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Lacunary Sequence Spaces of Interval Numbers

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Abstract : In this paper we introduce the concept of lacunary strongly convergence and lacunary statistical convergence of interval numbers. We prove some inclusion relations and study some of their properties.

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1 Introduction

Interval arithmetic was first suggested by Dwyer [1] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [2] in 1959 and Moore and Yang [3] in 1962. Furthermore, Moore and others [1, 4, 5, 6] have developed applications to differential equations.

Chiao [7] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Recently Şengönül and Eryilmaz [8] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space.

The idea of statistical convergence for ordinary sequences was introduced by Fast [9] in 1951. Schoenberg [10] studied statistical convergence as a summability method and listed some of elementary properties of statistical convergence. Both of these authors noted that if bounded sequence is statistically convergent, then it is Cesaro summable. Existing work on statistical convergence appears to have been restricted to real or complex sequence, but several authors extended the idea to apply to sequences of fuzzy numbers and also introduced and discussed the concept of statistically sequences of fuzzy numbers.

2 Preliminaries

Let $p = (p_k)$ be a positive sequence of real numbers. If $0 < h = \inf_k p_k \leq p_k \leq H = \sup_k p_k < \infty$ and $D = \max\left(1, 2^{H-1}\right)$, then for all $a_k, b_k \in \mathbb{C}$ for all $k \in \mathbb{N}$, we have

$$|a_k + b_k|^{p_k} \leq D\left(|a_k|^{p_k} + |b_k|^{p_k}\right).$$

By a lacunary sequence $\theta = (k_r)$, $r = 0, 1, 2, \ldots$, where $k_o = 0$, we shall mean an increasing sequence of non-negative with integers $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al. [11] as follows:

$$N_\theta = \left\{ x = (x_i) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} |x_i - L| = 0 \text{ for some } L \right\}.$$

We denote the set of all real valued closed intervals by $\mathbb{IR}$. Any elements of $\mathbb{IR}$ is called interval number and denoted by $\overline{x} = [x_l, x_r]$. Let x_l and x_r be first and last points of $\overline{x}$ interval number, respectively. For $\overline{x}_1, \overline{x}_2 \in \mathbb{IR}$, we have $\overline{x}_1 = \overline{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}, x_{1_r} = x_{2_r}$. $\overline{x}_1 + \overline{x}_2 = \left\{x \in \mathbb{R} : x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\right\}$, and if $\alpha \geq 0$, then $\alpha\overline{x} = \left\{x \in \mathbb{R} : \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\right\}$ and if $\alpha < 0$, then $\alpha\overline{x} = \left\{x \in \mathbb{R} : \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\right\}$,

$$\overline{x}_1.\overline{x}_2 = \{x \in \mathbb{R} : \min\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \leq x \leq \max\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\}\}.$$

The set of all interval numbers $\mathbb{IR}$ is a complete metric space defined by

$$d(\overline{x}_1, \overline{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\} \text{ [2]}.$$

In the special case $\overline{x}_1 = [a, a]$ and $\overline{x}_2 = [b, b]$, we obtain usual metric of $\mathbb{R}$.

Let's define transformation f from $\mathbb{N}$ to $\mathbb{R}$ by $k \to f(k) = \overline{x}$, $\overline{x} = (\overline{x}_k)$. Then $\overline{x} = (\overline{x}_k)$ is called sequence of interval numbers. The $\overline{x}_k$ is called k^{th} term of sequence $\overline{x} = (\overline{x}_k)$. w^i denotes the set of all interval numbers with real terms and the algebraic properties of w^i can be found in [8].

Now we give the definition of convergence of interval numbers:

Definition 2.1 ([7])**.** A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be *convergent to the interval number* $\overline{x}_o$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d(\overline{x}_k, \overline{x}_o) < \varepsilon$ for all $k \geq k_o$ and we denote it by $\lim_k \overline{x}_k = \overline{x}_o$.

Thus, $\lim_k \overline{x}_k = \overline{x}_o \Leftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$.

In this paper, we introduce and study the concepts of lacunary strongly convergence and lacunary statistically convergence for interval numbers.

3 Main Results

In this section we give some definition and prove the results of this paper.

