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A New Extension of Hermite-Hermite Matrix Polynomials and Their Properties

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Abstract : The main aim of this paper is to define and study of a new polynomial, say, two-index Hermite-Hermite matrix polynomials. An explicit representation, a matrix recurrence relation and matrix differential equations of these polynomials are presented. A new expansion of the matrix functions for a wide class of matrices in terms of two-index Hermite-Hermite matrix polynomials is proposed.

Keywords : Matrix functions; Two-index Hermite-Hermite matrix polynomials; Matrix recurrence relation; Two-index Hermite-Hermite matrix differential equation.
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1 Introduction and Preliminaries

Special functions, as a branch of mathematics are of utmost importance to scientists and engineers in many areas of applications [1]. Hermite and Chebyshev polynomials in [1] are among the most important special functions, with very diverse applications to physics, engineering and mathematical physics ranging from abstract number theory to problems of physics and engineering. Recently, the Hermite matrix polynomials have been introduced and studied in a number of papers [2–8]. In [9–17], extension to the matrix framework of the classical families of Hermite-Hermite, Hermite, Laguerre, Chebyshev and Gegenbauer polynomials have been proposed.

Our main aim in this paper deals with the introduction and study of Hermite matrix polynomials taking advantage of those recently treated in [2, 11, 12]. The organization of this paper is as follows. In Section 2 some new properties of two-index Hermite-Hermite matrix polynomials such as the three terms recurrence formula are established and deals with the two-index Hermite-Hermite matrix polynomials series expansions of $\exp(xA)$, $\cos(xA)$, $\sin(xA)$ $\cosh(xA)$ and $\sinh(xA)$ of an arbitrary matrix as well as with theirs finite series truncation. Finally, we study the case relevant to results are applied to certain differential equation in terms of Hermite-Hermite matrix polynomials.

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principal logarithm of z, then $z^{\frac{1}{2}}$ represents $\exp(\frac{1}{2}\log(z))$. If A is a matrix in $\mathbb{C}^{N\times N}$, its two-norm denoted by $||A||_2$ is defined by

$$||A||_2 = \frac{||Ax||_2}{||x||_2},$$

where for a vector y in $\mathbb{C}^N$, $||y||_2$ denotes the Euclidean norm of y, $||y||_2 = (y^T y)^{\frac{1}{2}}$. The set of all eigenvalues of A is denoted by $\sigma(A)$. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and if A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A) \subset \Omega$, then the matrix functional calculus [18] yields that

$$f(A)g(A) = g(A)f(A). \tag{1.1}$$

If A is a matrix with $\sigma(A) \subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A} = \exp(\frac{1}{2}\log(A))$ denotes the image by $z^{\frac{1}{2}} = \sqrt{z} = \exp(\frac{1}{2}\log(z))$ of the matrix functional calculus acting on the matrix A. We say that A is a positive stable matrix [9, 5, 7] if

$$Re(z) > 0 \quad \text{for all } z \in \sigma(A). \tag{1.2}$$

If $A(k,n)$ and $B(k,n)$ are matrices on $\mathbb{C}^{N\times N}$ for $n \geq 0$, $k \geq 0$, it follows in an analogous way to the proof of Lemma 11 of [1] that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]} A(k,n-mk), \;\; \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k) \tag{1.3}$$

for m is a positive integer, similarly to (1.3), we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+mk), \;\; \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k). \tag{1.4}$$

1.1 Extension of the Hermite Matrix Polynomials

One of the most direct ways of exploring generalized classes of Hermite matrix polynomials is to start from modified forms of the ordinary Hermite matrix polynomials generating matrix function. We consider therefore the generalized Hermite matrix polynomials $H_{n,m}(x,A)$ defined a two-index by the generating matrix function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n,m}(x,A) = \exp\left(xt\sqrt{mA} - t^m I\right) \tag{1.5}$$

with m being a positive integer. The matrix polynomials $H_{n,m}(x,A)$ are explicitly provided by

$$H_{n,m}(x,A) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k}{k!(n-mk)!} \left(x\sqrt{mA}\right)^{n-mk}, n \geq 0. \tag{1.6}$$

