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L_p-Approximation by Bézier Variant of Certain Summation-Integral Type Operators

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Abstract : The problem of L_p-aproximation has been received consideration of many authors. In this paper we establish the direct and inverse theorems for the Bézier variant $M_{n,\alpha}$ of certain summation-integral type operators M_n in L_p-norm using Ditzian-Totik modulus of smoothness. These operators include the well known Baskakov-Durrmeyer and Szász-Durrmeyer type operators as special cases.

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1 Introduction

For obvious reasons, summation type operators as such are not L_p- approximation methods. Nevertheless, several linear positive operators of summation type have been appropriately modified to become L_p-approximation method. The underlying idea behind such a modification is to replace, in the expression for the operator, the function value at a nodal point by an average value (in the sense of integration) of the function in an appropriate neighborhood of the point. The first such modification was made by Kantorovich [1] for the case of Bernstein polynomials. Another modification of Bernstein polynomials was introduced by Durrmeyer [2] and later studied extensively by Derrienic [3]. In 2000, Zeng and Chen [4]

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introduced the Durrmeyer-Bézier operators $D_{n,\alpha}$ as follows:

$$D_{n,\alpha} = (n+1)\sum_{k=0}^{1} Q_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k}(t)f(t)\,dt,$$

for f defined on $[0,1]$. where $p_{n,k}(x)$ are well known Bernstein basis functions and $Q_{n,k}^{(\alpha)}(x)$ are the Bézier basis functions introduced by Bézier [5]. The authors of [4] studied the rate of convergence of the operators $D_{n,\alpha}$ for functions of bounded variation. In the sequel Bézier variant of some well known operators were introduced (cf. [4, 6, 7]) and their rates of convergence for bounded variation functions have been investigated (cf. [8, 9]). In order to approximate Lebesgue integrable functions on the interval $[0,\infty)$, Gupta and Mohapatra [10] considered the operators

$$M_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x,c) \int_0^{\infty} b_{n,k}(t,c)f(t)\,dt,$$

where $p_{n,k}(x,c) = (-1)^k \frac{x^k}{k!}\varphi_{n,c}^{(k)}(x),\ b_{n,k}(x,c) = (-1)^{k+1}\frac{x^k}{k!}\varphi_{n,c}^{(k+1)}(x),\ x \in [0,\infty)$ and

$$\varphi_{n,c}(x) = \begin{cases} (1+cx)^{-n/c};\ c>0 \\ e^{-nx};\ c=0. \end{cases}$$

For $c>0$, the operators M_n reduce to Baskakov-Durrmeyer operators and when $c=0$ these become Szász-Durrmeyer type operators. Some approximation properties of these operators were studied in [11]. The rate of convergence by the operators M_n for the particular value $c=1$ was studied in [12].

Let $L_p[0,\infty)$ be the class of all $p-$Lebesgue integrable functions on the positive real line. For $f \in L_p[0,\infty)$, introducing the Bézier basis functions $Q_{n,k}^{\alpha}(x,c) = J_{n,k}^{\alpha}(x,c) - J_{n,k+1}^{\alpha}(x,c)$ $\alpha \geq 1$, the Bézier variant $M_{n,\alpha}$ of the operators M_n is defined by

$$M_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} Q_{n,k}^{\alpha}(x,c) \int_0^{\infty} b_{n,k}(t,c)f(t)\,dt,$$

where $J_{n,k}(x,c) = \sum_{\nu=k}^{\infty} p_{n,\nu}(x,c)$. For $\alpha = 1$, the operators $M_{n,\alpha}$ reduce to the operators M_n.

In order to make the paper self contained we give definitions of the unified $K-$functional and the Ditzian-Totik modulus of smoothness used in this paper.