Definition 3.1. Let $\theta = (k_r)$ be a lacunary sequence and $p = (p_k)$ be any sequence of strictly positive real numbers. A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be *lacunary strongly convergent* if there is a interval number $\overline{x}_o$ such that

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} [d(\overline{x}_k, \overline{x}_o)]^{p_k} = 0.$$

In this case we write $\overline{x}_k \to \overline{x}_o \left(\overline{N}_\theta^p\right)$ or $\overline{N}_\theta^p - \lim \overline{x}_k = \overline{x}_o$. We denote with $\overline{N}_\theta^p$ the set of all lacunary strongly convergent sequences of interval numbers. In the special case $\theta = (2^r)$, we shall write $\overline{N}^p$ instead of $\overline{N}_\theta^p$.

Definition 3.2. Let $\theta = (k_r)$ be a lacunary sequence. A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be *lacunary statistically convergent* to interval number $\overline{x}_o$ if for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} |\{k \in I_r : \ d(\overline{x}_k, \overline{x}_o) \geq \varepsilon\}| = 0.$$

In this case we write $\overline{x}_k \to \overline{x}_o (\overline{s}_\theta)$ or $\overline{s}_\theta - \lim \overline{x}_k = \overline{x}_o$. The set of all lacunary statistically convergent sequences of interval number sequences is denoted by $\overline{s}_\theta$. In the special case $\theta = (2^r)$, we shall write $\overline{s}$ instead of $\overline{s}_\theta$.

Theorem 3.3. *Let $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k)$ be sequences of interval numbers.*

(i) If $\overline{s}_\theta - \lim \overline{x}_k = \overline{x}_o$ and $\alpha \in \mathbb{R}$, then $\overline{s}_\theta - \lim \alpha\overline{x}_k = \alpha\overline{x}_o$.

(ii) If $\overline{s}_\theta - \lim \overline{x}_k = \overline{x}_o$ and $\overline{s}_\theta - \lim \overline{y}_k = \overline{y}_o$, then $\overline{s}_\theta - \lim (\overline{x}_k + \overline{y}_k) = \overline{x}_o + \overline{y}_o$.

Proof. (i) Let $\alpha \in \mathbb{R}$. We have $d(\alpha\overline{x}_k, \alpha\overline{x}_o) = |\alpha| \, d(\overline{x}_k, \overline{x}_o)$. For a given $\varepsilon > 0$

$$\frac{1}{h_r} |\{k \in I_r : \ d(\alpha\overline{x}_k, \alpha\overline{x}_o) \geq \varepsilon\}| = \frac{1}{h_r} \left| \left\{ k \in I_r : \ d(\overline{x}_k, \overline{x}_o) \geq \frac{\varepsilon}{|\alpha|} \right\} \right|.$$

Hence $\overline{s}_\theta - \lim \alpha\overline{x}_k = \alpha\overline{x}_o$.

(ii) Suppose that $\overline{s}_\theta - \lim \overline{x}_k = \overline{x}_o$ and $\overline{s}_\theta - \lim \overline{y}_k = \overline{y}_o$. We have

$$d(\overline{x}_k + \overline{y}_k, \overline{x}_o + \overline{y}_o) \leq d(\overline{x}_k, \overline{x}_o) + d(\overline{y}_k, \overline{y}_o).$$

Therefore given $\varepsilon > 0$, we have

$$\begin{aligned} &\frac{1}{h_r} |\{k \in I_r : d(\overline{x}_k + \overline{y}_k, \overline{x}_o + \overline{y}_o) \geq \varepsilon\}| \\ &\qquad \leq \frac{1}{h_r} |\{k \in I_r : d(\overline{x}_k, \overline{x}_o) + d(\overline{y}_k, \overline{y}_o) \geq \varepsilon\}| \\ &\qquad \leq \frac{1}{h_r} \left| \left\{ k \in I_r : \ d(\overline{x}_k, \overline{x}_o) \geq \frac{\varepsilon}{2} \right\} \right| + \frac{1}{h_r} \left| \left\{ k \in I_r : \ d(\overline{y}_k, \overline{y}_o) \geq \frac{\varepsilon}{2} \right\} \right|. \end{aligned}$$

