Thai Journal of Mathematics
Volume 10 (2012) Number 3 : 473–480
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Meir and Keeler Type Fixed Point Theorem for Set-Valued Generalized Contractions in Metrically Convex Spaces

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Abstract : A fixed point theorem for generalized set-valued contraction in metrically convex spaces has been proved which generalizes a fixed point theorem due to Rhoades [B.E. Rhoades, A fixed point theorem for some non-self mappings, Math. Japonica. 23 (4) (1978) 457–459]. An illustrative example is also discussed.

Keywords : metrically convex metric spaces; non-self mappings; set-valued mappings; metric convexity; Meir-Keeler type condition.
2010 Mathematics Subject Classification : 54H25; 47H10.

1 Introduction

Meir and Keeler [1] established that classical Banach contraction principle remains true for weakly uniformly strict contractions:
Given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta \text{ implies } d(Tx, Ty) < \epsilon. \tag{1.1}$$

In recent years this result due to Meir and Keeler [1] has been generalized,

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extended and improved in various ways and by now there exists a considerable literature in this direction for self mappings. To mention a few we cite [2–8].

In this note, we establish a Meir and Keeler [1] type fixed point theorem for set-valued generalized contraction in metrically convex spaces. In proving our result we follow the definition and convention of Assad and Kirk [9] and Nadler [10]. Before formulating our result, for the sake of completeness we state the following result due to Rhoades [11].

Theorem 1.1. *Let (X, d) be a complete metrically convex metric space and K a nonempty closed convex subset of X. Let $T : K \to X$ be a map satisfying:*

$$d(Tx, Ty) \leq M(x, y)$$

where

$$M(x, y) = h \ \max\left\{\frac{1}{2}d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{q}\right\} \tag{1.2}$$

for all $x, y \in K$, with $x \neq y$, where $0 < h < 1$, $q \geq 1 + 2h$, and

(i) $Tx \in K$ for each $x \in \delta K$.

Then T has a fixed point in K.

We now state relevant definition and lemmas which are used in the sequel.

Definition 1.2 ([9])**.** A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 1.3 ([9])**.** *Let K be a nonempty closed subset of a metrically convex metric space X. If $x \in K$ and $y \notin K$ then there exists a point $z \in \delta K$ (the boundary of K) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

In what follows, $CB(X)$ denotes the set of all closed and bounded subsets of (X, d), while $C(X)$ for collection of all compact subsets of (X, d). Also H denotes the Hausdoraff distance between two sets.

Lemma 1.4 ([10])**.** *Let $A, B \in CB(X)$. Then for all $\epsilon > 0$ and $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$. If $A, B \in C(X)$, then one can choose $b \in B$ such that $d(a, b) \leq H(A, B)$.*

2 Main Results

We prove the following.

Theorem 2.1. *Let (X,d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $T : K \to C(X)$ be a set-valued map which satisfies (i) and for a given $\epsilon > 0$ there exists $\delta(\epsilon) > 0, \delta(\epsilon)$ being a nondecreasing function of ϵ with $q \geq 1 + 2h$ where $0 < h < 1$ such that*

$$\epsilon \leq M(x,y) < \epsilon + \delta \text{ implies } H(Tx,Ty) < \epsilon. \tag{2.1}$$

Then T has a fixed point in K.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{x'_n\}$ in the following way. Let $x_0 \in K$. Define $x'_1 \in Tx_0$. If $x'_1 \in K$ then set $x'_1 = x_1$. If $x'_1 \notin K$ choose $x_1 \in \delta K$ so that

$$d(x_0,x_1) + d(x_1,x'_1) = d(x_0,x'_1).$$

Then $x_1 \in K$. By using Lemma 1.4, select $x'_2 \in Tx_1$ such that $d(x'_1,x'_2) \leq H(Tx_0,Tx_1)$. If $x'_2 \in K$ then $x'_2 = x_2$. Otherwise choose $x_2 \in \delta K$ such that

$$d(x_1,x_2) + d(x_2,x'_2) = d(x_1,x'_2).$$

Thus by induction, one obtains two sequences $\{x_n\}$ and $\{x'_n\}$ such that

(ii) $x'_{n+1} \in Tx_n$

(iii) $d(x'_{n+1},x'_n) \leq H(Tx_n,Tx_{n-1})$.

(iv) $x'_{n+1} \in K \Rightarrow x'_{n+1} = x_{n+1}$,

(v) $x'_{n+1} \notin K \Rightarrow x_{n+1} \in \delta K$ and

$$d(x_n,x_{n+1}) + d(x_{n+1},x'_{n+1}) = d(x_n,x'_{n+1}).$$

Now define

$$P = \{x_i \in \{x_n\} : x'_i = x_i, i = 1,2,3,...\}$$
$$Q = \{x_i \in \{x_n\} : x'_i \neq x_i, i = 1,2,3,...\}.$$

Obviously, the two consecutive terms cannot lie in Q.

