Thai Journal of Mathematics
Volume 10 (2012) Number 2 : 453–462
**http://thaijmath.in.cmu.ac.th**
ISSN 1686-0209

# Local Higher Derivations

**Tayebeh Lal Shateri**

Faculty of Mathematics and Computer Sciences
Hakim Sabzevari University, Sabzevar, Iran
e-mail : t.shateri@gmail.com

**Abstract :** In this paper, we study local higher derivations on some algebras generated by their idempotents. We prove that the space of all bounded higher derivations from these algebras into a unital algebra is reflexive.

**Keywords :** Subspace lattice; Topological algebra; Derivation; Higher derivation; Local higher derivation.
**2010 Mathematics Subject Classification :** 47Bxx.

## 1 Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. By as higher derivation of rank $j$ ($j$ might be $\infty$) we mean a family of linear mappings $\{D_n\}_{n=0}^{j}$ from $\mathcal{A}$ into $\mathcal{B}$ such that

$$D_n(ab) = \sum_{k=0}^{n} D_k(a)D_{n-k}(b), \ \ (a, b \in \mathcal{A}, \ n = 0, 1, 2, ..., j).$$

It is obvious that for a higher derivation $\{D_n\}_{n=0}^{j}$, $D_0$ is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ and $D_1$ is a $D_0$-derivation, that is $D_1(ab) = D_0(a)D_1(b) + D_1(a)D_0(b)$ $(a, b \in \mathcal{A})$. Note that $\mathcal{B}$ is a $\mathcal{A}$-bimodule with module operations $ax = D_0(a)x, xa = xD_0(a)$ $(a \in \mathcal{A}, x \in \mathcal{B})$. A higher derivation $\{D_n\}$ is said to be continuous if every $D_n$ is continuous. If $\mathcal{B} = \mathcal{A}$ and $D_0 = id_{\mathcal{A}}$, where $id_{\mathcal{A}}$ is the identity map on $\mathcal{A}$, then $D_1$ is a derivation and $\{D_n\}_{n=0}^{j}$ is called a *strongly* higher derivation. A standard example of a higher derivation of rank $j$ is $\{\frac{D^n}{n!}\}_{n=0}^{j}$ where $D : \mathcal{A} \to \mathcal{A}$ is a derivation. Higher derivations were introduced by Hasse and Schmidt [1]

**Copyright © 2012 by the Mathematical Association of Thailand. All rights reserved.**

and algebraists sometimes call them Hasse-Schmidt derivations. The reader may find more about higher derivations in [2–6]. Let $\{d_n\}_{n=0}^{j}$ be a family of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ for which $d_0$ is a homomorphism, then $\{d_n\}_{n=0}^{j}$ is called a *local higher derivation* if for every $a \in \mathcal{A}$ there exists a higher derivation $\{D_n^a\}_{n=0}^{j}$ from $\mathcal{A}$ into $\mathcal{B}$ such that $D_0^a = d_0$ and $d_n(a) = D_n^a(a)$ $(n \geq 1)$. In the context of derivations, the relation between local derivations and derivations is widely studied by Several authors [7–12]. Hadwin and Li [13] prove that every local derivation from an algebra $\mathcal{A}$ that is generated by it's idempotents, into any $\mathcal{A}$-bimodule is a derivation.