Thus, $\overline{s}_\theta - \lim (\overline{x}_k + \overline{y}_k) = \overline{x}_o + \overline{y}_o$. □

Theorem 3.4. *Let $\theta = (k_r)$ be a lacunary sequence and $\overline{x} = (\overline{x}_k)$ be a sequence of interval numbers. Then*

(i) For $\liminf_r q_r > 1$, then $\overline{x}_k \to \overline{x}_o(\overline{N}^p)$ implies $\overline{x}_k \to \overline{x}_o(\overline{N}^p_\theta)$;

(ii) For $\limsup_r q_r < \infty$, then $\overline{x}_k \to \overline{x}_o(\overline{N}^p_\theta)$ implies $\overline{x}_k \to \overline{x}_o(\overline{N}^p)$;

(iii) If $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, then $\overline{x}_k \to \overline{x}_o(\overline{N}^p)$ if and only if $\overline{x}_k \to \overline{x}_o(\overline{N}^p_\theta)$.

Proof. (i) Let $\overline{x}_k \to \overline{x}_o(\overline{N}^p)$ and $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r. Then we write

$$\frac{1}{h_r}\sum_{k\in I_r}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k} = \frac{1}{h_r}\sum_{k=1}^{k_r}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k} - \frac{1}{h_r}\sum_{k=1}^{k_{r-1}}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k}$$
$$= \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{k=1}^{k_r}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k}\right) - \frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k}\right).$$

Since $h_r = k_r - k_{r-1}$, we have $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{h_r}{k_{r-1}} \geq \frac{1}{\delta}$. The terms $\frac{1}{k_r}\sum_{k=1}^{k_r}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k}$ and $\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k}$ both converge to 0 as $r \to \infty$. Hence $\overline{x}_k \to \overline{x}_o(\overline{N}^p_\theta)$.

(ii) If $\limsup_r q_r < \infty$, then there exists $C > 0$ such that $q_r < C$ for all $r \geq 1$. Let $\overline{x}_k \to \overline{x}_o(\overline{N}^p_\theta)$ and $\varepsilon > 0$. There exists $B > 0$ such that for every $j \geq i$

$$A_j = \frac{1}{h_j}\sum_{k\in I_j}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k} < \varepsilon.$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, 3, \ldots$. Now let n be any integer with $k_{r-1} < n < k_r$,where $r \geq B$. Then

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k} &\leq \frac{1}{k_{r-1}}\sum_{k=1}^{k_r}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k}\\ &= \frac{1}{k_{r-1}}\sum_{k\in I_1}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k} + \frac{1}{k_{r-1}}\sum_{k\in I_2}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k}\\ &\quad + \cdots + \frac{1}{k_{r-1}}\sum_{k\in I_r}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k}\\ &= \frac{k_1}{k_{r-1}k_1}\sum_{k\in I_1}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k} + \frac{k_2-k_1}{k_{r-1}(k_2-k_1)}\sum_{k\in I_2}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k}\\ &\quad + \cdots + \frac{k_i-k_{i-1}}{k_{r-1}(k_i-k_{i-1})}\sum_{k\in I_i}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k}\\ &\quad + \cdots + \frac{k_r-k_{r-1}}{k_{r-1}(k_r-k_{r-1})}\sum_{k\in I_r}\left[d\left(\overline{x}_k,\overline{x}_o\right)\right]^{p_k}\end{aligned}$$

$$= \frac{k_1}{k_{r-1}}A_1 + \frac{k_2 - k_1}{k_{r-1}}A_2 + \cdots + \frac{k_i - k_{i-1}}{k_{r-1}}A_i + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}}A_r$$
$$\leq \left\{\sup_{j\geq 1} A_j\right\} \frac{k_i}{k_{r-1}} + \left\{\sup_{j\succeq i} A_j\right\} \frac{k_r - k_i}{k_{r-1}}$$
$$\leq K\frac{k_i}{k_{r-1}} + \varepsilon C.$$

Since $k_{r-1} \to \infty$ as $n \to \infty$, it follows that $\frac{1}{n}\sum_{k=1}^{n} \left[d\left(\overline{x}_k, \overline{x}_o\right)\right]^{p_k} \to 0$. Hence $\overline{x}_k \to \overline{x}_o(\overline{N}^p)$.