It is clear that

$$H_{-1,m}(x,A) = 0, \;\; H_{0,m}(x,A) = I, \;\; H_{1,m}(x,A) = x\sqrt{mA}.$$

which reduces to the ordinary case for $m = 2$ in [3, 4, 7]. Their recurrence properties can be derived either from (1.5) or from (1.6). It is indeed easy to prove that

$$\begin{aligned} &\frac{d}{dx} H_{n,m}(x,A) = n\sqrt{mA} H_{n-1,m}(x,A), \\ &H_{n+1,m}(x,A) = \left[x\sqrt{mA} - m\frac{d^{m-1}}{dx^{m-1}}\left(\sqrt{mA}\right)^{1-m}\right] H_{n,m}(x,A). \end{aligned} \tag{1.7}$$

The matrix differential equation satisfied by $H_{n,m}(x,A)$ can be straightforwardly deduced by introducing the shift operators

$$\widehat{P} = \frac{d}{dx}\left(\sqrt{mA}\right)^{-1}, \;\; \widehat{M} = x\sqrt{mA} - m\frac{d^{m-1}}{dx^{m-1}}\left(\sqrt{mA}\right)^{1-m} \tag{1.8}$$

which act on $H_{n,m}(x,A)$ according to the rules

$$\widehat{P} H_{n,m}(x,A) = nH_{n-1,m}(x,A), \;\; \widehat{M} H_{n,m}(x,A) = H_{n+1,m}(x,A). \tag{1.9}$$

Using the identity

$$\widehat{M}\widehat{P} H_{n,m}(x,A) = nH_{n,m}(x,A) \tag{1.10}$$

holds using the explicit definition of $\widehat{M}$ and $\widehat{P}$ given by (1.10), we find that $H_{n,m}(x,A)$ satisfies the following ordinary matrix differential equation of m^{th} order

$$\left[\frac{d^m}{dx^m} I - \frac{x}{m}\frac{d}{dx}\left(\sqrt{mA}\right)^m + \frac{n}{m}\left(\sqrt{mA}\right)^m\right] H_{n,m}(x,A) = 0. \tag{1.11}$$

We consider the operational definition of Hermite matrix polynomials [2] in the form

$$H_{n,m}(x,A) = \exp\left(-\frac{d^m}{dx^m}\left(\sqrt{mA}\right)^{-m}\right)\left(x\sqrt{mA}\right)^n. \tag{1.12}$$

Our aim is to prove some known properties as well as new expansions formulae related to these two-index Hermite-Hermite matrix polynomials. In the following, we will apply the above results to Hermite matrix polynomials and we will see that the results, summarized in this section, can be exploited to state quite general results.

2 Definition of Two-index Hermite-Hermite Matrix Polynomials

Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.2). The two-index Hermite-Hermite matrix polynomials are defined by the series

$$ {}_{H}H_{n,m}(x,A) = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k\left(\sqrt{mA}\right)^{n-mk}}{k!(n-mk)!}H_{n-mk,m}(x,A). \tag{2.1} $$

Using (1.4), (1.5) and (2.1), we arrange the series in the form

$$ \begin{aligned} \sum_{n=0}^{\infty}\frac{t^n}{n!}\,{}_{H}H_{n,m}(x,A) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k\left(\sqrt{mA}\right)^{n-mk}H_{n-mk,m}(x,A)}{k!(n-mk)!}t^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\sqrt{mA}\right)^{n}H_{n,m}(x,A)}{k!n!}t^{n+mk} \\ &= \sum_{n=0}^{\infty}\frac{\left(\sqrt{mA}\right)^{n}}{n!}t^nH_{n,m}(x,A)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t^{mk}I \\ &= \exp\left(xt\left(\sqrt{mA}\right)^2-\left(t\sqrt{mA}\right)^m\right)\exp\left(-t^mI\right) \\ &= \exp\left(xt\left(\sqrt{mA}\right)^2-t^m\left(\left(\sqrt{mA}\right)^m+I\right)\right). \end{aligned} \tag{2.2} $$

We obtain an explicit representation for the two-index Hermite-Hermite matrix polynomials by the generating matrix function in the form

$$ \begin{aligned} F(x,t,A) &= \sum_{n=0}^{\infty}\frac{{}_{H}H_{n,m}(x,A)t^n}{n!} \\ &= \exp\left(xt\left(\sqrt{mA}\right)^2-t^m\left(\left(\sqrt{mA}\right)^m+I\right)\right); \quad |t|<\infty \end{aligned} \tag{2.3} $$

where $F(x,t,A)$ regarded as a function of the complex variable t is an entire matrix, therefore has the Taylor series about $t=0$ and the series obtained converges for all values of x and t.

