac{\int_0^1|u_i''(x)|^{p_i}dx}{p_i},\quad J(u)=-\int_0^1 F(x,u_1(x),...,u_n(x))dx$$

and

$$\Psi(u)=-\int_0^1 G(x,u_1(x),...,u_n(x))dx$$

for each $u = (u_1, ..., u_n) \in X$. Since the critical points of the functional $\Phi + \lambda J + \mu\Psi$ are exactly the weak solutions of the system (3.1), to obtain our assertion it is enough to apply Theorem 2.1. To this end, we observe that, it is easy to verify the regularity assumptions on Φ, J and Ψ, as requested in Theorem 2.1. Moreover, as standard computations show, the condition $(B3)$ implies that $\Phi + \lambda J$ is coercive. Now, we claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} J(u) < r \frac{J(w)}{\Phi(w)}.$$

Moreover, since for $1 \leq i \leq n$,

$$\sup_{x \in [0,1]} |u_i(x)|^{p_i} \leq k \int_0^1 |u_i''(x)|^{p_i}$$

for each $u = (u_1, ..., u_n) \in X$ (see [14], Lemma 2), we have

$$\sup_{x \in [0,1]} \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \leq k \sum_{i=1}^n \frac{\int_0^1 |u_i''(x)|^{p_i} dx}{p_i}$$

for each $u = (u_1, ..., u_n) \in X$, and so for each $r > 0$

$$\begin{aligned} \Phi^{-1}(]-\infty, r]) &= \{u = (u_1, u_2, ..., u_n) \in X;\ \Phi(u) \leq r\} \\ &= \left\{ u = (u_1, u_2, ..., u_n) \in X;\ \sum_{i=1}^n \frac{\int_0^1 |u_i''(x)|^{p_i} dx}{p_i} \leq r \right\} \\ &\subseteq \left\{ u \in X;\ \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \leq kr \ \text{ for each } x \in [a, b] \right\}, \end{aligned}$$

which follows that

$$\begin{aligned} \sup_{u \in \Phi^{-1}(]-\infty, r])} J(u) &= \sup_{\sum_{i=1}^n \frac{\int_0^1 |u_i''(x)|^{p_i} dx}{p_i} \leq r} \int_0^1 F(x, u_1(x), ..., u_n(x)) dx \\ &\leq \int_0^1 \sup_{(t_1, ..., t_n) \in K(kr)} F(x, t_1, ..., t_n) dx, \end{aligned}$$

where $K(kr) = \{(t_1, ..., t_n) | \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \leq kr\}$.

Now, thanks to Lemma 3.1, there exist $r > 0$ and $w \in X$ such that

$$\begin{aligned} \int_0^1 \sup_{(t_1, ..., t_n) \in K(kr)} F(x, t_1, ..., t_n) dx &< \left(r \prod_{i=1}^n p_i \right) \frac{\int_0^1 F(x, w_1(x), ..., w_n(x)) dx}{\sum_{i=1}^n \prod_{j=1, j \neq i}^n p_j \int_0^1 |w_i''(x)|^{p_i} dx} \\ &= r \frac{\int_0^1 F(x, w_1(x), ..., w_n(x)) dx}{\sum_{i=1}^n \frac{\int_0^1 |w_i''(x)|^{p_i} dx}{p_i}} \\ &= r \frac{J(w)}{\Phi(w)}. \end{aligned}$$

So

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} J(u) < r\frac{J(w)}{\Phi(w)}.$$

Hence, our claim is proved. Fix ρ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} J(u) < \rho < r\frac{J(w)}{\Phi(w)},$$

from Proposition 2.2, with $x_0 = 0$ and $x_1 = w$ we obtain

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

Now, all the assumptions of Theorem 2.1, are satisfied. Hence, applying Theorem 2.1, we have the conclusion. □

We now want to point out a simple consequence of Theorem 3.2 in the case $p_i = 2$ for $1 \leq i \leq n$. Taking into account that, in this situation, $k = \frac{1}{8}$, we have the following result:

Theorem 3.3. *Let $F : [0,1]\times\mathbb{R}^n \to \mathbb{R}$ be a function such that $F(\cdot, t_1, ..., t_n)$ is continuous in $[0,1]$ for all $(t_1, ..., t_n) \in \mathbb{R}^n$, $F(x,\cdot, ...,\cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in [0,1]$ and $F(x, 0, ..., 0) = 0$ for every $x \in [0,1]$. Assume that there exist $n+2$ positive constants c, d and s_i for $1 \leq i \leq n$ with $n(32d)^2 > \frac{c}{2^{n+1}}$ and $s_i < 2$ and a positive function $\alpha \in L^1$ such that Assumption $(B1)$ in Lemma 3.1 and Assumption $(B3)$ in Theorem 3.2 hold. Furthermore, suppose that*

(B_2') $n(32d)^2\int_0^1 \sup_{(t_1,...,t_n)\in K(\frac{c}{2^n})} F(x, t_1, ..., t_n)dx < \frac{c}{2^{n+1}}\int_{\frac{1}{4}}^{\frac{3}{4}} F(x, d, ..., d)dx$
where $K(\frac{c}{2^n}) = \{(t_1, ..., t_n)| \sum_{i=1}^n \frac{|t_i|^2}{2} \leq \frac{c}{2^n}\}$ *(see* (2.4)*).*

Then, there exist a non-empty open interval $\Lambda \subseteq [0,+\infty[$ and a positive real number q with the following property: for every $\lambda \in \Lambda$ and an arbitrary function $G : [0,1]\times\mathbb{R}^n \to \mathbb{R}$ measurable in $[0,1]$ for all $(t_1, ..., t_n) \in \mathbb{R}^n$ and C^1 in $\mathbb{R}^n$ for every $x \in [0,1]$ satisfying (3.2), there is a $\delta > 0$ such that, for each $\mu \in [0,\delta]$ the systems

$$\begin{cases} u_i^{(4)} = \lambda F_{u_i}(x, u_1, ..., u_n) + \mu G_{u_i}(x, u_1, ..., u_n), \\ u_i(0) = u_i(1) = u_i''(0) = u_i''(1) = 0 \end{cases} \tag{3.4}$$

for $1 \leq i \leq n$, admits at least three weak solutions in $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1]) \times \cdots \times W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ whose norms are less than q.

Proof. The conclusion follows directly from Theorem 3.2 taking into account that from Assumption $(B'2)$ one arrives at Assumption $(B2)$. □

If $n = 1$, from Theorem 3.3 we have the following result.

Theorem 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist four positive constants c, d, μ and s with $c < 4(32d)^2$ and $s < 2$ such that*

(B_1'') *$f(t) \geq 0$ for each $t \in [-c, \max\{c, d\}]$;*

(B_2'') $1024\frac{\int_0^c f(\xi)d\xi}{c} < \frac{1}{8}\frac{\int_0^d f(\xi)d\xi}{d^2}$;

(B_3'') $\int_0^t f(\xi)d\xi \leq \mu(1 + |t|^s)$ *for all* $t \in R$.

Then, there exist a non-empty open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q with the following property: for every $\lambda \in \Lambda$ and an arbitrary L^1-Carathéodory function $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$, there is a $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the systems

$$\begin{cases} u^{(4)} = \lambda f(u) + \mu g(x, u), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{3.5}$$

admits at least three weak solutions in $W^{2,2}([0, 1]) \cap W_0^{1,2}([0, 1])$ whose norms are less than q.

We end this paper by giving the following example to illustrate Theorem 3.4.

Example 3.5. *Let d be a positive constant such that $8192e^{d-1} < d^{10}$ (for instance, d=4). Put $f(t) = (t^+)^{11}e^{-t^+}(12 - t^+)$ where $t^+ = \max\{t, 0\}$, one has*

$$\int_0^t f(\xi)d\xi = \begin{cases} 0 & \text{for all } t < 0, \\ t^{12}e^{-t} & \text{for all } t \geq 0. \end{cases}$$

It is easy to verify that with $c = 1$, μ sufficiently large and $s < 2$ the assumptions of Theorem 3.4 are satisfied, so there exist a non-empty open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q with the following property: for every $\lambda \in \Lambda$ and an arbitrary L^1-Carathéodory function $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$, there is a $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the system (3.5), in this case, admits at least three weak solutions in $W^{2,2}([0, 1]) \cap W_0^{1,2}([0, 1])$ whose norms are less than q.

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