Some Classical and Recent Results Concerning Renorming Theory

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Abstract : The problems connected with equivalent norms lie at the heart of Banach space theory. This is a short survey on some recent as well as classical results and open problems in renormings of Banach spaces.

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1 Introduction

Banach space theory is a classic topic in functional analysis. The study of the structure of Banach spaces provides a framework for many branches of mathematics like differential calculus, linear and nonlinear analysis, abstract analysis, topology, probability, harmonic analysis, etc. The geometry of Banach spaces plays an important role in Banach space theory. Since it is easier to do analysis on a Banach space which has a norm with good geometric properties than on a general space, we consider in this survey an area of Banach space theory known as *renorming theory*. Renorming theory is involved with problems concerning the construction of *equivalent norms* on a vector space with nice geometrical properties of convexity or differentiability. An excellent monograph containing the main advances on renorming theory until 1993 is [1].

We consider only Banach spaces over the reals. Given a Banach space X with norm $\|.\|$, we denote by $S(X)$ the unit sphere, and by X^* the dual space with

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(original) dual norm $\|.\|^*$. All undefined terms and notation are standard and can be found, for example, in [2–8].

2   Differentiable Norms

Differentiability of functions on Banach spaces is a natural extension of the notion of a directional derivative on $\mathbb{R}^n$. A function $f : X \to \mathbb{R}$ is said to be *Gâteaux differentiable* at $x \in X$ if there exists a functional $g \in X^*$ such that $g(y) = \lim_{t \to 0} \frac{f(x+ty)-f(x)}{t}$, for all $y \in X$. In this case, g is called *Gâteaux derivative* of f. If the limit above exists uniformly for each $y \in S(X)$, then f is called *Fréchet differentiable* at x with *Fréchet derivative* g. In this paper, most of our attention will be concentrated on the differentiability of the norm. Two important more strong notions of differentiability are obtained as uniform versions of both Fréchet and Gâteaux differentiability. The norm $\|.\|$ on X is *uniformly Fréchet differentiable* if $\lim_{t \to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists uniformly for $(x, y) \in S(X) \times S(X)$. Also, it is *uniformly Gâteaux differentiable* if for each $y \in S(X)$, $\lim_{t \to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists uniformly in $x \in S(X)$. Clearly, Fréchet differentiability implies Gâteaux differentiability, but the converse is true only for finite-dimensional Banach spaces, in general. As an example, the mapping $f : L^1[0, \pi] \to \mathbb{R}$ defined by $f(x) = \int_0^\pi \sin(x(t))dt$ is every where Gâteaux differentiable, but nowhere Fréchet differentiable [9].

Around the year of 1940, Šmulyan proved his following fundamental dual characterization of differentiability of norms, which is used in many basic renorming results.

Theorem 2.1 ([6, Ch. VIII]). *For each $x \in S(X)$, the following are equivalent:*

(i) *$\|.\|$ is Fréchet differentiable at x.*

(ii) *For all $(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty \subseteq S(X^*)$, if $\lim_{n \to \infty} f_n(x) = 1$ and $\lim_{n \to \infty} g_n(x) = 1$ then $\lim_{n \to \infty} \|f_n - g_n\|^* = 0$.*

(iii) *Each $(f_n)_{n=1}^\infty \subseteq S(X^*)$ with $\lim_{n \to \infty} f_n(x) = 1$ is convergent in $S(X^*)$.*

As a direct application of Šmulyan's theorem, we have the following corollary:

Corollary 2.2 ([6, Ch. VIII]). *If the dual norm of X^* is Fréchet differentiable then X is reflexive.*

Proof. A celebrated and deep theorem of James state that X is reflexive if and only if each nonzero $f \in X^*$ attains its norm at some $x \in S(X)$. Let $f \in S(X^*)$ and choose $(x_n)_{n=1}^\infty \in S(X)$ such that $\lim_{n \to \infty} f(x_n) = 1$. By Theorem 2.1, $\lim_{n \to \infty} x_n = x \in S(X)$. Therefore $f(x) = f(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} f(x_n) = 1 = \|f\|^*$. If now $f \in X^*$ is non-zero, then $\frac{f}{\|f\|^*} \in S(X^*)$ and according to the reasoning above there exists $x \in S(X)$ such that $\frac{f}{\|f\|^*}(x) = 1$.   □