Now, let $\mathcal{A}$ and $\mathcal{B}$ be topological algebras, denote by $hder(\mathcal{A}, \mathcal{B})$ the set of all continuous higher derivations from $\mathcal{A}$ into $\mathcal{B}$. If $\mathcal{A}$ is a topological algebra we say that $\mathcal{A}$ is topologically generated by its idempotents, if the subalgebra of $\mathcal{A}$ generated by its idempotents is dense in $\mathcal{A}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Hausdorff topological linear spaces and let $\mathfrak{B}(\mathcal{X}, \mathcal{Y})$ be the space of continuous linear mappings from $\mathcal{X}$ into $\mathcal{Y}$. We said that a subset $\mathcal{S}$ of $\mathfrak{B}(\mathcal{X}, \mathcal{Y})$ is reflexive if $T \in \mathcal{S}$ whenever $T \in \mathfrak{B}(\mathcal{X}, \mathcal{Y})$ and $Tx \in [\mathcal{S}x]$ for any $x \in \mathcal{X}$, where $[\cdot]$ is the topological closure. By a *subspace lattice* on $\mathcal{X}$ we mean a collection $\mathcal{L}$ of closed subspaces of $\mathcal{X}$ containing $(0)$ and $\mathcal{X}$ such that for each family $\{L_\alpha\}$ of elements of $\mathcal{L}$ both $\bigcap L_\alpha$ and $\bigvee L_\alpha$ belong to $\mathcal{L}$, where $\bigvee$ denotes the closed linear span of $\{L_\alpha\}$. If $\mathcal{L}$ is a subspace lattice, the algebra of all operators on $\mathcal{X}$ that leave invariant each element of $\mathcal{L}$ is denoted by $alg\mathcal{L}$. A totally ordered subspace lattice $\mathcal{N}$ is called a *nest* and the associated reflexive algebra $alg\mathcal{N}$ is called a *nest algebra*.

In Section 2 we prove some algebraic results for a family of linear mappings which is needed in Section 3. In Section 3, we show that if a unital topological algebra $\mathcal{A}$ is generated by its idempotents, then a local higher derivation $\{d_n\}$ from $\mathcal{A}$ into a unital algebra $\mathcal{B}$ provided that $d_0$ is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$, is a higher derivation and $hder(\mathcal{A}, \mathcal{B})$ is reflexive. In particular for a unital algebra $\mathcal{A}$ and any $n \geq 2$, we have that every local higher derivation from $M_n(\mathcal{A})$ into itself is a higher derivation.

The results in Sections 2 and 3 are the same in [13, Section2], which are proved for local derivations and here we prove them for local higher derivations.

# 2 Algebraic Results

In this section we assume that all algebras are unital. Recall that if $\mathcal{A}$ and $\mathcal{B}$ are two algebras and $d_0$ is a homomorphism from $\mathcal{A}$ into $\mathcal{B}$, then $\mathcal{B}$ is a $\mathcal{A}$-bimodule with module operations $a.x = d_0(a)x, x.a = xd_0(a)$ $(a \in \mathcal{A}, x \in \mathcal{B})$. To show our main results, we need two lemmas.

**Lemma 2.1.** *Let $\{d_n\}_{n=0}^{j}$ be a family of linear mappings from an algebra $\mathcal{A}$ into an algebra $\mathcal{B}$ such that for all $n \in \mathbb{N}$, $d_n(1) = 0$ and $d_0$ is a homomorphism. Then for each $a \in \mathcal{A}$ and any idempotents $p, q \in \mathcal{A}$ the following are equivalent.*

*(i)* $(1-p)\big(d_n(paq) - \sum_{k=1}^{n-1} d_{n-k}(pa)d_k(q)\big)(1-q) = 0$

*(ii)* $d_n(paq) = \sum_{k=0}^{n-1}\big[d_{n-k}(pa) - pd_{n-k}(a)\big]d_k(q) + pd_n(aq)$.

*Proof.* It is clear that $(ii)$ implies $(i)$. Now suppose that $(i)$ is true. We denote $p^\perp = 1 - p$, then we have

$$p\left(d_n(paq) - \sum_{k=1}^{n-1} d_{n-k}(pa)d_k(q)\right)q^\perp = \left(d_n(paq) - \sum_{k=1}^{n-1} d_{n-k}(pa)d_k(q)\right)q^\perp. \quad (2.1)$$

Also the left side equals to

$$\begin{aligned} &p\left(d_n((1-p^\perp)aq) - \sum_{k=1}^{n-1} d_{n-k}(1-p^\perp a)d_k(q)\right)q^\perp \qquad (2.2)\\ &\quad = p\left(d_n(aq) - d_n(p^\perp aq) + \sum_{k=1}^{n-1} d_{n-k}(p^\perp ad_k(q) - \sum_{k=1}^{n-1} d_{n-k}(a)d_k(q)\right)q^\perp. \end{aligned}$$