Let $f \in L_p[0,\infty)$, $\varphi(x) = \sqrt{x(1+cx)}, 0<\lambda<1$, then

$$\begin{aligned}\omega_{\varphi^\lambda}(f,t)_p &= \sup_{\substack{0<h\leqslant t\\ x-h\varphi^\lambda(x)/2\geqslant 0}} \left\|\tilde{\Delta}_{h\varphi^\lambda(x)}f(x)\right\|_p \\ &= \sup_{\substack{0<h\leqslant t\\ x-h\varphi^\lambda(x)/2\geqslant 0}} \left\|f\left(x+\frac{h\varphi^\lambda(x)}{2}\right) - f\left(x-\frac{h\varphi^\lambda(x)}{2}\right)\right\|_p,\end{aligned}$$

where $\varphi(x)$ is an admissible weight function of Ditzian-Totik modulus of smoothness. The corresponding $K-$ functional is defined as

$$K_{\varphi^\lambda}(f,t)_p = \inf_{g\in W_\lambda} \left\{\|f-g\|_p + t\|\varphi^\lambda g'\|_p\right\}$$

where $W_\lambda = \left\{g : g \in AC_{\text{loc}}, \|\varphi^\lambda g'\|_p < \infty, \|g'\|_p < \infty\right\}$ and by AC_{loc} we mean the class of functions absolutely continuous on every finite subset of $[0,\infty)$. Moreover, the following equivalence is well known (cf.[13])

$$\omega_{\varphi^\lambda}(f,t)_p \sim K_{\varphi^\lambda}(f,t)_p$$

i.e. there exists constants $C_1, C_2 > 0$ such that $C_1\omega_{\varphi^\lambda}(f,t)_p \le C_2 K_{\varphi^\lambda}(f,t)_p$.

In Section 2, we give some lemmas which will be used in our main theorems. Subsequently, in Section 3 we establish our main theorem. The constant M is not the same at each occurrence.

2 Preliminaries

Lemma 2.1 ([14])**.** *For the functions $J_{n,k}(x,c)$ and $Q^\alpha_{n,k}(x,c)$, we have*

1. $1 = J_{n,0}(x,c) > J_{n,1}(x,c) > \cdots > J_{n,k}(x,c) > J_{n,k+1}(x) > \cdots$
2. $0 < Q^\alpha_{n,k}(x,c) < \alpha\, p_{n,k}(x,c),\ \alpha \ge 1,$
3. $M'_{n,\alpha}(1,x) = 0$
4. $\left|M'_{n,\alpha}(f,x)\right| \le \alpha\Big|\sum_{k=0}^\infty \left(J^{\alpha-1}_{n,k}(x,c) - J^{\alpha-1}_{n,k+1}(x)\right)J'_{n,k+1}(x) \times$
   $\times \int_0^\infty f(t)b_{n,k}(t,c)\,dt + M'_n(f,x)\Big|.$

The following Lemma is due to Berens and Lorentz:

Lemma 2.2 ([15])**.** *Let Ω be monotone increasing on $[0,c]$. Then $\Omega(t) = O(t^\alpha)$, $t \to 0+$, if for some $0 < \alpha < r$ and all $h, t \in [0,c]$*

$$\Omega(h) < M\left[t^\alpha + (h/t)^r\Omega(t)\right].$$

We prove a Bernstein type lemma for the operators $M_{n,\alpha}$ which is useful while establishing the inverse theorem.

Lemma 2.3. *If $f \in L_p[0,\infty), 1 \le p \le \infty$ $\varphi(x) = \sqrt{x(1+cx)}$, and $0 < \lambda < 1$, then there holds*

1. $\|\varphi^\lambda M'_{n,\alpha}(f,.)\|_p \le M\,\alpha\,\|\varphi^\lambda f'\|_p$ *and*
2. $\|\varphi^\lambda M'_{n,\alpha}(f,.)\|_p \le M\,\alpha\, n^{1-\lambda/2}\|f\|_p,$

where $M = M(c,\lambda)$ is independent of f and n.

Proof. The result for $p=\infty$ has been established in [14]. Since, the operators $M_{n,\alpha}$ are bounded, therefore in view of Riesz-Thorin interpolation theorem (see [16, p.231–233]) the lemma is proved for all $1\le p\le\infty$ if it is also proved for $p=1$. In view of Lemma 2.1, we get

$$M'_{n,\alpha}(f,x):=[E_1+E_2],\ \text{say},$$

where

$$E_1=\alpha\left|\sum_{k=0}^{\infty}\left(J_{n,k}^{\alpha-1}(x,c)-J_{n,k+1}^{\alpha-1}(x)\right)J'_{n,k+1}(x)\int_0^{\infty}\left(\int_x^t f'(u)\,du\right)b_{n,k}(t,c)\,dt\right|$$