2.1 Matrix Recurrence Relations

Some matrix recurrence relations will be established for the two-index Hermite matrix polynomials. First, we obtain

Theorem 2.1. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying (1.2). The two-index Hermite-Hermite matrix polynomials ${}_HH_{m,n}(x,A)$ satisfy the relations*

$$\frac{d^r}{dx^r}\ {}_HH_{n,m}(x,A) = \frac{\left(\sqrt{mA}\right)^{2r} n!}{(n-r)!}\ {}_HH_{n-r,m}(x,A), \quad 0\le r\le n. \tag{2.4}$$

Proof. Differentiating the identity (2.3) with respect to x yields

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d}{dx}\ {}_HH_{n,m}(x,A) = t\left(\sqrt{mA}\right)^2 \exp\left(xt\left(\sqrt{mA}\right)^2 - t^m\left(\left(\sqrt{mA}\right)^m + I\right)\right). \tag{2.5}$$

From (2.5) and (2.3), we have

$$\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d}{dx}\ {}_HH_{n,m}(x,A)t^n = \left(\sqrt{mA}\right)^2\sum_{n=0}^{\infty}\frac{1}{n!}\ {}_HH_{n,m}(x,A)t^{n+1}.$$

Hence, identifying the coefficients at t^n, we obtain

$$\frac{d}{dx}\ {}_HH_{n,m}(x,A) = n\left(\sqrt{mA}\right)^2\ {}_HH_{n-1,m}(x,A), \quad n\ge 1. \tag{2.6}$$

Iteration (2.6) for $0\le r\le n$ implies (2.4). Therefore, the expression (2.4) is established and the proof of Theorem 2.1 is completed. □

The above three-terms recurrence relation will be used in the following theorem.

Theorem 2.2. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.2). Then we have*

$$\begin{aligned} {}_HH_{n,m}(x,A) =& x\left(\sqrt{mA}\right)^2\ {}_HH_{n-1,m}(x,A)\\ &- \frac{m\,(n-1)!}{(n-m)!}\left(\left(\sqrt{mA}\right)^m + I\right) HH_{n-m,m}(x,A),\ n\ge m. \end{aligned} \tag{2.7}$$

Proof. Differentiating (2.3) with respect to x and t, we find respectively

$$\begin{aligned} \frac{\partial}{\partial x}F(x,t,A) &= t\left(\sqrt{mA}\right)^2\exp\left(xt\left(\sqrt{mA}\right)^2 - t^m\left(\left(\sqrt{mA}\right)^m + I\right)\right)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d}{dx}\ {}_HH_{n,m}(x,A)t^n \end{aligned}$$

and

$$\frac{\partial}{\partial t}F(x,t,A) = \left(x\left(\sqrt{mA}\right)^2 - mt^{m-1}\left(\left(\sqrt{mA}\right)^m + I\right)\right) \times \exp\left(xt\left(\sqrt{mA}\right)^2 - t^m\left(\left(\sqrt{mA}\right)^m + I\right)\right) = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\ {}_HH_{n,m}(x,A)t^{n-1}.$$

Therefore, $F(x,t,A)$ satisfies the partial matrix differential equation

$$x\left(\sqrt{mA}\right)^2 - mt^{m-1}\left(\left(\sqrt{mA}\right)^m + I)\right)\frac{\partial F}{\partial x} - t\left(\sqrt{mA}\right)^2\frac{\partial F}{\partial t} = 0$$

which, by virtue of (2.3), becomes

$$\left(\sqrt{mA}\right)^2\sum_{n=0}^{\infty}\frac{n}{n!}\ {}_HH_{n,m}(x,A)t^n = x\left(\sqrt{mA}\right)^2\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d}{dx}\ {}_HH_{n,m}(x,A)t^n - m\left(\left(\sqrt{mA}\right)^m + I\right)\sum_{n=0}^{\infty}\frac{d}{dx}\frac{1}{n!}\ {}_HH_{n,m}(x,A)t^{n+m-1}.$$