By (2.1) and (2.2), it follows that

$$p\left(d_n(aq) - \sum_{k=1}^{n-1} d_{n-k}(a)d_k(q)\right)q^\perp = \left(d_n(paq) - \sum_{k=1}^{n-1} d_{n-k}(pa)d_k(q)\right)q^\perp. \quad (2.3)$$

Hence by (2.3)

$$\begin{aligned} &d_n(paq) - \sum_{k=1}^{n-1}\big[d_{n-k}(pa) - pd_{n-k}(a)\big]d_k(q) - pd_n(aq)\\ &\quad = \left(d_n(paq) - \sum_{k=1}^{n-1}\big[d_{n-k}(pa) - pd_{n-k}(a)\big]d_k(q) - pd_n(aq)\right)(q+q^\perp)\\ &\quad = \left(d_n(paq) - \sum_{k=1}^{n-1}\big[d_{n-k}(pa) - pd_{n-k}(a)\big]d_k(q) - pd_n(aq)\right)q\\ &\quad = \left(d_n(pa(1-q^\perp)) - \sum_{k=1}^{n-1} d_{n-k}(pa)d_k(1-q^\perp) + p\sum_{k=1}^{n-1} d_{n-k}(a)d_k(q) - pd_n(aq)\right)q\\ &\quad = d_n(pa)q - d_n(paq^\perp)q + \sum_{k=1}^{n-1} d_{n-k}(pa)d_k(q^\perp)q\\ &\qquad - \sum_{k=1}^{n-1} d_{n-k}(pa)d_k(1)q + p\sum_{k=1}^{n-1} d_{n-k}(a)d_k(q)q - pd_n(aq)q\\ &\quad = d_n(pa)q - pd_n(aq^\perp)q + \sum_{k=1}^{n-1} d_{n-k}(pa)d_k(q^\perp)q - pd_n(aq)q\\ &\qquad + p\sum_{k=1}^{n-1} d_{n-k}(a)d_k(q)q - pd_n(aq)q\\ &\quad = d_n(pa)q - pd_n(a)q. \end{aligned}$$

This completes the proof. □

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. Suppose $\{d_n\}_{n=0}^{j}$ is a family of linear mappings from an algebra $\mathcal{A}$ into an algebra $\mathcal{B}$. We say that $\{d_n\}_{n=0}^{j}$ satisfies the condition $(*)$ if $d_0$ is a homomorphism and

$$d_n(paq) = \sum_{k=0}^{n-1}\big[d_{n-k}(pa) - pd_{n-k}(a)\big]d_k(q) + pd_n(aq) \text{ and } d_n(1) = 0 \ (n = 1, 2, ...)$$

for all idempotents $p, q$ in $\mathcal{A}$ and $a \in \mathcal{A}$.

**Lemma 2.2.** *Suppose that $\{d_n\}_{n=0}^{j}$ is a family of linear mappings from an algebra $\mathcal{A}$ into an algebra $\mathcal{B}$ satisfying the condition $(*)$. Then for any idempotents $p_1, ..., p_l, q_1, ..., q_m$ in $\mathcal{A}$ and every $a \in \mathcal{A}$*

$$\begin{aligned} d_n(p_1 \cdots p_l a q_1 \cdots q_m) = &\sum_{k=0}^{n-1}\big[d_{n-k}(p_1 \cdots p_l a) - p_1 \cdots p_l d_{n-k}(a)\big]d_k(q_1 \cdots q_m) \\ &+ p_1 \cdots p_l d_n(a q_1 \cdots q_m). \end{aligned} \tag{2.4}$$

*Proof.* First we show that

$$d_n(p_1 \cdots p_l aq) = \sum_{k=0}^{n-1}\big[d_{n-k}(p_1 \cdots p_l a) - p_1 \cdots p_l d_{n-k}(a)\big]d_k(q) + p_1 \cdots p_l d_n(aq). \tag{2.5}$$