Theorem 2.3. *The following assertions imply the reflexivity of X:*

(i) *The norm of X is uniformly Fréchet differentiable [5, page 434].*

(ii) *The third dual norm of X is Gâteaux differentiable [6, page 276].*

Theorem 2.4 ([6, page 275]). *If X is separable and the second dual norm of X is Gâteaux differentiable then X^* is separable.*

Since the separable and reflexive spaces contain numerous nice structural aspects, they have an important role in our investigation. In fact, there are many renorming characterizations of reflexivity and separability, such as follows:

Theorem 2.5 ([10]). *If X is reflexive then can be renormed in such a way that both X and X^* have Fréchet differentiable norm.*

There exist reflexive spaces which do not admit any equivalent uniformly Gâteaux differentiable norm. This example can be found in Kutzarova and Troyanski [11]. However, Šmulyan proved the following positive result. His norm was the predual norm to the norm defined on X^* by $\||f\||^2 = \|f\|^{*2} + \sum_{i=1}^{\infty} 2^{-i} f^2(x_i)$, where $(x_i)_{i=1}^{\infty}$ is dense in $S(X)$.

Theorem 2.6 ([1, Ch. II]). *Any separable space admits an equivalent uniformly Gâteaux differentiable norm.*

Theorem 2.7 (Kadec [12]). *If a separable space X admits an equivalent Fréchet differentiable norm then X^* is separable.*

Proof. Observe that the set $B = \{\|.\|' : x \in X, x \neq 0\}$ is norm-separable, where $\|x\|'$ denotes the derivative of $\|.\|$ at x. The set B contains all norm-attaining functionals, and is thus norm-dense in X^* by the Bishop-Phelps theorem. $\square$

The norm $\|.\|$ on X is called *2-rotund (resp. weakly 2-rotund)* if for every $(x_n)_{n=1}^{\infty} \subseteq S(X)$ such that $\lim_{m,n \to \infty} \|x_m + x_n\| = 0$, there is an $x \in X$ such that $\lim_{n \to \infty} x_n = x$ in the norm (resp. weak) topology of X.

By using Theorem 2.1, it is proved that if a norm on X is 2-rotund then its dual norm is Fréchet differentiable. Also, if a norm on X is weakly 2-rotund then its dual norm is Gâteaux differentiable [13].

Theorem 2.8 ([14, page 208]). *X is reflexive if and only if it admits an equivalent weakly 2-rotund norm.*

Theorem 2.9 ([14, page 208]). *A separable space X is reflexive if and only if X admits an equivalent 2-rotund norm.*

Note that it is not known if the separability of X has to be assumed in theorem above.

The space X is *Hilbert generated space* if there is a Hilbert space H and a bounded linear operator T from H into X such that $T(H)$ is dense in X.

Theorem 2.10 ([13]). *X is a subspace of a Hilbert generated space if and only if X admits an equivalent uniformly Gâteaux differentiable norm.*

3   Asplund Spaces

It is a well-known theorem that every continuous convex function on a separable space X is Gâteaux differentiable at the points of a G_δ-dense subset of X [5, page 384]. Let f be a continuous convex function on X. Then the set G of all points in X where f is Fréchet differentiable (possibly empty) is a G_δ set in X [5, page 357]. The space X is said to be *Asplund* if every continuous convex function on it is Fréchet differentiable at each point of a dense G_δ subset of X. There exist many well-known equivalent characterizations of the Asplund spaces. For example, X is Asplund if and only if Y^* is separable whenever Y is a separable subspace of X. Every Banach space with a Fréchet differentiable norm is Asplund [15] but, on the other hand, Haydon [16] constructed Asplund spaces admitting no Gâteaux differentiable norm.

Theorem 3.1 ([15, 17])**.** *For each separable space X the following are equivalent:*

(i) X^* *is separable.*

(ii) X *is Asplund.*

(iii) X *admits an equivalent Fréchet differentiable norm.*

(iv) *There is no equivalent rough norm on X.*

Recall that the norm $\|.\|$ on X is *rough* if for some $\varepsilon > 0$,

$$\limsup_{h \to 0} \frac{1}{\|h\|}(\|x + h\| + \|x - h\| - 2) \geq \varepsilon,$$

for every $x \in S(X)$.

