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A Non-uniform Concentration Inequality for Randomized Orthogonal Array Sampling Designs

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Abstract : Let $f : [0,1]^3 \to \mathbb{R}$ be a measurable function. In many computer experiments, we estimate the value of $\int_{[0,1]^3} f(x)dx$, which is the mean $\mu = E(f \circ X)$, where X is a uniform random vector on the unit hypercube $[0,1]^3$. In 1992 and 1993, Owen and Tang introduced randomized orthogonal arrays to choose the sampling points to estimate the integral.

In this paper, we give a non-uniform concentration inequality for randomized orthogonal array sampling designs.

Keywords : Computer experiment, orthogonal array sampling designs, concentration inequality.
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1 Introduction and Main Result

Let X be the uniform random vector on $[0,1]^d$ and $f : [0,1]^d \to \mathbb{R}$. The aim of many experiments is to know

$$\mu = \int_{[0,1]^d} f(x)dx = E(f \circ X)$$

but it is often expensive to compute.

For examples, consider an electrical circuit, the performance of which depends on a number of quantities(capacitances, resistances) that vary from circuit to circuit, the fluid flow problems or computer graphics. A mathematical model for the device is developed from which we can simulate the behavior of the device on a computer and we often want to compute the expected value of some measure of performance of the device, given by the function $E(f \circ X)$. So we have the problem of estimating the expected value of some function.

It is well known that as the dimension d increases, Monte Carlo methods are useful and competitive(see, Davis and Rabinowitz (1984), chap. 5.10, Niederreiter(1992), Evans and Swartz (2000)). Hence, Monte Carlo methods are usually used for high-dimensional problems. That is, n values of the input random vector, $X_1, X_2, \ldots, X_n$, are generated in some fashion such that the expected value $E(f \circ X)$ can be estimated by

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} f \circ X_i. \tag{1.1}$$

It is important to pick a sampling that allows us to estimate $E(f \circ X)$. There are many methods for choosing $X_1, X_2, ..., X_n$. For examples, *simple random sampling*, i.e., n iid random vectors with the distribution of X, *lattice sampling* (see Patterson(1954)), *Latin hypercube sampling*(see, McKay, Conover and Beckman(1979), Stein(1987), Owen(1992b), Loh(1996)), *the orthogonal arrays* (see, Owen(1992a), Tang(1993)), *scrambled net*(see, Owen(1997a), (1997b)). In this work, we investigate orthogonal arrays sampling.

An orthogonal array of strength t with index λ ($\lambda \geq 1$), is an $n \times d$ matrix with elements taken from the set $\{0, 1, \ldots, q-1\}$ such that for any $n \times t$ submatrix, each of the q^t possible rows appears the same number λ of times where d, n, q and t are positive integers with $t \leq d$ and $q \geq 2$. Of course $n = \lambda q^t$. A class of this arrays is denoted by $OA(n, d, q, t)$(see Raghavarao(1971) for more details).

In 1996, Loh(1996) considered the class $OA(n, 3, q, 2)$ when $n = q^2$ and constructed the sampling $X_1, X_2, \ldots, X_{q^2}$ on the unit cube $[0, 1]^3$ as follows: Let

(a) π_1, π_2, π_3 be random permutations of $\{0, 1, \ldots, q-1\}$,

(b) $U_{i_1,i_2,i_3,j}$ be $[0, 1]$ uniform random variables where $i_1, i_2, i_3 \in \{0, 1, \ldots, q-1\}$, $j \in \{1, 2, 3\}$, and

(c) $U_{i_1,i_2,i_3,j}$'s and π_k's be all stochastically independent.

An orthogonal array-based sample of size q^2, $\{X_1, X_2, \ldots, X_{q^2}\}$, is defined to be

$$\left\{X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})) : 1 \leq i \leq q^2\right\},$$

where, for each $i_1, i_2, i_3 \in \{0, 1, \ldots, q-1\}$ and $j \in \{1, 2, 3\}$,

$$\begin{aligned} X\big(i_1, i_2, i_3\big) &= \big(X_1(i_1, i_2, i_3), X_2(i_1, i_2, i_3), X_3(i_1, i_2, i_3)\big), \\ X_j(i_1, i_2, i_3) &= \frac{i_j + U_{i_1,i_2,i_3,j}}{q}, \end{aligned}$$

and $a_{i,j}$ is the $(i, j)^{th}$ element of some arbitary but fixed $A \in OA(q^2, 3, q, 2)$.

So the estimator $\hat{\mu}$ of μ in (1.1) can be expressed in the form of

$$\hat{\mu} = \frac{1}{q^2} \sum_{i=1}^{q^2} f \circ X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})).$$

Owen(1992a) gave an expression for the asymptotic variance σ^2 of $\hat{\mu}$.

Assume that $\sigma^2 > 0$, we define

$$W = \sigma^{-1}(\hat{\mu} - \mu). \tag{1.2}$$

Loh(1996) gave a uniform bound on the normal approximation of W and Laipaporn and Neammanee improved the bound to the rate $O(q^{-\frac{1}{2}})$ in 2006. Besides the normal approximation, they also investigate a concentration inequality of W.

Let X be a random variable. The function $Q_X : [0, \infty) \to \mathbb{R}$ which defined by

$$Q_X(\lambda) = \sup_x P(x \le X \le x + \lambda)$$

is called a *uniform (Lévy) concentration function* of X and the function $Q_X : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ which defined by

$$Q_X(x; \lambda) = P(x \le X \le x + \lambda)$$

is called a *non-uniform (Lévy) concentration function* of X.

Laipaporn and Neammanee(2006) gave a uniform concentration inequality for W as follows.

Theorem 1.1 *(A uniform concentration inequality)*
Assume that $E(f \circ X)^4 < \infty$. Then, as $q \to \infty$,

$$P(a \le W \le a + \lambda) \le 2\lambda\big(1 + \frac{1}{q-1}\big) + O(\frac{1}{\sqrt{q}}),$$

for any real numbers a and $\lambda \ge 0$.

In this paper, we generalize Theorem 1.1 to the non-uniform case. The main result is following.