and

$$E_2=M'_n\left(\int_x^t f'(u)\,du,x\right).$$

Now, we estimate E_1 as follows

$$\begin{aligned}E_1&=\alpha\left|\sum_{k=0}^{\infty}\left(J_{n,k}^{\alpha-1}(x,c)-J_{n,k+1}^{\alpha-1}(x)\right)J'_{n,k+1}(x)\int_0^{\infty}\left(\int_x^t f'(u)\,du\right)b_{n,k}(t,c)\,dt\right|\\&\le\alpha\left|\sum_{k=0}^{\infty}p_{n,k}(x,c)J'_{n,k+1}(x)\int_0^{\infty}\left(\int_x^t f'(u)\,du\right)b_{n,k}(t,c)\,dt\right|\\&\le\alpha\|\varphi^{\lambda}f'\|_1\left|\sum_{k=0}^{\infty}p_{n,k}(x,c)J'_{n,k+1}(x)\int_0^{\infty}\left(\varphi^{-\lambda}(x)+\frac{t^{\lambda/2}}{x^{\lambda/2}}\varphi^{-\lambda}(t)\right)b_{n,k}(t,c)\,dt\right|.\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi^{\lambda}(x)E_1&\le\alpha\|\varphi^{\lambda}f'\|_1\left|\sum_{k=0}^{\infty}p_{n,k}(x,c)J'_{n,k+1}(x)\int_0^{\infty}\left(1+\frac{(1+cx)^{\lambda/2}}{(1+ct)^{\lambda/2}}\right)b_{n,k}(t,c)\,dt\right|\\&=\alpha\|\varphi^{\lambda}f'\|_1\,[F_1+F_2]\ \text{say},\end{aligned}$$

where F_1 and F_2 are the corresponding to two terms under the integral sign. Now,

$$\begin{aligned}F_1&=\sum_{k=0}^{\infty}p_{n,k}(x,c)J'_{n,k+1}(x)\int_0^{\infty}b_{n,k}(t,c)\,dt\\&=\frac{1}{x}\sum_{k=0}^{\infty}(k+1)p_{n,k}(x,c)p_{n,k+1}(x,c)\\&=\frac{c^3(1+cx)^{-3-\frac{2n}{c}}}{\Gamma^2(n/c)}\sum_{k=0}^{\infty}\left(\frac{cx}{1+cx}\right)^{2k}\frac{\Gamma(k+n/c)\Gamma(k+n/c+1)}{(k!)^2}\\&=M\frac{n}{c}(1+cx)^{-3-\frac{2n}{c}}\end{aligned}$$

which implies $\|F_1\|_1 = Mn\int_0^\infty (1+cx)^{-3-\frac{2n}{c}}\,dx = MO(1),\ M = M(c,\lambda)$. In view of the convergence of the integral $\int_0^\infty (1+ct)^{-\lambda/2} b_{n,k}(t,c)\,dt$ for $0<\lambda<1$, we get

$$\begin{aligned}
F_2 &= \sum_{k=0}^{\infty} p_{n,k}(x,c) J'_{n,k+1}(x) x^{-\lambda/2} \int_0^\infty (1+ct)^{-\lambda/2} b_{n,k}(t,c)\,dt \\
&= \sum_{k=0}^{\infty} p_{n,k}(x,c) J'_{n,k+1}(x) x^{-\lambda/2} \frac{c^{k+2}\Gamma(1+n/c+\lambda/2)\Gamma(k+1+n/c)}{\Gamma(n/c)\Gamma(k+2+n/c+\lambda/2)} \\
&= \frac{\Gamma(1+n/c+\lambda/2)c^5}{\Gamma^3(n/c)} \sum_{k=0}^{\infty} \frac{(k+n/c)\Gamma^3(k+n/c)x^{k+1-\lambda/2}}{(k!)^2(1+cx)^{2k+3+2n/c}\Gamma(k+2+n/c+\lambda/2)}.
\end{aligned}$$