By using the canonical norm of $C([0,1])$ as a rough norm, we obtain that $C([0,1])$ does not admit any Fréchet differentiable norm.

4   Kadec-Klee Property

The norm $\|.\|$ on X has *weak-Kadec-Klee* property provided that whenever $(x_n)_{n=1}^\infty \subseteq X$ converges weakly to some $x \in X$ and $\lim_{n\to\infty} \|x_n\| = \|x\|$, then $\lim_{n\to\infty} \|x_n - x\| = 0$. Also, a dual norm $\|.\|_*$ on X^* has *weak*-Kadec-Klee* property if $\lim_{n\to\infty} \|f_n - f\|_* = 0$, whenever $(f_n)_{n=1}^\infty \subseteq X^*$ is weak* convergent to some $f \in X^*$ and $\lim_{n\to\infty} \|f_n\|_* = \|f\|_*$.

The weak-Kadec-Klee norms play an important role in geometric Banach space theory and its applications.

Theorem 4.1 ([5, page 422])**.**

(i) *Let X be a separable space. If X^* has the weak*-Kadec-Klee property then X^* is separable.*

(ii) *If X^* is separable then X admits an equivalent norm such that X^* has the weak*-Kadec-Klee property.*

5 Strictly Convex Spaces

One interesting and fruitful line of research, dating from the early days of Banach space theory, has been to relate analytic properties of a Banach space to various geometric conditions on that space. The simplest example of such a condition is that of strict convexity.

The space X (or the norm $\|.\|$ on X) is called *strictly convex* (R) if for $x, y \in S(X)$, $\|x+y\| = 2$ implies $x = y$. The following result is a consequence of Theorem 2.1 of Šmulyan:

Theorem 5.1 ([6, Ch. VIII]). *If a dual norm of X^* is strictly convex (Gâteaux differentiable) then its predual norm is Gâteaux differentiable (strictly convex).*

The converse implications in the theorem above are true for reflexive spaces, but not in general. Strict convexity is not preserved by equivalent norms. It is well known that $\|.\|_\infty$ and $\|.\|_2$ are equivalent norms on $\mathbb{R}^n$, $\|.\|_2$ is strictly convex but $\|.\|_\infty$ is not. A most common strictly convex renorming is based on the following simple observation. Let Y be a strictly convex space and $T : X \to Y$ a linear one-to-one bounded operator; then $\||x\|| = \|x\| + \|T(x)\|$, $x \in X$, is an equivalent strictly convex norm on X.

Theorem 5.2 ([12, 18]). *Any separable space X admits an equivalent norm whose dual norm is strictly convex.*

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be dense in $S(X)$. Define a norm $\||.\||$ on X^* by $\||f\||^2 = \|f\|^{*2} + \sum_{i=1}^{\infty} 2^{-i} f^2(x_i)$. It is not hard to show that $\||.\||$ is a weak* lower semi-continuous function on X^* equivalent with $\|.\|^*$. Hence $\||.\||$ is the dual of a norm $|.|$ equivalent with $\|.\|$, and also it is strictly convex. $\square$

6 Locally Uniformly Convex Spaces

The concept of a locally uniformly convex norm was introduced by Lovaglia in [19]. The space X (or the norm $\|.\|$ on X) is said to be *locally uniformly convex* (LUR) if

$$\lim_{n \to \infty} \left(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 \right) = 0 \implies \lim_{n \to \infty} \|x - x_n\| = 0,$$

for any sequence $(x_n)_{n=1}^{\infty}$ and x in X.

Lovaglia showed, as a straightforward consequence of Theorem 2.1, that the norm of a Banach space is Fréchet differentiable if the dual norm is LUR. The converse does not hold, even up to renormings. In fact, there exists a space with a Fréchet differentiable norm, which does not admit any equivalent norm with a strictly convex dual norm [1]. However, in the class of spaces with unconditional bases, we do have equivalence up to a renorming.Many efforts have been dedicated in the renorming theory to obtain sufficient conditions for a Banach space to admit an equivalent LUR norm. In 1979, Troyanski stated the first characterization of existence of LUR renormings.

Theorem 6.1 ([20]). *If X^* has a dual LUR norm then X admits an equivalent LUR norm.*

Theorem 6.2 ([5, page 387]). *Any space X with a Fréchet differentiable norm which has a Gâteaux differentiable dual norm admits an equivalent LUR norm.*

Theorem 6.3 (Kadec [18]).