Since $x\frac{d}{dx}\ {}_HH_{1,m}(x,A) = {}_HH_{1,m}(x,A)$, it follows that

$$n\left(\sqrt{mA}\right)^2\ {}_HH_{n,m}(x,A) = x\left(\sqrt{mA}\right)^2\frac{d}{dx}\ {}_HH_{n,m}(x,A) - \frac{m\,n!\left(\left(\sqrt{mA}\right)^m + I\right)}{(n-m+1)!}\frac{d}{dx}\ {}_HH_{n-m+1,m}(x,A). \tag{2.8}$$

Using (2.6) and (2.8), we get (2.7). The proof of Theorem 2.2 is completed. □

The above recurrence properties can be derived either from (2.1) or from (2.2). It is easy to prove that

$$\frac{d}{dx}\ {}_HH_{n,m}(x,A) = n\left(\sqrt{mA}\right)^2\ {}_HH_{n-1,m}(x,A) = mnA\ {}_HH_{n-1,m}(x,A),$$
$${}_HH_{m,n+1}(x,A) = \left[mxA - m\frac{d^{m-1}}{dx^{m-1}}\left(\left(\sqrt{mA}\right)^m + I\right)(mA)^{1-m}\right]{}_HH_{n,m}(x,A). \tag{2.9}$$

The matrix differential equation satisfied by ${}_HH_{n,m}(x,A)$ can be straightforwardly inferred by introducing the shift operators

$$\widetilde{P} = \frac{d}{dx}(mA)^{-1},\quad \widetilde{M} = mxA - m\frac{d^{m-1}}{dx^{m-1}}((\sqrt{mA})^m + I)(mA)^{m-1} \tag{2.10}$$

which act on ${}_H H_{n,m}(x,A)$ according to the rules

$$\widetilde{P}\ {}_H H_{n,m}(x,A) = n\ {}_H H_{n-1,m}(x,A),\ \ \widetilde{M}\ {}_H H_{n,m}(x,A) = {}_H\ H_{n+1,m}(x,A). \tag{2.11}$$

Using the identity

$$\widetilde{M}\widetilde{P}\ {}_H H_{n,m}(x,A) = n\ {}_H H_{n,m}(x,A) \tag{2.12}$$

from (2.12), we find that ${}_H H_{n,m}(x,A)$ satisfies the following ordinary matrix differential equation of the m^{th} order

$$\left[m\frac{d^m}{dx^m}\left(\left(\sqrt{mA}\right)^m + I\right) - x\frac{d}{dx}(mA)^m + n(mA)^m\right]\ {}_H H_{n,m}(x,A) = 0. \tag{2.13}$$

In the next result, the two-index Hermite-Hermite matrix polynomials appear as finite series solutions of the m^{th} order order matrix differential equation.

Corollary 2.3. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying (1.2). The two-index Hermite-Hermite matrix polynomials are solutions of the matrix differential equation*

$$\left[m\frac{d^m}{dx^m}\left(\left(\sqrt{mA}\right)^m + I\right) - x\frac{d}{dx}(mA)^m + n(mA)^m\right]\ {}_H H_{n,m}(x,A) = 0,\ n \geq 0. \tag{2.14}$$

Proof. Replacing n by $n-m+1$ in (2.6) gives

$$\frac{d}{dx}\ {}_H H_{n-m+1,m}(x,A)) = (n-m+1)(mA)\ {}_H H_{n-m,m}(x,A). \tag{2.15}$$

Substituting from (2.15) into (2.4) yields

$$\begin{aligned}\frac{d^m}{dx^m}\ {}_H H_{n,m}(x,A) &= \frac{n!}{(n-m)!}(mA)^m\ {}_H H_{n-m,m}(x,A)\\ &= \frac{n!}{(n-m+1)!}(mA)^{m-1}\frac{d}{dx}\ {}_H H_{n-m+1,m}(x,A).\end{aligned} \tag{2.16}$$

From (2.7), (2.15) and (2.16) we obtain (2.14). Thus the proof of Corollary 2.3 is completed. □

2.2 Expansion of Two-index Hermite-Hermite Matrix Polynomials

Now, we can use the expansion of two-index Hermite-Hermite matrix polynomials together with their properties to prove the following result.