Theorem 1.2 *(A non-uniform concentration inequality)*
Assume that $E(f \circ X)^4 < \infty$. Then, there exists a constant C such that

$$P(z \le W \le z + \lambda) \le \frac{C\lambda}{1+z} + \frac{1}{1+z} O(\frac{1}{\sqrt{q}}), \ \ \textit{as } q \to \infty,$$

for any real numbers $z, \ \lambda \ge 0$.

To prove Theorem 1.2, it suffices to prove the following theorem.

Theorem 1.3 *Assume that $E(f \circ X)^4 < \infty$. Then, there exists a constant C such that*

$$P(z \le W \le w) \le \frac{C}{1+z}(w - z) + \frac{1}{1+z} O(\frac{1}{\sqrt{q}}), \ \ \textit{as } q \to \infty,$$

for any real number $0 < z \le w$.

In this paper, we give auxiliary results in section 2 and the non-uniform concentration inequality was proved in section 3.

2 Auxiliary Results

In 1996, Loh defined a random function ρ_π

$$\big(i_1, i_2, \rho_\pi(i_1,i_2)\big) = \big(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})\big) \tag{2.1}$$

for some $i \in \{1, ..., q^2\}$ and showed that W in (1.2) can be rewritten as the form

$$W = \sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1} Y\big(i_1, i_2, \rho_\pi(i_1,i_2)\big) \tag{2.2}$$

where

$$\mu(i_1,i_2,i_3) = Ef\circ X(i_1,i_2,i_3),$$

$$\mu_j(i_j) = \frac{1}{q^2}\sum_{\substack{i_k=0\\ k\neq j}}^{q-1}[\mu(i_1,i_2,i_3)-\mu],$$

$$\mu_{k,l}(i_k,i_l) = \frac{1}{q}\sum_{\substack{i_j=0\\ j\neq k,l}}^{q-1}[\mu(i_1,i_2,i_3)-\mu-\mu_k(i_k)-\mu_l(i_l)],$$

$$Y(i_1,i_2,i_3) = \frac{1}{q^2\sqrt{\mathrm{Var}(\hat{\mu})}}\Big[f\circ X(i_1,i_2,i_3)-\mu-\sum_{j=1}^{3}\mu_j(i_j)-\sum_{1\le k<l\le 3}\mu_{k,l}(i_k,i_l)\Big],$$

$$\tilde{\mu}(i_1,i_2,i_3) = EY(i_1,i_2,i_3).$$

Let I and K be uniformly distributed random variables on $\{0,1,...,q-1\}$, (I,K) uniformly distributed on $\{(i,k)|i,k=0,1,\dots,q-1, i\neq k\}$ and assume that they are indenpendent of all π_1,π_2,π_3 and $U_{i_1,i_2,i_3,j}$'s defined in the previous section.
Let

$$\widetilde{W} = W - S_1 - S_2 + S_3 + S_4$$

where

$$S_1 = \sum_{i_2=0}^{q-1} Y\big(I, i_2, \rho_\pi(I,i_2)\big), \qquad S_2 = \sum_{i_2=0}^{q-1} Y\big(K, i_2, \rho_\pi(K,i_2)\big),$$

$$S_3 = \sum_{i_2=0}^{q-1} Y\big(I, i_2, \rho_\pi(K,i_2)\big), \quad \text{and} \quad S_4 = \sum_{i_2=0}^{q-1} Y\big(K, i_2, \rho_\pi(I,i_2)\big).$$

Note that $\big(W,\widetilde{W}\big)$ is an exchangeable pair in the sense that

$$P(W\le a, \widetilde{W}\le b) = P(\widetilde{W}\le a, W\le b)$$

for every $a, b \in \mathbb{R}$(see [8], p.1213).

For each $i, j, k \in \{0, 1, 2, \ldots, q-1\}$, and $z > 0$, we let

$$Y_z(i,j,k) = Y(i,j,k)I\Big(\big|Y(i,j,k)\big| > 1+z\Big),$$
$$\widehat{Y}_z(i,j,k) = Y(i,j,k)I\Big(\big|Y(i,j,k)\big| \le 1+z\Big),$$
$$\widehat{Y} = \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j))$$

and
$$\widetilde{Y} = \widehat{Y} - \widehat{S}_{1,z} - \widehat{S}_{2,z} + \widehat{S}_{3,z} + \widehat{S}_{4,z}$$

where
$$\widehat{S}_{1,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(I,j,\rho_\pi(I,j)), \quad \widehat{S}_{2,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(K,j,\rho_\pi(K,j)),$$
$$\widehat{S}_{3,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(I,j,\rho_\pi(K,j)), \quad \widehat{S}_{4,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(K,j,\rho_\pi(I,j)).$$

The following lemmas are important tools for proving Theorem 1.3.

Lemma 2.1

(i) *S_1, S_2, S_3, S_4 are identically distributed.*

(ii) *If $E(f \circ X)^r < \infty$ for any positive even integer r, then, for every $i = 1, 2, 3, 4$,*
$$ES_i^r = O(q^{-\frac{r}{2}}) \quad \text{as } q \to \infty.$$

Lemma 2.2

(i) *If $E(f \circ X)^2 < \infty$, then*
$$\frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY^2(i,j,k) = 1 + O(\frac{1}{q}) \quad \text{as } q \to \infty.$$

(ii) *If $E(f \circ X)^r < \infty$ for some positive even integer r, then*
$$\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY^r(i,j,k) = O(q^{3-r}) \quad \text{as } q \to \infty.$$

(iii) *If $E(f \circ X)^r < \infty$ for some positive even integer r, then*
$$E\Big(\widetilde{W} - W\Big)^r \le O(q^{-\frac{r}{2}}) \quad \text{as } q \to \infty.$$

The proof of Lemma 2.1 and 2.2 can be seen in [7].