Now we distinguish the following three cases.

Case 1. If $x_n, x_{n+1} \in P$, then

$$\begin{aligned} d(x_n,x_{n+1}) = H(Tx_{n-1},Tx_n) &\leq M(x_{n-1},x_n) \\ &\leq h \ \max\left\{\frac{1}{2}d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n),\right. \\ &\qquad \left.\frac{d(x_{n-1},Tx_n) + d(x_n,Tx_{n-1})}{q}\right\}, \\ &\leq h \ \max\left\{\frac{1}{2}d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}),\right. \\ &\qquad \left.\frac{d(x_{n-1},x_{n+1}) + d(x_n,x_n)}{q}\right\}, \\ &\leq h \ \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}. \end{aligned}$$

If $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$ then we get $d(x_n, x_{n+1}) \leq d(x_n, x_{n+1})$, which is a contradiction. Otherwise, if $d(x_n, x_{n+1}) \leq h\ d(x_{n-1}, x_n)$ then one obtains $d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) \leq h\ d(x_{n-1}, x_n)$.

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$ then

$$d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1}),$$

which in turn yields

$$d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}).$$

Now, proceeding as in Case 1, we have

$$d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) \leq h\ d(x_{n-1}, x_n).$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$ then $x_{n-1} \in P$. Since x_n is a convex linear combination of x_{n-1} and x'_n, it follows that

$$d(x_n, x_{n+1}) \leq \max\{d(x_{n-1}, x_{n+1}), d(x_{n+1}, x'_n)\}.$$

Now, if $d(x_{n-1}, x_{n+1}) \leq d(x'_n, x_{n+1})$, then proceeding as in Case 1, one obtains

$$d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) \leq h\ d(x_{n-1}, x_n).$$

Otherwise if $d(x'_n, x_{n+1}) \leq d(x_{n-1}, x_{n+1})$, then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_{n-1}, x_{n+1}) = H(Tx_{n-2}, Tx_n) \leq M(x_{n-2}, x_n) \\ &\leq h\ \max\left\{\frac{1}{2}d(x_{n-2}, x_n), d(x_{n-2}, Tx_{n-2}), d(x_n, Tx_n), \right. \\ &\qquad \left. \frac{d(x_{n-2}, Tx_n) + d(x_n, Tx_{n-2})}{q}\right\} \\ &\leq h\ \max\left\{\frac{1}{2}d(x_{n-2}, x_n), d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), \right. \\ &\qquad \left. \frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{q}\right\}. \end{aligned}$$

Since

$$\frac{1}{2}d(x_{n-2}, x_n) = \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}.$$

Therefore, one obtains

$$\begin{aligned} d(x_n, x_{n+1}) \leq h\ \max&\left\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \right. \\ &\left. \frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{q}\right\} \end{aligned}$$

which in turn yields

$$d(x_n, x_{n+1}) \leq \begin{cases} h\ d(x_{n-1}, x_n), \text{ if } d(x_{n-1}, x_n) \geq d(x_{n-2}, x_{n-1}) \\ \\ h\ d(x_{n-2}, x_{n-1}), \text{ if } d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1}). \end{cases}$$

Thus in all the cases, we have

$$d(x_n, x_{n+1}) \leq h\ \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}.$$

It can be easily shown by induction that for $n \geq 1$, we have

$$d(x_n, x_{n+1}) \leq h\ \max\{d(x_0, x_1), d(x_1, x_2)\}.$$

Thus $d(x_n, x_{n+1})$ is a decreasing sequence and tending to $t \in [0, \infty)$ as $n \to \infty$.

Let on contrary

$$d(x_n, x_{n+1})\ >\ t \text{ for } \ n = 0, 1, 2... . \tag{2.2}$$

Suppose $t > 0$. Then there exists a $\delta = \delta(\epsilon)$ and a positive integer k such that $t \leq d(x_k, x_{k+1}) < \delta + t$. Hence by (2.1), one obtains

$$d(x_{k+1}, x_{k+2}) = d(Tx_k, Tx_{k+1}) < t,$$

which contradicts (2.2) therefore $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Now we wish to show that the sequence $\{x_n\}$ is Cauchy. If it is not Cauchy then there exists $2\epsilon > 0$ such that $d(x_m, x_n) > 2\epsilon$. Choose $\delta > 0$ with $\delta < \epsilon$ for which (2.1) is satisfied. Since $d(x_n, x_{n+1}) \to 0$ there exists a positive integer $N = N(\delta)$ such that $d(x_i, x_{i+1}) \leq \frac{\delta}{6}$ for all $i \geq N$. With this choice of N, let us choose m, n with $m > n > N$ such that