Above series is convergent as follows easily from Raabe's test. Moreover, taking n/c common does not affect the convergence of the series. Thus, for large values of n we use Stirling's asymptotic formula and obtain the estimate

$$\begin{aligned}
\|F_2\|_1 &\le M\frac{\Gamma(1+n/c+\lambda/2)}{\Gamma^3(n/c)} \sum_{k=0}^{\infty} \frac{\left((k+n/c-1)^{(k+n/c-1/2)} e^{-k-n/c+1)}\right)^3}{(k+n/c+1+\lambda/2)^{k+n/c+3/2+\lambda/2}} \\
&\quad \times \frac{(k+n/c)(k+2n/c+\lambda/2)^{k+2n/c+1/2+\lambda/2} e^{-k-2n/c-\lambda/2}}{e^{-k-n/c-1-\lambda/2}(k!)^2(2k+2n/c+2)^{2k+2n/c+5/2}e^{-2k-2n/c-2}} \\
&\le M\frac{\Gamma(1+n/c+\lambda/2)}{\Gamma^3(n/c)} \frac{n^{2n/c-4}}{e^{2n/c+6}} \sum_{k=0}^{\infty} \left(\frac{n}{e}\right)^k \frac{1}{(k!)^2} \\
&\le M\frac{(n/c+\lambda/2)^{n/c+\lambda/2+1/2} e^{-n/c-\lambda/2} n^{2n/c-4}}{\left((n/c-1)^{n/c-1/2}e^{-n/c+1}\right)^3 e^{2n/c+6}} \\
&\le Mn^{\lambda/2-2} = MO\left(n^{-3/2}\right).
\end{aligned}$$

Similarly, it follows by direct calculations that $\|E_2\|_p \le C\,\|\varphi^\lambda f'\|_p$. Collecting E_1, E_2 the lemma follows for $p=1$.

Now, in order to establish second inequality we again use Lemma 2.1. Thus, we get

$$\begin{aligned}
&|\varphi^\lambda(x) M'_{n,\alpha}(f,x)| \\
&\quad \le \sum_{k=0}^{\infty} \left| \int_0^\infty b_{n,k}(t,c) f(t)\,dt \right| \varphi^\lambda(x) J'_{n,k+1}(x)\,dx \left\{ J_{n,k}^{\alpha-1}(x,c) - J_{n,k+1}^{\alpha-1}(x) \right\} \\
&\qquad + \sum_{k=0}^{\infty} \left| \int_0^\infty b_{n,k}(t,c) f(t)\,dt \right| \varphi^\lambda(x) J_{n,k}^{\alpha-1}(x,c) \left| p'_{n,k}(x,c) \right| dx \\
&\quad = A_1 + A_2,\ \text{say}.
\end{aligned}$$

Since $J'_{n,k+1}(x) = \frac{k+1}{x}p_{n,k+1}(x,c)$, we have

$$\begin{aligned}\|A_1\|_1 &\leq M\sum_{k=0}^{\infty}\Big\|p_{n,k}(x,c)J'_{n,k+1}(x)\varphi^{\lambda}(x)\Big\|_1\Big|\int_0^{\infty} b_{n,k}(t,c)f(t)\,dt\Big| \\ &\leq M\|f\|_1\sum_{k=0}^{\infty}\Big\|p_{n,k}(x,c)\frac{k+1}{x}p_{n,k+1}(x,c)\varphi^{\lambda}(x)\Big\|_1 \\ &\leq M\|f\|_1\frac{c^3\Gamma(2+2n/c-\lambda)}{\Gamma^2(n/c)}\sum_{k=0}^{\infty}\frac{c^{2k}(k+n/c)\Gamma^2(k+n/c)\Gamma(2k+1+\lambda/2)}{(k!)^2\Gamma(2k+3+2n/c-\lambda/2)}.\end{aligned}$$

The series on the right is convergent. We apply Stirling's asymptotic formula $\Gamma(s+1)\simeq\sqrt{2\pi}\,s^{s+1/2}e^{-s}$, to obtain

$$\begin{aligned}&\|A_1\|_1 \\ &\leq M\|f\|_1\sum_{k=0}^{\infty}\frac{c^{2k}(k+n/c)\Gamma(2k+1+\lambda/2)\Big((k+n/c-1)^{k+n/c-1/2}e^{-k-n/c+1}\Big)^2}{(k!)^2\Big((n/c-1)^{n/c-1/2}e^{-n/c+1}\Big)^2} \\ &\quad\times\frac{(1+2n/c-\lambda)^{3/2+2n/c-\lambda}e^{-1-2n/c+\lambda}}{(2k+2+2n/c-\lambda/2)^{2k+5/2+2n/c-\lambda/2}e^{-2k-2-2n/c+\lambda/2}} \\ &\leq M\|f\|_1 n^{-\lambda/2},\ M=M(c,\lambda).\end{aligned}$$