From Lemma 2.2(ii), we note that

$$\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} E|Y^m(i,j,k)Y_z^n(i,j,k)| \le \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \frac{E|Y(i,j,k)|^{m+n+t}}{(1+z)^t} \le \frac{O(q^{3-m-n-t})}{(1+z)^t} \tag{2.3}$$

for any integers m, n and t which $m \ge 0, n, t > 0$ and $m+n+t$ is an even number.

Lemma 2.3 *Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable function. Then*

(i) $E\widehat{Y}g(\widehat{Y}) = \frac{q-1}{4}E(\widetilde{Y} - \widehat{Y})(g(\widetilde{Y}) - g(\widehat{Y})) + \widetilde{\Delta}g(\widehat{Y})$
where

$$\widetilde{\Delta}g(\widehat{Y}) = \frac{1}{q}Eg(\widehat{Y})\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k). \tag{2.4}$$

(ii) *If g is a continuous and piecewise continuously differentiable function, then*

$$E\widehat{Y}g(\widehat{Y}) = E\int_{-\infty}^{\infty} g'(\widehat{Y}+t)K(t)dt + \widetilde{\Delta}g(\widehat{Y})$$

where

$$K(t) = \frac{q-1}{4}\Big(\widetilde{Y} - Y\Big)\Big(\mathbb{I}\big(0 \le t \le \widetilde{Y} - Y\big) - \mathbb{I}\big(\widetilde{Y} - Y \le t < 0\big)\Big)$$

and $\mathbb{I}$ is the indicator function.

(iii) $|\widetilde{\Delta}\widehat{Y}| \le O(\frac{1}{q})$ *where* $\widetilde{\Delta}\widehat{Y}$ *is defined in (2.4) for $g(x) = x$.*

Proof. (i) Let $\mathcal{B}$ be σ-algebra generated by π_i's and $U_{i_1,i_2,i_3,j}$'s. Note that

$$\begin{aligned} E^{\mathcal{B}}\big[\widetilde{Y} - \widehat{Y}\big] &= E^{\mathcal{B}}\Big[-\widehat{S}_{1,z} - \widehat{S}_{2,z} + \widehat{S}_{3,z} + \widehat{S}_{4,z}\Big] \\ &= -\frac{2}{q}\sum_{i=0}^{q-1} E^{\mathcal{B}}\Big[\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j))\Big] \\ &\quad + \frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{\substack{k=0\\k\neq i}}^{q-1} E^{\mathcal{B}}\Big[\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(k,j))\Big] \end{aligned}$$

$$= -\frac{2}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j)) + \frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{\substack{k=0\\k\neq i}}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(k,j))$$
$$= (-\frac{2}{q} - \frac{2}{q(q-1)})\widehat{Y} + \frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(k,j))$$
$$= -\frac{2}{q-1}\widehat{Y} + \frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)$$

and we can show that $(\widetilde{Y},\widehat{Y})$ is an exchangeable pair by using the same technique for proving the exchangeability of $(W,\widetilde{W})$. From these facts and the fact that

$$F(w,\tilde{w}) = (\tilde{w}-w)(g(\tilde{w})+g(w)),$$

is anti-symmetric,(see [18] page 9), we have $EF(\widetilde{Y},\widehat{Y}) = 0$. Hence,

$$\begin{aligned}
0 &= E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})+g(\widehat{Y})\big)\\
&= E\big(\widetilde{Y}-\widehat{Y}\big)(2g(\widehat{Y})) + E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big)\\
&= 2Eg(\widehat{Y})E^{\mathcal{B}}\big[\widetilde{Y}-\widehat{Y}\big] + E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big)\\
&= 2Eg(\widehat{Y})\big\{-\frac{2}{q-1}\widehat{Y} + \frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\big\}\\
&\quad + E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big)\\
&= -\frac{4}{q-1}E\widehat{Y}g(\widehat{Y}) + \frac{4}{q(q-1)}Eg(\widehat{Y})\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\\
&\quad + E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big)
\end{aligned}$$

which implies

$$E\widehat{Y}g(\widehat{Y}) = \frac{q-1}{4}E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big) + \widetilde{\Delta}g(\widehat{Y}).$$

(ii) Follows directly from 1 and the fact that

$$\begin{aligned}
\frac{q-1}{4}E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big) &= \frac{q-1}{4}E\big(\widetilde{Y}-\widehat{Y}\big)\int_0^{\widetilde{Y}-\widehat{Y}} g'(\widehat{Y}+t)dt\\
&= E\int_{-\infty}^{\infty} g'(\widehat{Y}+t)K(t)dt + \widetilde{\Delta}g\big(\widehat{Y}\big).
\end{aligned}$$

(iii) Note that

$$\begin{aligned}
\widetilde{\Delta}\widehat{Y} &= \frac{1}{q}E\widehat{Y}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k) \\
&= \frac{1}{q^2}E\big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\big)\big(\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1}\widehat{Y}_z(l,m,n)\big) \\
&= \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1}E\big\{Y(i,j,k)Y(l,m,n) - Y(i,j,k)Y_z(l,m,n) \\
&\quad - Y(l,m,n)Y_z(i,j,k) + Y_z(i,j,k)Y_z(l,m,n)\big\} \\
&= \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}E\big\{Y^2(i,j,k) - 2Y(i,j,k)Y_z(i,j,k) + Y_z^2(i,j,k)\big\} \\
&\quad + \frac{1}{q^2}E\sum_{i,j,k}\sum_{\substack{l,m,n \\ (l,m,n)\neq(i,j,k)}}\big\{Y(i,j,k)Y(l,m,n) - 2Y(i,j,k)Y_z(l,m,n) \\
&\quad + Y_z(i,j,k)Y_z(l,m,n)\big\} \\
&= \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY^2(i,j,k) - \frac{2}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY(i,j,k)Y_z(i,j,k) \\
&\quad + \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY_z^2(i,j,k) \\
&\quad + \frac{1}{q^2}\sum_{i,j,k}\sum_{\substack{l,m,n \\ (l,m,n)\neq(i,j,k)}}\Big\{\tilde{\mu}(i,j,k)\tilde{\mu}(l,m,n) - 2\tilde{\mu}(i,j,k)EY_z(l,m,n) \\
&\quad + EY_z(i,j,k)Y_z(l,m,n)\Big\} \\
&= A_1 + A_2 + A_3 + A_4 + A_5 \qquad (2.5)
\end{aligned}$$