If $l = 1$ the condition $(*)$ implies the result. Suppose (2.5) holds for $l = j$. If $l = j + 1$, by the condition $(*)$ it follows that

$$\begin{aligned} &d_n(p_1 \cdots p_{j+1} aq) \\ &= \sum_{k=0}^{n-1}\big[d_{n-k}(p_1 \cdots p_{j+1} a) - p_1 d_{n-k}(p_2 \cdots p_{j+1} a)\big]d_k(q) + p_1 d_n(p_2 \cdots p_{j+1} aq) \\ &= \sum_{k=0}^{n-1}\big[d_{n-k}(p_1 \cdots p_{j+1} a) - p_1 d_{n-k}(p_2 \cdots p_{j+1} a)\big]d_k(q) \\ &\quad + p_1\left(\sum_{k=0}^{n-1}\big[d_{n-k}(p_2 \cdots p_{j+1} a) - p_2 \cdots p_{j+1} d_{n-k}(a)\big]d_k(q) - p_2 \cdots p_{j+1} d_n(aq)\right) \\ &= \sum_{k=0}^{n-1}\big[d_{n-k}(p_1 \cdots p_{j+1} a) - p_1 \cdots p_{j+1} d_{n-k}(a)\big]d_k(q) + p_1 \cdots p_{j+1} d_n(aq). \end{aligned}$$

Now we prove (2.4) is true. For $m = 1$, (2.5) implies (2.4). Now assume (2.4) for $m = j$. If $m = j + 1$, by the induction assumption, the condition $(*)$ and (2.5), we have

$$
\begin{aligned}
&d_n(p_1\cdots p_l a q_1\cdots q_{j+1})\\
&=\sum_{i=0}^{n-1}\big[d_{n-i}(p_1\cdots p_l a q_1\cdots q_j)-p_1\cdots p_l d_{n-i}(aq_1\cdots q_j)\big]d_i(q_{j+1})\\
&\quad+p_1\cdots p_l d_n(aq_1\cdots q_{j+1})\\
&=\sum_{i=0}^{n-1}\Big[\sum_{k=0}^{n-i-1}(d_{n-i-k}(p_1\cdots p_l a)d_k(q_1\cdots q_j)-p_1\cdots p_l d_{n-i-k}(a)d_k(q_1\cdots q_j))\\
&\quad-p_1\cdots p_l d_{n-i}(aq_1\cdots q_j)+p_1\cdots+p_l d_{n-i}(aq_1\cdots q_j)\Big]d_i(q_{j+1})\\
&\quad+p_1\cdots p_l d_n(aq_1\cdots q_{j+1})\\
&=\Big[\sum_{k=0}^{n-1}\big(d_{n-k}(p_1\cdots p_l a)-p_1\cdots p_l d_{n-k}(a)\big)d_k(q_1\cdots q_j)\Big]q_{j+1}\\
&\quad+\Big[\sum_{k=0}^{n-2}\big(d_{n-2-k}(p_1\cdots p_l a)-p_1\cdots p_l d_{n-2-k}(a)\big)d_k(q_1\cdots q_j)\Big]d_1(q_{j+1})\\
&\quad+\cdots+\Big[\sum_{k=0}^{1}\big(d_1(p_1\cdots p_l a)-p_1\cdots p_l d_{n-2}(a)\big)d_1(q_1\cdots q_j)\Big]d_{n-2}(q_{j+1})\\
&\quad+\big[d_1(p_1\cdots p_l a)-p_1\cdots p_l d_1(a)\big](q_1\cdots q_j)d_{n-1}(q_{j+1})+p_1\cdots p_l d_n(aq_1\cdots q_{j+1})\\
&=\big(d_n(p_1\cdots p_l a)-p_1\cdots p_l d_n(a)\big)d_0(q_1\cdots q_{j+1})\\
&\quad+\big(d_{n-2-k}(p_1\cdots p_l a)-p_1\cdots p_l d_{n-2-k}(a)\big)\sum_{k=0}^{1}d_{1-k}(q_1\cdots q_j)d_k(q_{j+1})\\
&\quad+\cdots+\big(d_2(p_1\cdots p_l a)-p_1\cdots p_l d_2(a)\big)\sum_{k=0}^{n-2}d_{n-2-k}(q_1\cdots q_j)d_k(q_{j+1})\\
&\quad+\big(d_1(p_1\cdots p_l a)-p_1\cdots p_l d_1(a)\big)\sum_{k=0}^{n-1}d_{n-1-k}(q_1\cdots q_j)d_k(q_{j+1})\\
&\quad+p_1\cdots p_l d_n(aq_1\cdots q_{j+1})\\
&=\sum_{k=0}^{n-1}\big[d_{n-k}(p_1\cdots p_l a)-p_1\cdots p_l d_{n-k}(a)\big]d_(q_1\cdots q_{j+1})+p_1\cdots p_l d_n(aq_1\cdots q_{j+1})
\end{aligned}
$$