We have $\varphi^2(x)p'_{n,k}(x,c) = (n+c)(\frac{k}{n+c}-x)p_{n,k}(x,c)$ and $\sum_{k=0}^{\infty}(\frac{k}{n+c}-x)p_{n,k}(x,c) = \frac{-[n(1+cx)+c^2x]}{(n+c)(1+cx)}$, Therefore,

$$\begin{aligned}\|A_2\|_1 &\leq M\|f\|_1\Big(\sum_{k=0}^{\infty}n(1+cx)\varphi^{\lambda-2}p_{n,k}(x,c)+\sum_{k=0}^{\infty}c^2x\varphi^{\lambda-2}p_{n,k}(x,c)\Big) \\ &= A_3+A_4,\ \text{say}.\end{aligned}$$

We obtain estimate for A_3 as

$$\begin{aligned}\|A_3\|_1 &\leq M\frac{n\|f\|_1}{\Gamma(n/c)}\sum_{k=0}^{\infty}\frac{\Gamma(k+n/c)}{k!}c^{k+1}\int_0^{\infty}\frac{x^{k+\lambda/2-1}}{(1+cx)^{k+1+n/c-\lambda/2}}\,dx \\ &\leq M\frac{n\|f\|_1}{\Gamma(n/c)}\sum_{k=0}^{\infty}\frac{(n/c+k-1)^{n/c+k-1/2}e^{-2n/c-k+\lambda+1/2}c^{k+1}\Gamma(k+\lambda/2)}{k!(k+n/c-\lambda/2)^{k+n/c-\lambda/2+1/2}e^{-k-n/c+\lambda/2}} \\ &\leq M\frac{n\|f\|_1 n^{n/c-\lambda/2-1/2}}{(n/c-1)^{n/c-1/2}} \\ &\leq M\|f\|_1 n^{1-\lambda/2}.\end{aligned}$$

Similarly, $\|A_3\|_1 \leq M\|f\|_1 n^{1-\lambda/2}$. Combining these estimates the second inequality (ii) is established. □

3 Main Results

The main result of the present paper is the following:

Theorem 3.1. *Let $f \in L_p[0,\infty)$, $1 \leq p \leq \infty$, $\varphi(x) = \sqrt{x(1+cx)}$, $0 < \lambda < 1, c \geq 0$ and $0 < \gamma < 1$. Then, there holds the implication (i) $\Leftrightarrow$ (ii) in the following statements:*

1. $\left\|M_{n,\alpha}(f) - f\right\|_p = O\left(n^{\lambda/2-1}\right)^{\gamma}$
2. $\omega_{\varphi^\lambda}(f,x)_p = O(x^\gamma)$.

Proof. **Direct Part:**
It is sufficient to consider the case $p = 1$ only . The case $p = \infty$ was studied in [14]. The result for $1 \leq p \leq \infty$ then follows from Riesz-Thorin interpolation theorem [16]. By the definition of $K_{\varphi^\lambda}(f,t)$ for fixed n, x, λ, we can choose $g = g_{n,x,\lambda} \in W_\lambda$ such that

$$\|f-g\|_p + \frac{\alpha}{n^{1-\lambda/2}}\left\|\varphi^\lambda g'\right\|_p \leq K_{\varphi^\lambda}\left(f, \frac{\alpha}{n^{1-\lambda/2}}\right)_p .$$

Since, $M_{n,\alpha}$ is constant preserving, we can write

$$\|M_{n,\alpha}(f,x) - f(x)\|_p \leq C\|f-g\|_p + \|M_{n,\alpha}(g,x) - g(x)\|_p. \tag{3.1}$$

For the case $p = \infty$ the result has been established in [14]. We take the two term Taylor's expansion $g(t) = g(x) + R(g,t,x)$, where $R(g,t,x) = \int_x^t g'(\tau)\,d\tau$.