where

$$A_1 = \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY^2(i,j,k)$$

$$A_2 = -\frac{2}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY(i,j,k)Y_z(i,j,k)$$

$$A_3 = \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1} EY_z(i,j,k)Y_z(l,m,n)$$

$$A_4 = \frac{1}{q^2}\sum_{\substack{i,j,k \\ (i,j,k)\neq}}\sum_{\substack{l,m,n \\ (l,m,n)}} \tilde{\mu}(i,j,k)\tilde{\mu}(l,m,n)$$

and

$$A_5 = -\frac{2}{q^2}\sum_{i,j,k}\sum_{\substack{l,m,n \\ (l,m,n)\neq(i,j,k)}} \tilde{\mu}(i,j,k)EY_z(l,m,n).$$

From Lemma 2.2(i) and (2.3) we have

$$|A_1| = O(\frac{1}{q}), \quad |A_2| = \frac{1}{(1+z)^2}O(\frac{1}{q^3}) \tag{2.6}$$

and

$$\begin{aligned} |A_3| &= \frac{1}{q^2}\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY_z(i,j,k)\Big)^2 \\ &\le q\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY_z^2(i,j,k) \\ &= \frac{1}{(1+z)^4}O(\frac{1}{q^2}). \end{aligned} \tag{2.7}$$

Now, we consider $|A_4|$ and $|A_5|$. We note that

$$\sum_{i_j=0}^{q-1} \tilde{\mu}(i_1,i_2,i_3) = 0 \quad \text{for} \quad j = 1,2,3. \tag{2.8}$$

(see [8], p.1212). Hence from (2.8) and Lemma 2.2(ii) we have

$$\begin{aligned} |A_4| &= \frac{1}{q^2}\Big|\sum_{i,j,k}\sum_{\substack{l,m,n \\ (l,m,n)\neq(i,j,k)}} \tilde{\mu}(i,j,k)\tilde{\mu}(l,m,n)\Big| \\ &= \frac{1}{q^2}\Big| -\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \tilde{\mu}^2(i,j,k)\Big| \\ &\le \frac{1}{q^2}\Big|\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY^2(i,j,k)\Big| \\ &= O(\frac{1}{q}), \end{aligned} \tag{2.9}$$

and

$$
\begin{aligned}
|A_5| &= \frac{2}{q^2}\Big|\sum_{i,j,k}\sum_{\substack{l,m,n\\(l,m,n)\neq(i,j,k)}}\tilde{\mu}(i,j,k)EY_z(l,m,n)\Big|\\
&= \frac{2}{q^2}\Big|-\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1}\tilde{\mu}(l,m,n)EY_z(l,m,n)\Big|\\
&\le \frac{1}{q^2}\Big|\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1}\tilde{\mu}^2(l,m,n)+\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1}EY_z^2(l,m,n)\Big|\\
&\le \frac{2}{q^2}\Big|\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1}EY^2(l,m,n)\Big|\\
&= O(\frac{1}{q}). \qquad (2.10)
\end{aligned}
$$

Then (iii) follows from (2.5)-(2.10). □

Lemma 2.4

(i) *If* $E(f\circ X)^2<\infty$, *then* $\big|E\widehat{Y}^2-1\big|\le O(\frac{1}{q})$ *as* $q\to\infty$.

(ii) *If* $E(f\circ X)^r<\infty$, *then, for any even number* $r\ge 2$, $E|\widetilde{Y}-\widehat{Y}|^r\le O(\frac{1}{q^{\frac{r}{2}}})$ *as* $q\to\infty$.

(iii) *If* $E(f\circ X)^{r+1}<\infty$, *then, for any odd number* $r\ge 3$, $E|\widetilde{Y}-\widehat{Y}|^r\le O(\frac{1}{q^{\frac{r}{2}}})$ *as* $q\to\infty$.

Proof. (i) Let

$$
\begin{aligned}
S_{1,z} &= \sum_{j=0}^{q-1}Y_z(I,j,\rho_\pi(I,j)), \quad S_{2,z}=\sum_{j=0}^{q-1}Y_z(K,j,\rho_\pi(K,j))\\
S_{3,z} &= \sum_{j=0}^{q-1}Y_z(I,j,\rho_\pi(K,j)), \quad S_{4,z}=\sum_{j=0}^{q-1}Y_z(K,j,\rho_\pi(I,j)).
\end{aligned}
$$

We can use the same argument of Lemma 2.1(i) to show that

$$S_{1,z},S_{2,z},S_{3,z} \text{ and } S_{4,z}$$

are identically distributed.

Note that

$$
\begin{aligned}
ES_1^2 &= \frac{1}{q}\sum_{i=0}^{q-1} E\Big(\sum_{j=0}^{q-1} Y(i,j,\rho_\pi(i,j))\Big)^2 \\
&= \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} EY^2(i,j,\rho_\pi(i,j)) \\
&\quad + \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j_1=0}^{q-1}\sum_{\substack{j_2=0\\ j_2\neq j_1}}^{q-1} EY(i,j_1,\rho_\pi(i,j_1))Y(i,j_2,\rho_\pi(i,j_2)) \\
&= \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY^2(i,j,k) \\
&\quad + \frac{1}{q^2(q-1)}\sum_{i=0}^{q-1}\sum_{j_1=0}^{q-1}\sum_{\substack{j_2=0\\ j_2\neq j_1}}^{q-1}\sum_{k_1=0}^{q-1}\sum_{\substack{k_2=0\\ k_2\neq k_1}}^{q-1} EY(i,j_1,k_1)Y(i,j_2,k_2) \\
&= \frac{1}{q}\left(1+O\left(\frac{1}{q}\right)\right) + \frac{1}{q^2(q-1)}\sum_{i=0}^{q-1}\sum_{j_1=0}^{q-1}\sum_{k_1=0}^{q-1}\tilde{\mu}(i,j_1,k_1)\Big(\sum_{\substack{j_2=0\\ j_2\neq j_1}}^{q-1}\sum_{\substack{k_2=0\\ k_2\neq k_1}}^{q-1}\tilde{\mu}(i,j_2,k_2)\Big) \\
&= \frac{1}{q}\left(1+O\left(\frac{1}{q}\right)\right) + \frac{1}{q^2(q-1)}\sum_{i=0}^{q-1}\sum_{j_1=0}^{q-1}\sum_{k_1=0}^{q-1}\tilde{\mu}(i,j_1,k_1)\Big(-\sum_{\substack{j_2=0\\ j_2\neq j_1}}^{q-1}\tilde{\mu}(i,j_2,k_1)\Big) \\
&= \frac{1}{q}\left(1+O\left(\frac{1}{q}\right)\right) + \frac{1}{q^2(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\tilde{\mu}^2(i,j,k) \\
&\le \frac{1}{q}\left(1+O\left(\frac{1}{q}\right)\right) + \frac{1}{q^2(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY^2(i,j,k) \\
&= \frac{1}{q} + O\left(\frac{1}{q^2}\right),
\end{aligned} \tag{2.11}
$$