  (i) *If X is separable then X admits an equivalent LUR norm.*

  (ii) *If X^* is separable then X admits an equivalent norm whose dual norm is LUR.*

Proof. We show the second statement. Suppose that $\{x_i\}_{i=1}^{\infty}$ be dense in $S(X)$ and $\{f_i\}_{i=1}^{\infty}$ be dense in $S(X^*)$. For $i \in \mathbb{N}$, put $F_i = \operatorname{span}\{f_1, f_2, ..., f_i\}$. Define a norm $|||.|||$ on X^* by

$$|||f|||^2 = \|f\|^{*2} + \sum_{i=1}^{\infty} 2^{-i}\operatorname{dist}(f, F_i)^2 + \sum_{i=1}^{\infty} 2^{-i}f^2(x_i).$$

It is not hard to show that $|||.|||$ is a weak* lower semicontinuous function on X^* equivalent with $\|.\|^*$. Hence $|||.|||$ is the dual of a norm $|.|$ equivalent with $\|.\|$, and also it is LUR. $\qquad\square$

The theorem above shows that, in particular, every separable space admits an equivalent strictly convex norm. By using Theorems 3.1 and 6.3, we see that if X is an Asplund space then X^* admits an equivalent LUR norm. The next theorem is a powerful result of Troyanski [21]:

Theorem 6.4. *X admits an equivalent LUR norm if and only if it admits an equivalent weak-Kadec-Klee norm and an equivalent strictly convex norm.*

Theorem 6.5 ([22]). *Let Y be a closed subspace of X such that both Y and X/Y admit equivalent norms whose dual norms are LUR. Then X admits an equivalent norm whose dual norm is LUR.*

Let us mention here that the analogue of Theorem 6.5 for Fréchet differentiable norms is still an open question. Talagrand [23] proved that the corresponding result for Gâteaux differentiable norms is false. Here we offer a characterization of LUR spaces in terms of Lipschitz separated spaces:

Given a positive scalar M, we will let $L_{X,M}$ be the space of all functions $f : X \to \mathbb{R}$ such that $|f(x) - f(y)| \leq \|x - y\|$ for each $x, y \in X$ and $\sup\{|f(x)| : x \in X\} \leq M$ endowed with the metric $\rho(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$. With this metric, $L_{X,M}$ is a complete metric space. Given a closed nonempty subset $Y \subseteq X$ and $f \in L_{X,M}$, we let $L_{f,M} = \{\widetilde{f} \in L_{X,M} : \widetilde{f}_{|Y} = f\}$. We say X is *Lipschitz separated* if for every proper closed subspace $Y \subseteq X$ and every $f \in L_{Y,M}$, we have $\sup_{\widetilde{f} \in L_{f,M}} \widetilde{f}(x) > \inf_{\widetilde{f} \in L_{f,M}} \widetilde{f}(x)$ for all $x \in X \setminus Y$.

Theorem 6.6 ([24]). *The separable space X can be equivalently renormed so that it is LUR but not Lipschitz separated.*

Theorem 6.7 ([24]). *Any space X with a separable dual admits an equivalent norm under which X is Lipschitz separated but not LUR.*

7   Uniformly Convex Spaces

A Banach space is strictly convex if the midpoint of each chord of the unit ball lies beneath the surface. In 1936, Clarkson introduced the stronger notion of uniform convexity. A Banach space is uniformly convex if the midpoints of all chords of the unit ball whose lengths are bounded below by a positive number are uniformly buried beneath the surface. The class of uniformly convex Banach spaces is very interesting and has numerous applications. The space X (or the norm $\|.\|$ on X) is said to be *uniformly convex* (UR) if for all sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty} \subseteq X$

$$\lim_{n\to\infty} \left(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2\right) = 0 \implies \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

For example, any Hilbert space is uniformly convex and it can be shown that L^p spaces are uniformly convex whenever $1 < p < \infty$. We have (UR)$\Rightarrow$(LUR)$\Rightarrow$(R) but the converse is not true. For example, define a norm $\|\|.\|\|$ on $C([0,1])$ by $\|\|f\|\|^2 = \|f\|_{\infty}^2 + \|f\|_2^2$, where $\|.\|_{\infty}$ denotes the standard supremum norm of $C([0,1])$ and $\|.\|_2$ denotes the canonical norm of $L^2[0,1]$. Then $\|\|.\|\|$ is strictly convex but not LUR on $C([0,1])$. There is a complete duality between uniform convexity and uniform Fréchet differentiability.