Theorem 2.4. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying (1.2). Then, we have*

$$(mxA)^n = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{\left(\left(\sqrt{mA}\right)^m + I\right)^k}{k!(n-mk)!}\ {}_H H_{n-mk,m}(x,A),\ \ -\infty < x < \infty. \tag{2.17}$$

Proof. By (1.3) and (2.2) we can write

$$\begin{aligned}\exp\left(mxtA\right) &= \sum_{n=0}^{\infty}\frac{(mxA)^n}{n!}t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left(\left(\sqrt{mA}\right)^m+I\right)^k \ {}_HH_{n,m}(x,A)}{n!k!}t^{n+mk}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}\frac{\left(\left(\sqrt{mA}\right)^m+I\right)^k \ {}_HH_{n-mk,m}(x,A)}{k!(n-mk)!}t^n.\end{aligned} \tag{2.18}$$

Expanding the left-hand side of (2.18) into powers of t and identifying the coefficients of t^n on both sides gives (2.17). Therefore, the expression (2.17) is established and the proof of Theorem 2.4 is completed. □

2.3 Two-index Hermite-Hermite Matrix Polynomials Series Expansions

It is well-known that the matrix exponential plays an important role in many different fields. Using two-index Hermite-Hermite matrix polynomial series we propose new expansions of the matrices $\exp(xB)$, $\sin(xB)$, $\cos(xB)$, $\cosh(xB)$ and $\sinh(xB)$ for matrices satisfying the spectral property

$$|Re(x)| > |Im(x)| \quad \text{for all } x \in \sigma(B). \tag{2.19}$$

Theorem 2.5. *Let B be a matrix in $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ satisfying the condition (2.19). Then*

$$\exp\left(xB\right) = \exp\left(\left(\sqrt{B}\right)^m+I\right)\sum_{n=0}^{\infty}\frac{1}{n!}\ {}_HH_{n,m}\left(x,\frac{1}{m}B\right), -\infty < x < \infty, \tag{2.20}$$

$$\cos(xB) = \exp\left((-1)^{\frac{m}{2}}\left(\left(\sqrt{B}\right)^m+I\right)\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\ {}_HH_{2n,m}\left(x,\frac{1}{m}B\right), -\infty < x < \infty, \tag{2.21}$$

$$\sin(xB) = \exp\left((-1)^{\frac{m}{2}}\left(\left(\sqrt{B}\right)^m+I\right)\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\ {}_HH_{2n+1,m}\left(x,\frac{1}{m}B\right), \tag{2.22}$$

$-\infty < x < \infty$,

$$\cosh(xB) = \exp\left(\left(\sqrt{B}\right)^m+I\right)\sum_{n=0}^{\infty}\frac{1}{(2n)!}\ {}_HH_{2n,m}\left(x,\frac{1}{m}B\right), -\infty < x < \infty \tag{2.23}$$

and

$$\sinh(xB) = \exp\left(\left(\sqrt{B}\right)^m+I\right)\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\ {}_HH_{2n+1,m}\left(x,\frac{1}{m}B\right), -\infty < x < \infty. \tag{2.24}$$

Proof. Let $A = \frac{1}{m}B$. By the spectral mapping theorem [3, 4, 6, 18] and (2.19), it follows that

$$\sigma(A) = \left\{\frac{b}{m}; b \in \sigma(B)\right\},\ Re\left(\frac{b}{m}\right) = \frac{1}{m}\{Re(b) - Im(b)\} > 0,\ b \in \sigma(B). \tag{2.25}$$

Thus A is a positive stable matrix and taking $t = 1$ in (2.2), $B = mA$ gives

$$\exp\left(xB - \left(\left(\sqrt{B}\right)^m + I\right)\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\ {}_H H_{n,m}\left(x, \frac{1}{m}B\right). \tag{2.26}$$

Therefore, (2.20) follows.