and, for each $r, s \in \mathbb{N}$ such that $r+s$ is an even number

$$
\begin{aligned}
E|S_{1,z}|^r &= E\left|\sum_{j=0}^{q-1} Y_z(I,j,\rho_\pi(I,j))\right|^r \\
&\le q^{r-1}\sum_{j=0}^{q-1} E\left|Y_z^r(I,j,\rho_\pi(I,j))\right|
\end{aligned} \tag{2.12}
$$

$$= q^{r-3}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} E|Y_z^r(i,j,k)|$$
$$\leq q^{r-3}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \frac{EY^{r+s}(i,j,k)}{(1+z)^s}$$
$$\leq \frac{1}{(1+z)^s}\, O\left(\frac{1}{q^s}\right). \tag{2.13}$$

Hence, by Lemma 2.1(i) and (2.11),

$$ES_1^2 = ES_2^2 = ES_3^2 = ES_4^2 = \frac{1}{q} + O\left(\frac{1}{q^2}\right) \tag{2.14}$$

and from (2.12) when we choose $r = 2$ and $s = 4$,

$$ES_{1,z}^2 = ES_{2,z}^2 = ES_{3,z}^2 = ES_{4,z}^2 \leq \frac{1}{(1+z)^4}\, O\left(\frac{1}{q^4}\right). \tag{2.15}$$

From (2.14) and (2.15) ,

$$E|S_1 S_{1,z}| \leq \left\{ES_1^2\right\}^{\frac{1}{2}}\left\{ES_{1,z}^2\right\}^{\frac{1}{2}}$$
$$= \frac{1}{(1+z)^2}\, O\left(\frac{1}{q^2\sqrt{q}}\right). \tag{2.16}$$

By using the same argument as in (2.16) and the fact that $S_{1,z}, S_{2,z}, S_{3,z}$ and $S_{4,z}$ have the same distribution, we can conclude that

$$E|S_i S_{j,z}| \leq \frac{1}{(1+z)^2}\, O\left(\frac{1}{q^2\sqrt{q}}\right) \tag{2.17}$$

for $i,j = 1,2,3,4$. Next, we will bound ES_iS_j for $1 \leq i < j \leq 4$.
Let $\mathcal{A}$ be the $\sigma-$algebra generated by

$$\{\big(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})\big), U_{\pi_1(a_{i,1}),\pi_2(a_{i,2}),\pi_3(a_{i,3}),j} : 1 \leq i \leq q^2, 1 \leq j \leq 3\}.$$

From [8], p.1220-1221, we have $E\big|E^{\mathcal{A}}S_iS_j\big| = O\left(\frac{1}{q^2}\right)$ for $1 \leq i < j \leq 4$. Thus

$$|ES_iS_j| \leq E\big|E^{\mathcal{A}}S_iS_j\big| = O\left(\frac{1}{q^2}\right). \tag{2.18}$$

Note from Lemma 2.3(i) that

$$E\widehat{Y}^2 = \frac{q-1}{4}E\big(\widetilde{Y} - \widehat{Y}\big)^2 + \widetilde{\Delta}\widehat{Y} \tag{2.19}$$

and by (2.14) we have

$$
\begin{aligned}
E\big(\widetilde{Y}-\widehat{Y}\big)^2 &= E\big(\widehat{S}_{1,z}+\widehat{S}_{2,z}-\widehat{S}_{3,z}-\widehat{S}_{4,z}\big)^2\\
&= E\big(S_1+S_2-S_3-S_4-(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z})\big)^2\\
&= E\big(S_1+S_2-S_3-S_4\big)^2\\
&\quad -2E\big(S_1+S_2-S_3-S_4\big)\big(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z}\big)\\
&\quad +E\big(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z}\big)^2\\
&= \sum_{k=1}^{4} ES_k^2 + 2E\{S_1S_2-S_1S_3-S_1S_4-S_2S_3-S_2S_4+S_3S_4\}\\
&\quad -2E\big(S_1+S_2-S_3-S_4\big)\big(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z}\big)\\
&\quad +E\big(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z}\big)^2\\
&= \frac{4}{q}+O\left(\frac{1}{q^2}\right)+2E\{S_1S_2-S_1S_3-S_1S_4-S_2S_3-S_2S_4+S_3S_4\}\\
&\quad -2E\big(S_1+S_2-S_3-S_4\big)\big(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z}\big)\\
&\quad +E\big(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z}\big)^2. \qquad (2.20)
\end{aligned}
$$