note that the last equality follows from the identity

$$d_{n-i}(q_1\cdots q_{j+1})=\sum_{k=0}^{n-i}d_{n-i-k}(q_1\cdots q_j)d_k(q_{j+1})\ \ (0\le i\le n)$$

and the result proves. □

# 3 Local Higher Derivations on Topological Algebras

In this section we assume that all algebras are unital topological algebras. Recall that an algebra $\mathcal{A}$ is called a *topological algebra* if $\mathcal{A}$ satisfies

(i) $\mathcal{A}$ is a topological vector space, and

(ii) with the product topology of $\mathcal{A} \times \mathcal{A}$, the map $f : (x, y) \to xy$ is continuous.

Let $M$ be an $\mathcal{A}$-module and let $\tau$ be an ideal of $\mathcal{A}$. We say that $\tau$ is a separating set of $M$, if for every $m, n \in M$, $m\tau = \{0\}$ implies $m = 0$ and $\tau n = \{0\}$ implies $n = 0$.

The following theorem generalizes [13, Theorem 2.7] for higher derivations.

**Theorem 3.1.** *Let $\tau$ be a separating set of the algebra $\mathcal{B}$. Suppose that $\tau$ is contained in the algebra generated by all idempotents in $\mathcal{A}$. If $\{d_n\}_{n=0}^{j}$ is a family of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ satisfying the condition $(*)$, then $\{d_n\}_{n=0}^{j}$ is a higher derivation.*

*Proof.* Since $\tau$ is contained in the algebra generated by all idempotents in $\mathcal{A}$, the condition $(*)$ follows that for each $x, y \in \tau$,

$$d_n(xy) = \sum_{k=0}^{n} d_{n-k}(x)d_k(y) \ \ (n = 1, 2, ...). \tag{3.1}$$

Since $\tau$ is an ideal of $\mathcal{A}$, by $(*)$ for any $a \in \mathcal{A}$ we have

$$\begin{aligned} d_n(xay) = d_n\big((xa)y\big) &= \sum_{k=0}^{n} d_{n-k}(xa)d_k(y) \\ &= \sum_{k=0}^{n-1} \big[d_{n-k}(xa)d_k(y) - xd_{n-k}(a)\big] d_k(y) + xd_n(ay). \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), it follows that

$$xd_n(ay) = xad_n(y) + \sum_{k=0}^{n-1} xd_{n-k}(a)d_k(y)x\big(ad_n(y) + \sum_{k=0}^{n-1} d_{n-k}(a)d_k(y)\big). \tag{3.3}$$

Since $\tau$ is a separating set of the algebra $\mathcal{B}$, (3.3) implies that

$$d_n(ay) = ad_n(y) + \sum_{k=0}^{n-1} d_{n-k}(a)d_k(y) = \sum_{k=0}^{n} d_{n-k}(a)d_k(y). \tag{3.4}$$