Theorem 7.1 (Lindenstrauss [25]). *For any space X, the dual norm of X^* is uniformly convex if and only if its predual norm is uniformly Fréchet differentiable. Also, the dual norm of X^* is uniformly Fréchet differentiable if and only if its predual norm is uniformly convex.*

One of the first theorems to relate the geometry of the norm to linear topological properties is the following:

Theorem 7.2 ([3, pages 37-50]). *Any uniformly convex Banach space is reflexive.*

Proof. Assume that the norm of X is uniformly convex. Then the dual norm of X^* is uniformly Fréchet differentiable by Theorem 7.1. Therefore X^* is reflexive by Theorem 2.3 and thus X is reflexive. □

The theorem above shows that any Hilbert space is a reflexive Banach space which is a well-known result in functional analysis. Note that the class of uniformly convex Banach spaces does not coincide with the all reflexive Banach spaces: an example of a reflexive Banach space which is not uniformly convex can be given. Notice that, the space $C([0,1])$ is a separable non reflexive space. Consequently, $C([0,1])$ admits no equivalent uniformly convex norm, although, by Theorem 6.3, it does admit an equivalent LUR norm.

Theorem 7.3 ([6, Ch. XI])**.** *Any space that admits an equivalent WUR norm is an Asplund space.*

The norm $\|.\|$ on X is *weakly uniformly convex* (WUR) if for all sequences $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty \subseteq X$ with $\lim_{n\to\infty} 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 = 0$, then $\lim_{n\to\infty} x_n - y_n = 0$, in the weak topology of X. Using Theorems 3.1 and 7.3, we have the following corollary.

Corollary 7.4. *If the norm of a separable space X is WUR then X^* is separable.*

8 Super-Reflexive Spaces

Given Banach space Y, we say that Y is *finitely representable* in X if for every $\varepsilon > 0$ and for every finite-dimensional subspace Z of Y, there is an isomorphism T of Z onto $T(Z) \subseteq X$ such that $\|T\|\|T^{-1}\| < 1 + \varepsilon$. The space X is said to be *super-reflexive* if every finitely representable space in X is reflexive. Clearly, every super-reflexive space is reflexive. One of the well-known super-reflexive spaces are Hilbert spaces. The L^p spaces for $1 < p < \infty$ are other examples of super-reflexive spaces. But there are many other super-reflexive spaces. This class is mapped by the following equivalence.

Theorem 8.1 ([5, page 436])**.** *The following assertions are equivalent:*

   (i) X is super-reflexive.

   (ii) X admits an equivalent uniformly convex norm.

   (iii) X admits an equivalent uniformly Fréchet differentiable norm.

   (iv) X admits an equivalent norm which is uniformly convex and uniformly Fréchet differentiable.

9 Mazur Intersection Property

In 1933, it was Mazur who first studied Banach spaces which have the so called *Mazur intersection property* (MIP): every bounded closed convex set can be represented as an intersection of closed balls. A systematic study of this topic was initiated by Phelps [26]. In 1978, Giles, Gregory and Sims gave some characterizations of this property [27]. They raised the question whether every Banach space with the MIP is an Asplund space. They also characterized the associated property for a dual space, called the *weak* Mazur intersection property*: every bounded weak* closed convex set can be represented as an intersection of closed dual balls. Associated with MIP, we have also the following concepts:

A set C in X is a *Mazur set* if given $f \in X^*$ with sup $f(C) < \lambda$, then there exists a closed ball D such that $C \subseteq D$ and sup $f(D) < \lambda$. The space X is called a *Mazur space* provided that any intersection of closed balls in X is a Mazur set.