(iii) It follows from (i) and (ii). □

Theorem 3.5. *Let $\theta = (k_r)$ be a lacunary sequence and $\overline{x} = (\overline{x}_k)$ be a sequence of interval numbers. Then*

(i) $\overline{x}_k \to \overline{x}_o(\overline{N}_\theta^p)$ implies $\overline{x}_k \to \overline{x}_o(\overline{s}_\theta)$;

(ii) $\overline{x} = (\overline{x}_k) \in \overline{m}$ and $\overline{x}_k \to \overline{x}_o\left(\overline{s}_\theta\right)$ imply $\overline{x}_k \to \overline{x}_o(\overline{N}_\theta^p)$;

(iii) If $\overline{x} = (\overline{x}_k) \in \overline{m}$, then $\overline{x}_k \to \overline{x}_o(\overline{N}_\theta^p)$ and $\overline{x}_k \to \overline{x}_o\left(\overline{s}_\theta\right)$;

where $\overline{m} = \{\overline{x} = (\overline{x}_k) : \ \sup_k d\left(\overline{x}_k, \overline{x}_o\right) < \infty\}$.

Proof. (i) Let $\varepsilon > 0$ and $\overline{x}_k \to \overline{x}_o(\overline{N}_\theta^p)$. Then we write

$$\frac{1}{h_r}\sum_{k\in I_r}\left[d\left(\overline{x}_k, \overline{x}_o\right)\right]^{p_k} = \frac{1}{h_r}\sum_{\substack{k\in I_r\\ d(\overline{x}_k,\overline{x}_o)\geq\varepsilon}}\left[d\left(\overline{x}_k, \overline{x}_o\right)\right]^{p_k} + \frac{1}{h_r}\sum_{\substack{k\in I_r\\ d(\overline{x}_k,\overline{x}_o)<\varepsilon}}\left[d\left(\overline{x}_k, \overline{x}_o\right)\right]^{p_k}$$
$$\geq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ d(\overline{x}_k,\overline{x}_o)\geq\varepsilon}}\left[d\left(\overline{x}_k, \overline{x}_o\right)\right]^{p_k} \geq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ d(\overline{x}_k,\overline{x}_o)\geq\varepsilon}}\varepsilon^{p_k}$$
$$\geq \frac{1}{h_r}\left|\{k \in I_r : \ d\left(\overline{x}_k, \overline{x}_o\right) \geq \varepsilon\}\right| \min\left(\varepsilon^h, \varepsilon^H\right).$$

Hence $\overline{x}_k \to \overline{x}_o\left(\overline{s}_\theta\right)$.

(ii) Suppose that $\overline{x} = (\overline{x}_k) \in \overline{m}$ and $\overline{x}_k \to \overline{x}_o\left(\overline{s}_\theta\right)$. Since $\overline{x} = (\overline{x}_k) \in \overline{m}$, there is a constant $C > 0$ such that $d\left(\overline{x}_k, \overline{x}_o\right) \leq C$. Given $\varepsilon > 0$, we have

$$\frac{1}{h_r}\sum_{k\in I_r}\left[d\left(\overline{x}_k, \overline{x}_o\right)\right]^{p_k} = \frac{1}{h_r}\sum_{\substack{k\in I_r\\ d(\overline{x}_k,\overline{x}_o)\geq\varepsilon}}\left[d\left(\overline{x}_k, \overline{x}_o\right)\right]^{p_k} + \frac{1}{h_r}\sum_{\substack{k\in I_r\\ d(\overline{x}_k,\overline{x}_o)<\varepsilon}}\left[d\left(\overline{x}_k, \overline{x}_o\right)\right]^{p_k}$$
$$\leq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ d(\overline{x}_k,\overline{x}_o)\geq\varepsilon}}\max\left(C^h, C^H\right) + \frac{1}{h_r}\sum_{\substack{k\in I_r\\ d(\overline{x}_k,\overline{x}_o)<\varepsilon}}\varepsilon^{p_k}$$
$$\leq \max\left(C^h, C^H\right)\frac{1}{h_r}\left|\{k \in I_r : \ d\left(\overline{x}_k, \overline{x}_o\right) \geq \varepsilon\}\right| + \max\left(\varepsilon^h, \varepsilon^H\right).$$

Thus we obtain $\overline{x}_k \to \overline{x}_o(\overline{N}_\theta^p)$.