If $a, b \in \mathcal{A}$ and $y \in \tau$, by (3.4) we have that

$$\begin{aligned} d_n(bay) &= \sum_{k=0}^{n} d_{n-k}(b) d_k(ay) = \sum_{k=0}^{n} d_{n-k}(b) \sum_{i=0}^{k} d_{k-i}(a) d_i(y) \\ &= \left( \sum_{k=0}^{n} d_{n-k}(b) d_k(a) \right) y + \sum_{k=1}^{n} \sum_{i=0}^{n-k} d_{n-k-i}(b) d_i(a) d_k(y) \end{aligned} \tag{3.5}$$

on the other hand

$$d_n(bay) = \sum_{k=0}^{n} d_{n-k}(ba) d_k(y) = d_n(ba)y + \sum_{k=1}^{n} d_{n-k}(b) \sum_{i=0}^{k} d_{k-i}(a) d_i(y). \tag{3.6}$$

Since $\tau$ is a separating set, by (3.5) and (3.6) it follows that

$$d_n(ba) = \sum_{k=0}^{n} d_{n-k}(b) d_k(a).$$

Therefore $\{d_n\}_{n=0}^{j}$ is a higher derivation. □

**Corollary 3.2.** *Let $\{I_\gamma : \gamma \in \Gamma\}$ is a collection of two-sided ideals in $\mathcal{A}$ such that*

*(i) $\mathcal{A}/I_\gamma$ is generated by its idempotents,*

*(ii) $\cap_{\gamma \in \Gamma} = 0$.*

*If $\{d_n\}_{n=0}^{j}$ is family of linear mappings from $\mathcal{A}$ into an algebra $\mathcal{B}$ satisfying the condition $(*)$ and $d_n(I_\gamma) \subseteq I_\gamma$, for each $n$. Then $\{d_n\}_{n=0}^{j}$ is a higher derivation.*

*Proof.* For any $n$ and each $\gamma$ in $\Gamma$ and, $d_n$ induces a linear mapping $d_n^\gamma$ on $\mathcal{A}/I_\gamma$ satisfying the condition $(*)$. By Theorem 3.1 and assumptions it follows that $\{d_n^\gamma\}$ is a higher derivation. Therefore for any $a, b \in \mathcal{A}$ we have that $d_n(ab) - \sum_{k=0}^{n} d_{n-k}(a) d_k(b) \in I_\gamma$, for each $\gamma \in \Gamma$. By $(ii)$, it follows that $d_n(ab) = \sum_{k=0}^{n} d_{n-k}(a) d_k(b)$. □

**Remark 3.3.** *Let $\mathcal{A}$, $\mathcal{B}$ and $\tau$ be as in Theorem 3.1 and let $\{d_n\}_{n=0}^{j}$ be a local higher derivation from $\mathcal{A}$ into $\mathcal{B}$. Then $\{d_n\}_{n=0}^{j}$ satisfies in condition $(*)$, in fact [13, Theorem 2.7] implies that $d_1$ is a derivation. Suppose $j = 2$ and $p, q$ are idempotents in $\mathcal{A}$ and $a \in \mathcal{A}$. Then there exists a higher derivation $\{D_n^{paq}\}_{n=0}^{2}$ from $\mathcal{A}$ into $\mathcal{B}$ such that $D_0^{paq} = d_0$ and $d_n(paq) = D_n^{paq}(paq)$ $(1 \le n \le 2)$. In fact $D_1^{paq} = d_1$, because $d_1$ is a derivation. Hence we have*

$$\begin{aligned} (1-p)\big(d_2(paq) - d_1(pa)d_1(q)\big)(1-q) &= (1-p)\big(D_2^{paq}(paq) - D_1(pa)D_1(q)\big)(1-q) \\ &= (1-p)\big(paD_2^{paq}(q) + D_2^{paq}(pa)q\big)(1-q) \\ &= 0 \end{aligned}$$

*by Lemma 2.1 the assertion proves.*

With the help of Remark 3.3, the following theorem can be derived along the same argument in the proof of Theorem 3.1.