Theorem 9.1 ([28])**.** *Any space with Fréchet differentiable norm satisfies the MIP. Also, every space whose dual satisfies the MIP is reflexive and each reflexive space with a Fréchet differentiable norm is a Mazur space. Finally, Mazur spaces with the MIP are Asplund and Gâteaux differentiable.*

Theorem 9.2 ([29])**.** *A Mazur space with the MIP admits an equivalent Fréchet differentiable norm.*

Theorem 9.3 ([22])**.** *Let Y be a subspace of X such that both Y^* and $(X/Y)^*$ can be renormed to have the weak* Mazur intersection property. Then X^* can be renormed to have the weak* Mazur intersection property.*

10 Weakly Compactly Generated Spaces

The space X is said to be *weakly compactly generated* (WCG) if X is the closed linear span of a weakly compact set $K \subseteq X$. The class of WCG spaces has been intensively studied during the last forty years and now is in the core of modern Banach space theory [3–5]. Recall that the space X is separable if there exists a countable set $\{x_n\}_{n=1}^{\infty}$ with $\overline{\{x_n\}_{n=1}^{\infty}} = X$. An important characterization of reflexivity is the result that X is reflexive if and only if $B(X)$, the closed unit ball of X, is weakly compact. Notice that if X is reflexive, then one may take $K = B(X)$ in the definition above, whereas if X is separable, with $\{x_n\}_{n=1}^{\infty}$ dense in the $S(X)$, we can take $K = \{n^{-1}x_n\}_{n=1}^{\infty} \bigcup \{0\}$. In this way we see that both separable and reflexive spaces are WCG.

Theorem 10.1.

   (i) *If X is WCG then X admits an equivalent norm that is simultaneously LUR and Gâteaux differentiable [30, 31].*

   (ii) *If X^* is WCG then X admits an equivalent norm $|.|$ the dual norm of which is LUR. In particular, $|.|$ is Fréchet differentiable [32].*

The first part of theorem above shows that every reflexive space admits an equivalent norm with weak-Kadec-Klee property.

Corollary 10.2 ([5, page 589])**.** *If X^* is WCG then X is Asplund.*

The corollary above shows that any reflexive Banach space is Asplund.

Theorem 10.3 ([30])**.** *If X is WCG then X^* admits an equivalent strictly convex dual norm.*

Theorem 10.4 ([33])**.** *If X is WCG and Asplund then X^* admits an equivalent LUR dual norm.*

If M is a bounded total set in X (i.e., a bounded set M in X such that $\overline{span}M = X$), we will say that the norm of X is *dually M-2-rotund* if $(f_n)_{n=1}^{\infty}$ is convergent to some $f \in B(X^*)$ uniformly on M whenever $f_n \in S(X^*)$ are such that $\lim_{m,n\to\infty} \|f_m + f_n\|^* = 0$.

Theorem 10.5 ([13]). *X is WCG if and only if X admits an equivalent dually M-2-rotund norm for some bounded total set in X.*

11   Vašák Spaces

A class of spaces wider than WCG spaces, known as weakly countably determined or Vašák spaces, was originally defined and investigated by Vašák. The space X is *Vašák* if there is a sequence $(B_n)_{n=1}^\infty$ of *weak** compact sets in X^{**} such that given $x \in X$ and $u \in X^{**}\backslash X$, there is $n \in \mathbb{N}$ such that $x \in B_n$ and $u \notin B_n$.

Theorem 11.1 ([6, Ch. XI]). *If X^* is Vašák then X admits an equivalent Fréchet differentiable norm.*

Theorem 11.2 ([34]). *Every Vašák space has an equivalent norm the dual norm of which is strictly convex.*

Many of the renorming results for WCG spaces are actually valid for Vašák spaces. For example, any Vašák space admits a Gâteaux differentiable norm [1, Ch. VII]. Further details can be found in [4, Ch. VII].

12   Uniform Eberlein Compact Spaces

A compact space K is said to be *uniform Eberlein* if K is homeomorphic to a weakly compact subset of a Hilbert space in its weak topology.

Theorem 12.1 ([5, page 624]).