Considering (2.17) for the positive stable matrix $A = \frac{1}{m}B$, we obtain that

$$(xB)^{2n} = (2n)! \sum_{k=0}^{[\frac{2n}{m}]} \frac{\left(\left(\sqrt{B}\right)^m + I\right)^k}{k!(2n-mk)!}\ {}_H H_{2n-mk,m}\left(x, \frac{1}{m}B\right).$$

Taking into account the series expansion of $\cos(xB)$ and (1.4), we can write

$$\begin{aligned}
\cos(xB) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(xB)^{2n} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{2n}{m}]} \frac{(-1)^n \left(\left(\sqrt{B}\right)^m + I\right)^k}{k!(2n-mk)!}\ {}_H H_{2n-mk,m}\left(x, \frac{1}{m}B\right) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n+\frac{mk}{2}} \left(\left(\sqrt{B}\right)^m + I\right)^k}{k!(2n)!}\ {}_H H_{2n,m}\left(x, \frac{1}{m}B\right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{\frac{mk}{2}} \left(\left(\sqrt{B}\right)^m + I\right)^k}{k!} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\ {}_H H_{2n,m}\left(x, \frac{1}{m}B\right) \\
&= \exp\left((-1)^{\frac{mk}{2}} \left(\left(\sqrt{B}\right)^m + I\right)\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\ {}_H H_{2n,m}\left(x, \frac{1}{m}B\right).
\end{aligned}$$

Therefore, (2.21) follows. By similar arguments we can prove the relations (2.22), (2.23) and (2.24). Moreover, the convergence of the matrix series appearing in (2.20)-(2.23) and (2.24) to the respective matrix function $\exp(xB)$, $\sin(xB)$, $\cos(xB)$, $\sinh(xB)$ and $\cosh(xB)$ is uniform in any bounded interval of the real axis. Therefore, the result is established. □

Remark 2.6. *The series developments given by (2.20)-(2.24) have one important advantage as compared to the Taylor series, from the computational point of view. In fact, the advantage follows from the fact that it is not necessary to compute the powers B^n of the matrix B, as well as from the fact that using relationship (2.7), the two-index Hermite-Hermite matrix polynomials can be computed recurrently in terms of ${}_H H_{0,m}(x, \frac{1}{m}B) = I$ and ${}_H H_{1,m}(x, \frac{1}{m}B) = xB$.*

In the next theorem we obtain another representation for the two-index Hermite-Hermite matrix polynomials.

Theorem 2.7. *Suppose that A is a matrix in $\mathbb{C}^{N\times N}$ satisfying (1.2). Then the two-index Hermite-Hermite matrix polynomials have the representation*

$$ {}_{H}H_{n,m}(x,A) = \exp\left(-\frac{d^m}{dx^m}(mA)^{-m}\right)\left(\sqrt{mA}\right)^n H_{n,m}(x,A). \tag{2.27}$$

Proof. It is clear by (1.7) and (2.1) that

$$\begin{aligned}
&\exp\left(-\frac{d^m}{dx^m}(mA)^{-m}\right)\left(\sqrt{mA}\right)^n H_{n,m}(x,A)\\
&\quad = \sum_{n=0}^{\infty}\frac{(-1)^k}{k!}\frac{d^{mk}}{dx^{mk}}(mA)^{-mk}\left(\sqrt{mA}\right)^n H_{n,m}(x,A)\\
&\quad = n!\sum_{k=0}^{\infty}\frac{(-1)^k\left(\sqrt{mA}\right)^{-mk}}{k!(n-mk)!}\left(\sqrt{mA}\right)^n H_{n-mk,m}(x,A)\\
&\quad = n!\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k\left(\sqrt{mA}\right)^{n-mk}}{k!(n-mk)!}H_{n-mk,m}(x,A) = {}_{H}H_{n,m}(x,A).
\end{aligned}$$

Therefore, the result is established. □

The use of the inverse of (2.27) allows to conclude that

$$H_{n,m}(x,A) = \exp\left(\frac{d^m}{dx^m}(mA)^{-m}\right)\left(\sqrt{mA}\right)^{-n} {}_{H}H_{n,m}(x,A). \tag{2.28}$$

Using (2.9) and substituting for n the values $0,1,...,n-1$, we get

$$ {}_{H}H_{n,m}(x,A) = \left[mxA - m\frac{d^{m-1}}{dx^{m-1}}\left(\left(\sqrt{mA}\right)^m + I\right)(mA)^{m-1}\right]^n {}_{H}H_{0,m}(x,A). \tag{2.29}$$

The two-index Hermite-Hermite matrix polynomials are a particular case of ${}_{H}H_{n,m}(x,A)$, accordingly, ${}_{H}H_{n,m}(x,A) = {}_{H}H_{n,2}(x,A)$ in [14]. These last identities indicate that the method described in this paper can go beyond the specific problem addressed here and can be exploited in a wider context.

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