**Theorem 3.4.** *Let $\tau$ be a separating set of the algebra $\mathcal{B}$. Suppose that $\tau$ is contained in the algebra generated by all idempotents in $\mathcal{A}$. If $\{d_n\}$ is a local higher derivation from $\mathcal{A}$ into $\mathcal{B}$, then $\{d_n\}$ is a higher derivation.*

In the following, we give some applications of Theorem 3.4.

**Corollary 3.5.** *Let $\mathcal{A}$ be an algebra and let $\mathcal{B}$ be a unital algebra such that for every unital algebra $\mathcal{C}$, $\mathcal{C} \otimes \mathcal{B}$ is generated by it's idempotents. Then every local higher derivation from $\mathcal{A} \otimes \mathcal{B}$ into itself is a higher derivation. In particular for $2 \leq n$, every local higher derivation from the matrix algebra $M_n(\mathcal{A})$ into itself is a higher derivation.*

Note that by [13, Proposition 2.2], $M_n(\mathcal{A})$ is generated by it's idempotents.

**Corollary 3.6.** *If for any $a, b \in \mathcal{A}$, there exists a unital subalgebra $\mathcal{B}$ of $\mathcal{A}$ containing $a$ and $b$ such that $\mathcal{B}$ is isomorphic to a matrix algebra, then every local higher derivation from $\mathcal{A}$ into itself is a higher derivation.*

Now we consider the local higher derivations on a reflexive subalgebra in a factor von Neumann algebra. The proof of the following corollaries uses Theorem 3.1 and arguments similar to those in the proof of [13, Theorem 2.17, Theorem 2.18].

**Corollary 3.7.** *Suppose that $\mathcal{L}$ is a subspace lattice in a factor von Neumann algebra $\mathcal{M}$ on $H$ with $\bigcap\{L \in \mathcal{L} : \ 0 \subset L\} \neq 0$ and $\bigvee\{L \in \mathcal{L} : \ L \subset H\} \neq H$. If $\{d_n\}_{n=0}^{j}$ is a family of linear mappings from $\mathcal{M} \cap alg\mathcal{L}$ into $\mathcal{M}$ satisfying the condition $(*)$ (in particular, if $\{d_n\}_{n=0}^{j}$ is a local higher derivation), then $\{d_n\}_{n=0}^{j}$ is a higher derivation.*

**Corollary 3.8.** *Let $\mathcal{N}$ be a nest in a factor von Neumann algebra $\mathcal{M}$ on $H$. If $\{d_n\}_{n=0}^{j}$ is a family of linear mappings from $\mathcal{M} \cap alg\mathcal{N}$ into $\mathcal{M}$ satisfying the condition $(*)$ (in particular, if $\{d_n\}_{n=0}^{j}$ is a local higher derivation), then $\{d_n\}_{n=0}^{j}$ is a higher derivation.*

**Theorem 3.9.** *Suppose that $\mathcal{A}$ is topologically generated by its idempotents. If $\{d_n\}_{n=0}^{j}$ is a family of continuous linear mappins from $\mathcal{A}$ into a topological algebra $\mathcal{B}$ satisfying the condition $(*)$ (in particular, if $\{d_n\}_{n=0}^{j}$ is a local higher derivation), then $\{d_n\}_{n=0}^{j}$ is a higher derivation.*

*Proof.* If $a = \sum_{i=1}^{m} \alpha_i \prod_{j=1}^{t_i} p_j^{(i)}, \ b = \sum_{s=0}^{l} \beta_s \prod_{k=1}^{u_s} q_k^{(s)}$, where $p_j^{(i)}, \ q_k^{(s)}$ are idempotents of $\mathcal{A}$ and $\alpha_i, \ \beta_s \in \mathbb{C}$, Lemma 2.2 implies that $d_n(ab) = \sum_{k=0}^{n} d_{n-k}(a) d_k(b)$. Since $\{d_n\}_{n=0}^{j}$ is continuous and $\mathcal{A}$ is topologically generated by its idempotents, the result follows. □