$$\begin{aligned}
&|M_{n,\alpha}(g,x) - g(x)| \\
&\quad\leq \alpha \sum_{k=0}^{\infty} p_{n,k}(x,c) \int_0^\infty \left|\int_x^t g'(u)\,du\right| b_{n,k}(t)\,dt \\
&\quad\leq \alpha \sum_{k=0}^{\infty} p_{n,k}(x,c) \int_0^\infty \left(\frac{1}{\varphi^\lambda(x)} + \frac{1}{(x(1+ct))^{\lambda/2}}\right) b_{n,k}(t)\,dt \left|\int_x^t \varphi^\lambda(u) g'(u)\,du\right| \\
&\quad\leq \alpha\|\varphi^\lambda g'\|_1 \sum_{k=0}^{\infty} p_{n,k}(x,c) \int_0^\infty \left(\frac{1}{\varphi^\lambda(x)} + \frac{1}{(x(1+ct))^{\lambda/2}}\right) b_{n,k}(t)\,dt \\
&\quad= J_1 + J_2, \text{ say}
\end{aligned}$$

where, J_1, J_2 are two terms corresponding to two terms in above integral. Now, in view of $\int_0^\infty b_{n,k}(t,c)\,dt = 1$ and the convergence of the integral $\int_0^\infty p_{n,k}(x,c)\varphi^{-\lambda}(x)\,dx$ for $0 < \lambda < 1$, we get

$$\begin{aligned}
\|J_1\|_1 &\leq \alpha\|\varphi^\lambda g'\|_1 \sum_{k=0}^{\infty} \int_0^\infty p_{n,k}(x,c)\varphi^{-\lambda}(x)\,dx \\
&\leq \alpha\|\varphi^\lambda g'\|_1 \frac{\Gamma(n/c+\lambda)}{\Gamma(n/c)} \sum_{k=0}^{\infty} \frac{\Gamma(k+1-\lambda/2)\Gamma(k+n/c)}{\Gamma(k+n/c+\lambda/2+1)k!}
\end{aligned}$$

$$\leq M\alpha\|\varphi^{\lambda}g'\|_1 \frac{\Gamma(n/c+\lambda)}{\Gamma(n/c)} \sum_{k=0}^{\infty} \frac{\Gamma(k+1-\lambda/2)}{k!}$$
$$\times \frac{(k+n/c-1)^{(k+n/c-1/2)}e^{-k-n/c+1}}{(k+n/c+\lambda/2)^{(k+n/c+1/2+\lambda/2)}e^{-k-n/c+\lambda/2}}$$
$$\leq M\alpha\|\varphi^{\lambda}g'\|_1 \frac{(n/c+\lambda-1)^{n/c+\lambda-1/2}n^{-1-\lambda/2}e^{-n/c-\lambda+1}}{(n/c-1)^{n/c-1/2}e^{-n/c+1}}$$
$$\leq \frac{M\alpha\|\varphi^{\lambda}g'\|_1}{n^{1-\lambda/2}}, \quad M = M(c,\lambda).$$

In order to find estimate for J_2 we proceed as in the estimate of F_2 in Lemma 2.3. Thus, we obtain

$$|J_2| \leq \alpha\|\varphi^{\lambda}g'\|_1 \sum_{k=0}^{\infty} p_{n,k}(x,c)x^{-\lambda/2} \int_0^{\infty} (1+ct)^{-\lambda/2} b_{n,k}(t,c)\,dt$$
$$\leq \alpha\|\varphi^{\lambda}g'\|_1 \sum_{k=0}^{\infty} p_{n,k}(x,c)x^{-\lambda/2} \frac{\Gamma(k+1+n/c)c^{k+2}\Gamma(1+n/c+\lambda/2)}{\Gamma(n/c)\Gamma(k+2+n/c+\lambda/2)}$$