Hence, by (2.19) and (2.20),

$$
\begin{aligned}
E\widehat{Y}^2 &= (1-\frac{1}{q})+O\left(\frac{1}{q}\right)\\
&\quad +\frac{q-1}{2}E\{S_1S_2-S_1S_3-S_1S_4-S_2S_3-S_2S_4+S_3S_4\}\\
&\quad -\frac{q-1}{2}E\big(S_1+S_2-S_3-S_4\big)\big(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z}\big)\\
&\quad +\frac{q-1}{4}E\big(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z}\big)^2+\widetilde{\Delta}\widehat{Y},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&|E\widehat{Y}^2-1|\\
&\le O\left(\frac{1}{q}\right)+q\Big(\sum_{1\le i<j\le 4}E|S_iS_j|+\sum_{i=1}^{4}\sum_{j=1}^{4}E|S_iS_{j,z}|+\sum_{i=1}^{4}ES_{i,z}^2\Big)+|\widetilde{\Delta}\widehat{Y}|\\
&= O\left(\frac{1}{q}\right)+q\Big(\sum_{1\le i<j\le 4}E|S_iS_j|+\sum_{i=1}^{4}\sum_{j=1}^{4}E|S_iS_{j,z}|+\sum_{i=1}^{4}ES_{i,z}^2\Big)
\end{aligned}
$$

where we have used Lemma 2.3(iii) in the last equality.

From this fact, (2.15), (2.17) and (2.18) we obtain (i).

(ii) From (2.12), we choose $r=s$, we have $E|S_{1,z}|^r \le \frac{1}{(1+z)^r}O\left(\frac{1}{q^r}\right)$.

Since $S_{1,z}, S_{2,z}, S_{3,z}$ and $S_{4,z}$ have the same distribution,

$$E|S_{1,z}|^r = E|S_{2,z}|^r = E|S_{3,z}|^r = E|S_{4,z}|^r \le \frac{1}{(1+z)^r} O\left(\frac{1}{q^r}\right). \tag{2.21}$$

Hence 2 follows from Lemma 2.2(3), (2.21) and the fact that

$$\begin{aligned} E|\widehat{Y} - \widetilde{Y}|^r &= E|\widehat{S}_{1,z} + \widehat{S}_{2,z} - \widehat{S}_{3,z} - \widehat{S}_{4,z}|^r \\ &= E|S_1 + S_2 - S_3 - S_4 - (S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z})|^r \\ &= E|(\widetilde{W} - W) - (S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z})|^r \\ &\le C\{E|\widetilde{W} - W|^r + E|S_{1,z}|^r + E|S_{2,z}|^r + E|S_{3,z}|^r + E|S_{4,z}|^3\} \end{aligned}$$

for some constant C.

(iii) Can be shown by using the same argument as in 2. □

Lemma 2.5 *Let* $\gamma = \max\left(\frac{q}{4}E|\widehat{Y} - \widetilde{Y}|^3,\ \frac{1}{\sqrt{q}}\right)$ *and*

$$U_\gamma = \sum_{i\neq k}\left|\sum_{j=0}^{q-1}\{\widehat{Y}_z(i,j,\rho_\pi(i,j)) + \widehat{Y}_z(k,j,\rho_\pi(k,j)) - \widehat{Y}_z(i,j,\rho_\pi(k,j)) - \widehat{Y}_z(k,j,\rho_\pi(i,j))\}\right|$$

$$\times \min\left(\gamma, \sum_{i\neq k}\left|\sum_{j=0}^{q-1}\{\widehat{Y}_z(i,j,\rho_\pi(i,j)) + \widehat{Y}_z(k,j,\rho_\pi(k,j)) - \widehat{Y}_z(i,j,\rho_\pi(k,j)) - \widehat{Y}_z(k,j,\rho_\pi(i,j))\}\right|\right).$$

If $(1+z)\gamma < 1$ *and* $E(f \circ X)^4 < \infty$ *then, as* $q \to \infty$*,*

(i) $EU_\gamma \ge 3q + O(1)$,

(ii) $Var(U_\gamma) \le \frac{1}{1+z} O(q\sqrt{q})$.

Proof. First, from Lemma 2.4(iii), we note that $\gamma \le O\left(\frac{1}{\sqrt{q}}\right)$.

(i) From (2.20), we have

$$E(\widetilde{Y} - \widehat{Y})^2 = \frac{4}{q} + O\left(\frac{1}{q^2}\right) + EM_1 \tag{2.22}$$

where

$$\begin{aligned} M_1 =& 2\{S_1S_2 - S_1S_3 - S_1S_4 - S_2S_3 - S_2S_4 + S_3S_4\} \\ &- 2(S_1 + S_2 - S_3 - S_4)(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z}) \\ &+ (S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z})^2. \end{aligned}$$

By (2.15),(2.17) and (2.18), we have

$$q(q-1)|EM_1| \leq O(1). \tag{2.23}$$

Hence, by (2.22), (2.23) and the fact that $\min(a,b) \geq b - \dfrac{b^2}{4a}$ for any $a, b > 0$,

$$\begin{aligned} EU_\gamma &= q(q-1)E(\widehat{Y} - \widetilde{Y}) \min(\gamma, |\widehat{Y} - \widetilde{Y}|) \\ &\geq q(q-1)E(\widehat{Y} - \widetilde{Y})^2 - \frac{q(q-1)}{4\gamma} E|\widehat{Y} - \widetilde{Y}|^3 \\ &\geq 3q + O(1). \end{aligned}$$

(ii) For each $i, k \in \{0, 1, ..., q-1\}$ and random permutations β, α on $\{0, 1, ..., q-1\}$, we let

$$s_\gamma[(i,k),(\beta,\alpha)] = \Big|\sum_{j=0}^{q-1}\big\{\widehat{Y}_z(i,j,\beta(j)) + \widehat{Y}_z(k,j,\alpha(j)) - \widehat{Y}_z(i,j,\alpha(j)) - \widehat{Y}_z(k,j,\beta(j))\big\}\Big|$$

$$\times \min\Big(\gamma, \Big|\sum_{j=0}^{q-1}\big\{\widehat{Y}_z(i,j,\beta(j)) + \widehat{Y}_z(k,j,\alpha(j)) - \widehat{Y}_z(i,j,\alpha(j)) - \widehat{Y}_z(k,j,\beta(j))\big\}\Big|\Big),$$

$$\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] = s_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] - Es_\gamma[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]$$

$$T_\gamma = \sum_{i \neq k} \hat{s}_\gamma[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]$$

and

$$\begin{aligned} \widetilde{T}_\gamma = T_\gamma &- \hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))] \\ &- \hat{s}_\gamma[(\bar{L},\bar{M}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))] \\ &+ \hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))] \\ &+ \hat{s}_\gamma[(\bar{L},\bar{M}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))], \end{aligned}$$

where $\bar{I}, \bar{K}, \bar{L}$ and $\bar{M}$ be uniformly distributed random vectors on $\{0, 1, \ldots, q-1\}$ which satisfy the followings:

(i) $(\bar{I}, \bar{K})$ and $(\bar{L}, \bar{M})$ are uniformly distributed random vectors on

$$\Big\{(i,k) | i, k = 0, 1, \ldots, q-1 \text{ and } i \neq k\Big\}.$$

(ii) $\Big[(\bar{I}, \bar{K}), (\bar{L}, \bar{M})\Big]$ is uniformly distributed on

$$\Big\{\Big[(i,k),(l,m)\Big] | i,k,l,m = 0,1,...,q-1 \text{ and } i \neq k, l \neq m \text{ and } (i,k) \neq (l,m)\Big\}.$$

(iii) $(\bar{I},\bar{K}),(\bar{L},\bar{M})$ and $U_{i_1,i_2,i_3,j}$'s, π_1,π_2,π_3 are mutually independent.

Note that

$$P\Big([(\bar{I},\bar{K}),(\bar{L},\bar{M})]=\big[(i,k),(l,m)\big]\Big)=\frac{1}{q(q-1)\big[q(q-1)-1\big]}$$

for any $i,k,l,m=0,1,...,q-1$ and $i\neq k, l\neq m$ and $(i,k)\neq(l,m)$, $\big(T_\gamma,\widetilde{T}_\gamma\big)$ is an exchangeable pair and $Var(U_\gamma)=ET_\gamma^2$.

By the same argument of Lemma 2.3(i), we have

$$\begin{aligned} ET_\gamma^2 &= \frac{q(q-1)}{4}E(\widetilde{T}_\gamma-T_\gamma)^2 \\ &\quad+\frac{1}{q(q-1)}\sum_{i\neq k}\sum_{l\neq m}E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]T_\gamma. \end{aligned} \tag{2.24}$$

Note that

$$\begin{aligned} &E(\widetilde{T}_\gamma-T_\gamma)^2 \\ &=E\Big\{\hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))]+\hat{s}_\gamma[(\bar{L},\bar{M}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))] \\ &\quad-\hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))]-\hat{s}_\gamma[(\bar{L},\bar{M}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))]\Big\}^2 \\ &\leq C\Big\{E\Big|\hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))]\Big|^2+E\Big|\hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))]\Big|^2\Big\} \\ &\leq C\gamma^2\Big\{E\Big|\sum_{j=0}^{q-1}\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{L},j))+\widehat{Y}(\bar{K},j,\rho_\pi(\bar{M},j)) \\ &\quad-\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{M},j))-\widehat{Y}_z(\bar{K},j,\rho_\pi(\bar{L},j))\Big|^2 \\ &\quad+E\Big|\sum_{j=0}^{q-1}\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{I},j))+\widehat{Y}_z(\bar{K},j,\rho_\pi(\bar{K},j)) \\ &\quad-\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{K},j))-\widehat{Y}_z(\bar{K},j,\rho_\pi(\bar{I},j))\Big|^2\Big\} \\ &\leq C\gamma^2\Big\{E\Big|\sum_{j=0}^{q-1}\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{L},j))\Big|^2+E\Big|\sum_{j=0}^{q-1}\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{I},j))\Big|^2 \\ &\quad+E\Big|\sum_{j=0}^{q-1}\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{K},j))\Big|^2\Big\} \end{aligned}$$

$$
\begin{aligned}
&\le C\gamma^2\Big\{E\Big|\sum_{j=0}^{q-1} Y(\bar{I},j,\rho_\pi(\bar{L},j))\Big|^2 + E|\sum_{j=0}^{q-1} Y_z(\bar{I},j,\rho_\pi(\bar{L},j))\Big|^2 \\
&\quad + E\Big|\sum_{j=0}^{q-1} Y(\bar{I},j,\rho_\pi(\bar{I},j))\Big|^2 + E\Big|\sum_{j=0}^{q-1} Y_z(\bar{I},j,\rho_\pi(\bar{I},j))\Big|^2 \\
&\quad + E\Big|\sum_{j=0}^{q-1} Y(\bar{I},j,\rho_\pi(\bar{K},j))\Big|^2 + E\Big|\sum_{j=0}^{q-1} Y_z(\bar{I},j,\rho_\pi(\bar{K},j))\Big|^2\Big\} \\
&\le C\gamma^2\big(2ES_3^2 + ES_1^2 + q\sum_{j=0}^{q-1}\frac{E|Y(I,j,\rho_\pi(L,j))|^4}{(1+z)^2} \\
&\quad + q\sum_{j=0}^{q-1}\frac{E|Y(I,j,\rho_\pi(I,j))|^4}{(1+z)^2} + q\sum_{j=0}^{q-1}\frac{E|Y(I,j,\rho_\pi(K,j))|^4}{(1+z)^2}\big) \\
&\le C\gamma^2\Big\{O\left(\frac{1}{q}\right) + \frac{1}{q}\sum_{i,j,k}\frac{E|Y(i,j,k)|^4}{(1+z)^2}\Big\} \\
&= \gamma^2\Big\{O\left(\frac{1}{q}\right) + \frac{1}{(1+z)^2}\,O\left(\frac{1}{q^2}\right)\Big\}. \qquad (2.25)
\end{aligned}
$$