By [13, Proposition 2.3] and Theorem 3.9, it follows that

**Corollary 3.10.** *Suppose that $\mathcal{N}$ is a nest in a von Neumann algebra $\mathcal{M}$ and $\mathcal{A} = \mathcal{M} \cap alg\mathcal{N}$. If $\{d_n\}_{n=0}^{j}$ is a $w^*$-continuous local higher derivation from $\mathcal{A}$ into $\mathcal{M}$, then $\{d_n\}_{n=0}^{j}$ is a higher derivation.*

**Corollary 3.11.** *Let $\mathcal{A}$ and $\mathcal{B}$ be as in Theorem 3.1. Then $hder(\mathcal{A}, \mathcal{B})$ is reflexive.*

*Proof.* Suppose that $\{d_n\}_{n=0}^{j}$ is a family of continuous linear mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for each $x \in \mathcal{A}$ and $n \in \mathbb{N}$, $d_n(x) \in [hder(\mathcal{A}, \mathcal{B})x]$. Then there exists a sequence $\{\Delta_m^n\}$ (depending on $x$) in $hder(\mathcal{A})$ such that $\lim_{m\to\infty} \Delta_m^n(x) = d_n(x)$. Let $p, q$ be idempotents of $\mathcal{A}$. For any $a \in \mathcal{A}$, take $x = paq$. It follows that

$$(1-p)\left(d_n(paq) - \sum_{k=1}^{n-1} d_{n-k}(pa)d_k(q)\right)(1-q)$$
$$= \lim_{m\to\infty}(1-p)\left(\Delta_m^n(paq) - \sum_{k=1}^{n-1} \Delta_m^{n-k}(pa)\Delta_m^k(q)\right)(1-q) = 0.$$

Lemma 2.1 and Theorem 3.1 imply that $\{d_n\}_{n=0}^{j}$ is a higher derivation, hence $hder(\mathcal{A}, \mathcal{B})$ is reflexive. □

# References


[1] H. Hasse, F.K. Schmidt, Noch eine Begrüdung der theorie der höheren differential quotienten in einem algebraischen funtionenkörper einer unbestimmeten, J. Reine Angew. Math. 177 (1937) 215–237.

[2] S. Hejazian, T.L. Shatery, Automatic continuity of higher derivations on $JB^*$-algebras, Bull. Iranian Math. Soc. 33 (1) (2007) 11–23.

[3] S. Hejazian, T.L. Shatery, Higher derivations on Banach algebras, J. Anal. Appl. 6 (2008) 1–15.

[4] S. Hejazian, T.L. Shateri, A characterization of higher derivations, to appear in Italian Journal of pure and applied mathematics.

[5] N.P. Jewell, Continuity of module and higher derivations, Pacific J. Math. 68 (1977) 91–98.

[6] K.W. Jun, Y.W. Lee, The image of a continuous strong higher derivation is contained in the radical, Bull. Korean Math. Soc. 33 (2) (1996) 229–232.

[7] R. Crist, Local derivations on operator algebras, J. Func. Anal. 135 (1996) 72–92.

[8] R. Crist, Local automorphisms, Proc. Amer. Math. Soc. 128 (1999) 1409–1414.

[9] D. Hadwin, J. LI, Local derivations and local automorphisms, J. Math. Anal. Appl. 290 (2004) 702–714.

[10] B. Johnson, Local derivations on $C^*$-algebras are derivations, Trans. Amer. Math. Soc. 353 (2001) 313–325.

[11] R. Kadison, Local derivations, J. Algebra. 130 (1990) 494–509.

[12] D. Larson, A. Sourour, Local derivations and local automorphisms, Proc. Sympos. Pure Math. 51 (1990) 187–194.

[13] D. Hadwin, J. Li, Local derivations and local automorphisms on some algebras, J. Operator thoery, 60 (1) (2008) 29–44.

(Received 21 May 2011)
(Accepted 15 November 2011)