Therefore,

$$\|J_2\|_1 \leq \alpha\|\varphi^{\lambda}g'\|_1 \sum_{k=0}^{\infty} \frac{\Gamma(k+1+n/c)c^{k+2}\Gamma(1+n/c+\lambda/2)}{\Gamma(n/c)\Gamma(k+2+n/c+\lambda/2)} \int_0^{\infty} p_{n,k}(x,c)x^{-\lambda/2}\,dx$$
$$\leq \alpha\|\varphi^{\lambda}g'\|_1 \frac{(n/c+\lambda/2)\Gamma^2(n/c+\lambda/2)}{\Gamma^2(n/c)} \sum_{k=0}^{\infty} \frac{\Gamma(k+n/c)\Gamma(k+1-\lambda/2)}{\Gamma(k+1)\Gamma(k+2+n/c+\lambda/2)}$$
$$\leq M\alpha\|\varphi^{\lambda}g'\|_1 \frac{(n/c+\lambda/2)\Gamma^2(n/c+\lambda/2)}{\Gamma^2(n/c)} \sum_{k=0}^{\infty} n^{-\lambda/2-2\frac{\Gamma(k+1-\lambda/2)}{\Gamma(k+1)}}$$
$$\leq M\alpha\|\varphi^{\lambda}g'\|_1 \frac{1}{n^{1-\lambda/2}}.$$

Combining the estimates for J_1 and J_2, we get following

$$\|M_{n,\alpha}(g,x)-g(x)\|_1 \leq M\alpha \frac{1}{n^{1-\lambda/2}} \|\varphi^{\lambda}g'\|_1$$

which on substituting in (3.1) gives

$$\|M_{n,\alpha}(f,x)-f(x)\|_1 \leq M\omega_{\varphi^{\lambda}}\Big(f, \frac{\alpha}{n^{1-\lambda/2}}\Big).$$

Inverse Part:
We make use of the weighted Steklov type average function S_δ defined as follows

$$S_\delta(x) := \frac{1}{\delta\varphi^{\lambda}(x)} \int_{\frac{-\delta}{2}\varphi^{\lambda}(x)}^{\frac{\delta}{2}\varphi^{\lambda}(x)} f(x+u)\,du, \quad 0<\lambda<1.$$

Then, it follows that

1. $\|S_\delta - f\|_1 \leq \omega_{\varphi^\lambda}(f,\delta)_1$;
2. $\|S'_\delta\|_1 \leq \delta^{-1}\omega_{\varphi^\lambda}(f,\delta)_1$.

(see [17, p. 117]). We get

$$\left\|\tilde{\Delta}_{h\varphi^\lambda(x)} f(x)\right\|_1 \leq \left\|\tilde{\Delta}_{h\varphi^\lambda(x)}\left(f(x) - M'_{n,\alpha}(f,x)\right)\right\|_1 + \left\|\tilde{\Delta}_{h\varphi^\lambda(x)} M'_{n,\alpha}(,x)\right\|_1$$
$$\leq M\left(n^{\frac{\lambda}{2}-1}\right)^\gamma + \left\|\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} M'_{n,\alpha}(f - S_\delta, x+u)\,du\right\|_1$$
$$+ \left\|\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} M'_{n,\alpha}(S_\delta, x+u)\,du\right\|_1.$$

Using Bernstein type inequalities, and properties of S_δ functon, we get the estimate

$$\left|\tilde{\Delta}_{h\varphi^\lambda(x)} M_{n,\alpha}(f,x)\right| \leq M\left(n^{\frac{\lambda}{2}-1}\right)^\gamma + h\varphi^\lambda(x)\left(|M'_{n,\alpha}(f - S_\delta, x)| + |M'_{n,\alpha}(S_\delta, x)|\right)$$

Therefore,

$$\left\|\tilde{\Delta}_{h\varphi^\lambda(x)} M_{n,\alpha}(f,x)\right\|_1 \leq M\left(n^{\frac{\lambda}{2}-1}\right)^\gamma + \left(\frac{h}{n^{\lambda/2-1}}\right)\left(\|f - S_\delta\|_1 + \frac{1}{n^{1-\lambda/2}}\|\varphi^\lambda S'_\delta\|_1\right)$$
$$\omega_{\varphi^\lambda}(f,h) \leq M\left(n^{\frac{\lambda}{2}-1}\right)^\gamma + \left(\frac{h}{n^{\lambda/2-1}}\right)\omega_{\varphi^\lambda}\left(f, \frac{1}{n^{1-\lambda/2}}\right).$$

Using Lemma 2.2, we finally get $\omega_{\varphi^\lambda}(f,x)_1 = O\left(x^\gamma\right),\ 0 < \gamma < 1$. This completes the proof. □

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