  (i) *$(B(X^*), w^*)$ is uniform Eberlein compact if and only if X admits an equivalent uniformly Gâteaux differentiable norm.*

  (ii) *Let K be a compact space. $C(K)$ admits an equivalent uniformly Gâteaux differentiable norm if and only if K is uniform Eberlein.*

13   Bases and Renorming Theory

A *Schauder* basis for X is a sequence $(x_n)_{n=1}^\infty$ of vectors in X such that every vector in X has a unique representation of the form $\sum_{n=1}^\infty a_n x_n$ with each a_n a scalar and where the sum is converges in the norm topology. Recall that a series $\sum_{n=1}^\infty x_n$ is said to be *unconditionally convergent* if the series $\sum_{n=1}^\infty x_{n_i}$ converges for every choice of $n_1 < n_2 < n_3 < \cdots$. A Shauder basis $(x_n)_{n=1}^\infty$ for X is said to be *unconditional* if for every $x \in X$, its expansion in terms of the basis $\sum_{n=1}^\infty a_n x_n$ converges unconditionally.

Theorem 13.1 ([35]). *Let X have an unconditional basis. Then X admits an equivalent norm with an LUR dual norm whenever X admits an equivalent Fréchet differentiable norm.*

A biorthogonal system $\{x_i; f_i\}_{i \in I}$ in $X \times X^*$ $\big($i.e., $f_i(x_j) = \delta_{ij}$ (the Kronecker delta) for $i, j \in I\big)$ is called *fundamental* provided that $\overline{span}(x_i)_{i \in I} = X$. A fundamental biorthogonal system $\{x_i; f_i\}_{i \in I}$ is a *Markushevich* basis if $(f_i)_{i \in I}$ separates the points of X. A Markushevich basis $\{x_i; f_i\}_{i \in I}$ is called *shrinking* if $\overline{span}(f_i)_{i \in I} = X^*$. Clearly, every Schauder basis is Markushevich. An example of a Markushevich basis that is not a Schauder basis is the sequence of trigonometric polynomials $\{e^{i2\pi nt} : n = 0, \pm 1, \pm 2, ...\}$ in the $\widetilde{C}([0,1])$ of complex continuous functions on $[0,1]$ whose values at $0, 1$ are equal, with the sup-norm. If X^* is separable then X has a shrinking Markushevich basis [5, page 231].

A compact space K is called a *Corson compact space* if K is homeomorphic to a subset C of $[-1,1]^\Gamma$, for some set Γ, such that each point in C has only a countable number of nonzero coordinates. For example, any metrizable compact is a Corson compact or any weakly compact set in a Banach space is a Corson compact or the dual ball for a Vašák space in its weak* topology is a Corson compact [1, Ch. VI]. A Banach space X is called *weakly Lindelöf determined* (WLD) if $(B(X^*), w^*)$ is a Corson compact. Every Vašák space is WLD.

Theorem 13.2 ([14, page 211]). *For any space X the following are equivalent:*

(i) *X has a shrinking Markushevich basis.*

(ii) *X is WCG and Asplund.*

(iii) *X is WLD and Asplund.*

(iv) *X is WLD and admits an equivalent norm whose dual norm is LUR.*

(v) *X is WLD and admits an equivalent Fréchet differentiable norm.*

Theorem 13.3 ([28, 36]). *Let X have a fundamental biorthogonal system $\{x_i; f_i\}_{i \in I} \subseteq X \times X^*$. Then the subspace $Y = \mathrm{span}(x_i)_{i \in I}$ admits an equivalent LUR norm.*

Theorem 13.4 ([36]). *Let X have a fundamental biorthogonal system. Then X^* admits an equivalent norm with the weak* Mazur intersection property (every bounded weak* closed convex set can be represented as an intersection of closed dual balls).*

Proof. A dual Banach space has the weak* Mazur intersection property provided its predual has a dense set of LUR points. Let us consider a biorthogonal system $\{x_i; f_i\}_{i \in I} \subseteq X \times X^*$ such that $X = \overline{span}(x_i)_{i \in I}$ and put $Y = \mathrm{span}(x_i)_{i \in I}$. Using Theorem 13.3, we obtain an equivalent LUR norm $|.|$ on Y. Let $\|.\|$ be the norm $|.|$ extended to X. Then, the unit ball of $\|.\|$ is the closure of the unit ball of $|.|$. We claim that $\|.\|$ is LUR at each point of Y. Take $y \in Y \setminus \{0\}$ and a sequence $(x_n)_{n \in \mathbb{N}}$ in X so that $\lim_{n \to \infty} 2\|y\|^2 + 2\|x_n\|^2 - \|y + x_n\|^2 = 0$. If we choose $y_n \in Y$ with $\|y_n - x_n\| < \frac{1}{n}$, then $\lim_{n \to \infty} 2|y|^2 + 2|y_n|^2 - |y + y_n|^2 = 0$ and, hence, $\lim_{n \to \infty} \|x_n - y\| = \lim_{n \to \infty} |y_n - y| = 0$. $\qquad\square$

In the theorem above, in fact, we prove that every Banach space with a fundamental biorthogonal system admits an equivalent norm with a dense set of locally uniformly convex points.