(iii) It follows from (i) and (ii). □

Theorem 3.6. *Let $\theta = (k_r)$ be a lacunary sequence and $\overline{x} = (\overline{x}_k)$ be a sequence of interval numbers. Then*

(i) For $\liminf_r q_r > 1$, then $\overline{x}_k \to \overline{x}_o\,(\overline{s})$ implies $\overline{x}_k \to \overline{x}_o\,(\overline{s}_\theta)$;

(ii) For $\limsup_r q_r < \infty$, then $\overline{x}_k \to \overline{x}_o\,(\overline{s}_\theta)$ implies $\overline{x}_k \to \overline{x}_o\,(\overline{s})$;

(iii) If $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, then $\overline{x}_k \to \overline{x}_o\,(\overline{s})$ if and only if $\overline{x}_k \to \overline{x}_o\,(\overline{s}_\theta)$.

Proof. (i) Suppose that $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1+\delta$ for sufficiently large r, which implies

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$$

if $\overline{x}_k \to \overline{x}_o\,(\overline{s})$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r}\left|\{k \leq k_r : d(\overline{x}_k, \overline{x}_o) \geq \varepsilon\}\right| &\geq \frac{1}{k_r}\left|\{k \in I_r : d(\overline{x}_k, \overline{x}_o) \geq \varepsilon\}\right| \\ &\geq \frac{\delta}{1+\delta}.\frac{1}{h_r}\left|\{k \in I_r : d(\overline{x}_k, \overline{x}_o) \geq \varepsilon\}\right|.\end{aligned}$$

Hence $\overline{x}_k \to \overline{x}_o\,(\overline{s}_\theta)$.

(ii) If $\limsup_r q_r < \infty$, then there exists $C > 0$ such that $q_r < C$ for all $r \geq 1$. Let $\overline{x}_k \to \overline{x}_o\,(\overline{s}_\theta)$ and set $A_r = \left|\{k \in I_r : d(\overline{x}_k, \overline{x}_o) \geq \varepsilon\}\right|$. Then there exists an $r_o \in \mathbb{N}$ such that

$$\frac{A_r}{h_r} < \varepsilon \text{ for all } r > r_o. \tag{3.1}$$

Now let $N = \max\{A_r :\ 1 \leq r \leq r_o\}$ and choose n such that $k_{r-1} < n \leq k_r$. Then we have

$$\begin{aligned}\frac{1}{k_r}\left|\{k \leq n : d(\overline{x}_k, \overline{x}_o) \geq \varepsilon\}\right| &\leq \frac{1}{k_{r-1}}\left|\{k \leq k_r : d(\overline{x}_k, \overline{x}_o) \geq \varepsilon\}\right| \\ &= \frac{1}{k_{r-1}}\left\{A_1 + A_2 + \cdots + A_{r_o} + A_{(r_o+1)} + \cdots + A_r\right\} \\ &\leq \frac{N}{k_{r-1}}r_o + \frac{1}{k_{r-1}}\left\{h_{r_o+1}\frac{A_{(r_o+1)}}{h_{r_o+1}} + \cdots + h_r\frac{A_r}{h_r}\right\} \\ &\leq \frac{N}{k_{r-1}}r_o + \frac{1}{k_{r-1}}\left(\sup_{r>r_o}\frac{A_r}{h_r}\right)\{h_{r_o+1} + \cdots + h_r\} \\ &\leq \frac{N}{k_{r-1}}r_o + \varepsilon\frac{k_r - k_{r_o}}{k_{r-1}}, \quad by\,(3.1) \\ &\leq \frac{N}{k_{r-1}}r_o + \varepsilon q_r \\ &\leq \frac{N}{k_{r-1}}r_o + \varepsilon C.\end{aligned}$$

Thus we obtain $\overline{x}_k \to \overline{x}_o\,(\overline{s})$.

(iii) It follows from (i) and (ii). □

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