$$d(x_m, x_n) \geq 2\epsilon > \epsilon + \delta. \tag{2.3}$$

By (2.3), $m - n > 6$, otherwise

$$d(x_m, x_n) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+4}, x_{n+5}) \quad \leq \frac{5\delta}{6} < \delta,$$

a contradiction. Now suppose that $d(x_n, x_{m-1}) \leq \epsilon + \frac{\delta}{3}$. Then

$$d(x_n, x_m) \leq d(x_n, x_{m-1}) + d(x_{m-1}, x_m) \leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} \quad < \epsilon + \delta,$$

a contradiction. Similarly, suppose $d(x_n, x_{m-2}) \leq \epsilon + \frac{\delta}{3}$. Then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{m-2}) + d(x_{m-2}, x_{m-1}) + d(x_{m-1}, x_m) \\ &\leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} < \epsilon + \delta. \end{aligned}$$

Let for the smallest integer $j \in (m, n)$ with $d(x_n, x_j) > \epsilon + \frac{\delta}{3}$, whereas

$$d(x_n, x_j) \leq d(x_n, x_{j-1}) + d(x_{j-1}, x_j) \leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < \epsilon + \frac{2\delta}{3}.$$

Thus there exists a $j \in (n, m)$ such that

$$\epsilon + \frac{\delta}{3} < d(x_n, x_j) < \epsilon + \frac{2\delta}{3}.$$

Then

$$\begin{aligned} d(x_n, x_j) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{j+1}) + d(x_{j+1}, x_j) \\ &\leq \frac{\delta}{6} + \epsilon + \frac{\delta}{6} = \epsilon + \frac{\delta}{3}, \end{aligned}$$

which is indeed a contradiction, therefore one may conclude that the sequence $\{x_n\}$ is Cauchy and it converges to a point z in X.

Now, we assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is contained in P. Using (2.1), one can write

$$H(Tx_{n_{k-1}}, Tz) \leq h \ \max\left\{\frac{1}{2}d(x_{n_{k-1}}, z), d(x_{n_{k-1}}, Tx_{n_{k-1}}), d(z, Tz), \frac{d(z, Tx_{n_{k-1}}) + d(x_{n_{k-1}}, Tz)}{q}\right\}$$

which on letting $k \to \infty$ we get $H(Tz, z) \leq hd(Tz, z)$, yielding thereby $z \in Tz$. This completes the proof. □

Remark 2.2. *By setting* $\delta(\epsilon) = \frac{2(1-h)\epsilon}{h}, 0 < h < 1$ *in the Theorem 2.1 then* $\delta(\epsilon)$ *is nondecreasing function of* $\epsilon > 0$, *one obtains*

$$\epsilon^{'} < \epsilon = \epsilon^{'} + \frac{1}{2}\delta(\epsilon^{'}) < \epsilon^{'} + \delta(\epsilon^{'})$$

by choosing $\epsilon^{'} = h\epsilon$. *The condition* (2.1) *of Theorem 2.1 reduces to* (1.2) *due to Rhoades [11].*

Finally, we furnish an example to discuss the validity of the hypotheses of Theorem 2.1 proved in this note which also establish the genuineness of our result.

Example 2.3. *Let* $X = R$ *with Euclidean metric and* $K = [0, 16] \cup \{-4\}$. *Define* $T : K \to X$ *as*

$$Tx = \begin{cases} -\frac{x}{4}, & \text{if } 0 \leq x \leq 16 \\ 1, & \text{if } x = -4. \end{cases}$$

Since δK*(boundary of* K*)* $= \{-4, 0, 16\}$*. Also* $-4 \in \delta K \Rightarrow T(-4) = 1 \in K$*,* $0 \in \delta K \Rightarrow T0 = 0 \in K$*,* $16 \in \delta K \Rightarrow T16 = -4 \in K$*. Moreover, if* $0 \leq x, y \leq 16$*, then*

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{4}|x-y| = \frac{1}{2}\left(\frac{1}{2}d(x,y)\right) \\ &\leq h \ \max\left\{\frac{1}{2}d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{q}\right\}. \end{aligned}$$

Next, if $x \in [0, 16]$ *and* $y = -4$ *then*

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{4}|x+y| = \frac{1}{2}\left(\frac{1}{2}d(x,y)\right) \\ &\leq h \ \max\left\{\frac{1}{2}d(x,y), d(x,Tx), (y,Ty), \frac{d(x,Ty)+d(y,Tx)}{q}\right\} \end{aligned}$$

which shows that the contraction condition (2.1) *is satisfied for every* $x, y \in K$*. Thus all the conditions of the Theorem 2.1 are satisfied and* 0 *is the fixed point of* T*.*

Acknowledgement : Authors express their sincere thanks to the learned referee for their critical reading and his kind suggestions towards the improvement of entire manuscript.

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(Received 23 February 2011)
(Accepted 13 January 2012)