Theorem 13.5 ([22]). *Let X^* be a dual Banach space with a fundamental biorthogonal system $\{x_i; f_i\}_{i \in I} \subseteq X^* \times X$. Then X admits an equivalent norm with the MIP.*

14   Some Interesting Problems

According to the author's knowledge and taste, the following problems in this area arise:

(Q1) If the space X has the Radon-Nikodym property (i.e., for every $\varepsilon > 0$ every bounded subset of X has a non-empty slice of diameter less than ε), does it follow that X admits an equivalent weak-Kadec-Klee norm? Does it admits an equivalent strictly convex norm? Is it true that X admits an equivalent LUR norm?

(Q2) Does every Asplund space admit an equivalent SSD norm? (If X admits an equivalent SSD norm, then X is Asplund (G. Godefroy)). Recall that the norm $\|.\|$ on X is called strongly subdifferentiale (SSD) if for each $x \in X$, the one-sided limit $\lim_{t \to 0^+} \frac{\|x+ty\|-\|x\|}{t}$ exists uniformly for y in $S(X)$. Note that the norm $\|.\|$ is Fréchet differentiable if and only if it is Gâteaux differentiable and at the same time SSD.

(Q3) Assume that a Banach space X admits an equivalent Gâteaux differentiable norm and that X admits also an equivalent SSD norm. Does X admit an equivalent Fréchet differentiable norm?

(Q4) Assume that X is a nonseparable non Asplund space. Does X admit an equivalent norm that is nowhere SSD except at the origin? For separable non Asplund space the answer is yes.

(Q5) Assume that the norm of a separable Banach space X has the property that its restriction to every infinite dimensional closed subspace $Y \subseteq X$ has a point of Fréchet differentiability on Y. Is then X^* necessarily separable?

(Q6) Assume that X is Vašák. Does X admit an equivalent norm that has the following property: $(f_n)_{n=1}^\infty$ is weak convergent to some $f \in B(X^*)$ whenever $f_n \in S(X^*)$ are such that $\lim_{n,m \to \infty} \|f_n + f_m\|^* = 2$?

(Q7) Assume that X has an unconditional basis and admits a Gâteaux differentiable norm. Does X admit a norm the dual norm of which is strictly convex?

(Q8) (Godefroy) Assume an Asplund space X has a Markushevich basis $\{x_i, f_i\}_{i \in I}$ with span$\{f_i\}_{i \in I}$ norming X^*. Is X WCG?

(Q9) Assume X admits an equivalent Fréchet differentiable norm. Does X admits an equivalent LUR norm?

(Q10) It is proved in [24] that every weakly uniformly convex Banach space is Lipschitz separated. Can a Lipschitz separated Banach space be equivalently renormed with a weakly uniformly convex norm? A related question is if Lipschitz separated Banach space necessarily an Asplund space?

(Q11) Is it true that an equivalent Fréchet differentiable norm in a subspace of a separable and reflexive Banach space can be extended to an equivalent Fréchet differentiable norm in the whole space?

(Q12) Assume that for every nonempty closed, bounded and convex subset A of X^* there exists $x \in X$ which attains its supremum on A. Is X Asplund?

(Q13) Is it true that an equivalent Fréchet differentiable norm in a subspace of a separable and reflexive Banach space can be extended to an equivalent Fréchet differentiable norm in the whole space?

(Q14) A separable Banach space X is reflexive if and only if X admits an equivalent 2-rotund norm. Is it true in general for nonseparable spaces?

(Q15) Assume that the norm of a separable Banach space X is such that the restriction of it to every subspace of X is Fréchet differentiable at a point. Must X^* be separable?

(Q16) Let X be a WLD space and X admits a Gâteaux differentiable norm. Does X admit a norm whose dual norm is strictly convex?

(Q17) Let X be a WLD space. Is every convex continuous function on X Gâteaux differentiable at some points?

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