From this fact, $(1+z)\gamma < 1$ and (2.24), if we can show that

$$
\sum_{i\neq k}\sum_{l\neq m} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot)),\rho_\pi(m,\cdot)]T_\gamma \le \gamma^2 O(q^4), \qquad (2.26)
$$

then

$$
Var(U_\gamma) = ET_\gamma^2 \le \frac{\gamma}{1+z}O(q^2).
$$

To prove (2.26), we note that

$$
\begin{aligned}
&E\Big[\sum_{i\neq k}\sum_{l\neq m}\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]T\Big] \\
&= E\big\{\sum_{i\neq k}\sum_{l\neq m}\sum_{u\neq v}\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\big\} \\
&= \sum_A E\big\{\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\big\}, \qquad (2.27)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \Big\{(i,k,l,m,u,v)\Big|\ \ i,k,l,m,u,v,r,s\in\{0,1,...,q-1\} \\
&\qquad \text{and } i\neq k,\ l\neq m,\ u\neq v\Big\} \\
&= \bigcup_{i=1}^{7} A_i
\end{aligned}
$$

and
$$A_1 = \Big\{(i,k,l,m,u,v) \in A \big|\ \ u = l,\ u \neq m,\ v \neq l,\ v = m\Big\}$$
$$A_2 = \Big\{(i,k,l,m,u,v) \in A \big|\ \ u \neq l,\ u = m,\ v = l,\ v \neq m\Big\}$$
$$A_3 = \Big\{(i,k,l,m,u,v) \in A \big|\ \ u \neq l,\ u \neq m,\ v \neq l,\ v = m\Big\}$$
$$A_4 = \Big\{(i,k,l,m,u,v) \in A \big|\ \ u \neq l,\ u = m,\ v \neq l,\ v \neq m\Big\}$$
$$A_5 = \Big\{(i,k,l,m,u,v) \in A \big|\ \ u \neq l,\ u \neq m,\ v = l,\ v \neq m\Big\}$$
$$A_6 = \Big\{(i,k,l,m,u,v) \in A \big|\ \ u = l,\ u \neq m,\ v \neq l,\ v \neq m\Big\}$$
$$A_7 = \Big\{(i,k,l,m,u,v) \in A \big|\ \ u \neq l,\ u \neq m,\ v \neq l,\ v \neq m\Big\}.$$

We first consider the sum on A_1. Note that

$$\begin{aligned}
&\sum_{A_1} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \\
&= \sum_{i\neq k}\sum_{l\neq m} \hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] \\
&= \sum_{i\neq k}\sum_{l\neq m} E\Big\{s_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] - Es_\gamma[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]\Big\} \\
&\qquad \times \Big\{s_\gamma[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] - Es_\gamma[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\Big\} \\
&\leq \mathcal{R}_{11} + \mathcal{R}_{12}, \qquad (2.28)
\end{aligned}$$

where

$$\mathcal{R}_{11} = \sum_{i\neq k}\sum_{l\neq m} E\Big\{s_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]s_\gamma[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\Big\}$$
$$\mathcal{R}_{12} = \sum_{i\neq k}\sum_{l\neq m} Es_\gamma[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]Es_\gamma[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))].$$

By Lemma 2.2(ii) and Lemma 2.4(ii), we note that

$$\begin{aligned}
\mathcal{R}_{11} &\leq \sum_{i\neq k}\sum_{l\neq m} E\big\{s_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\big\}^2 \\
&\quad + \sum_{i\neq k}\sum_{l\neq m} E\big\{s_\gamma[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\big\}^2 \\
&\leq \sum_{i\neq k}\sum_{l\neq m} E\big\{s_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\big\}^2 \\
&\quad + q^2 \sum_{l\neq m} E\big\{s_\gamma[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\big\}^2
\end{aligned}$$

$$
\begin{aligned}
&\le \gamma^2 \sum_{i\neq k}\sum_{l\neq m} E\Big\{\big|\sum_{i_2=0}^{q-1} \widehat{Y}_z(i,i_2,\rho_\pi(l,i_2)) + \widehat{Y}_z(k,i_2,\rho_\pi(m,i_2)) \\
&\quad - \widehat{Y}_z(i,i_2,\rho_\pi(m,i_2)) - \widehat{Y}_z(k,i_2,\rho_\pi(l,i_2))\big|\Big\}^2 \\
&\quad + \gamma^2 q^2 \sum_{i\neq k} E\Big\{\big|\sum_{i_2=0}^{q-1} \widehat{Y}_z(i,i_2,\rho_\pi(i,i_2)) + \widehat{Y}_z(k,i_2,\rho_\pi(k,i_2)) \\
&\quad - \widehat{Y}_z(i,i_2,\rho_\pi(k,i_2)) - \widehat{Y}_z(k,i_2,\rho_\pi(i,i_2))\big|\Big\}^2 \\
&\le 16\gamma^2 q^2 \sum_{i=0}^{q-1}\sum_{l=0}^{q-1} E\Big\{\big|\sum_{i_2=0}^{q-1} \widehat{Y}_z(i,i_2,\rho_\pi(l,i_2))\Big\}^2 + \gamma^2 q^4 E(\widetilde{Y}-\widehat{Y})^2 \\
&\le 16\gamma^2 q^2 \sum_{i=0}^{q-1}\sum_{l=0}^{q-1} E\big\{\sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(l,i_2)) - \sum_{i_2=0}^{q-1} Y_z(i,i_2,\rho_\pi(l,i_2))\big\}^2 \\
&\quad + \gamma^2 O(q^3) \\
&\le 16\gamma^2 q^2 \sum_{i=0}^{q-1}\sum_{l=0}^{q-1} E\big\{\sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(l,i_2))\big\}^2 \\
&\quad + 16\gamma^2 q^2 \sum_{i=0}^{q-1}\sum_{l=0}^{q-1} E\big\{\sum_{i_2=0}^{q-1} Y_z(i,i_2,\rho_\pi(l,i_2))\big\}^2 \\
&\quad + \gamma^2 O(q^3) \\
&\le 16\gamma^2 q^2 E\big\{\sum_{i_2=0}^{q-1} Y(\bar{I},i_2,\rho_\pi(\bar{L},i_2))\big\}^2 \\
&\quad + \gamma^2 q^3 \sum_{i=0}^{q-1}\sum_{l=0}^{q-1}\sum_{i_2=0}^{q-1} EY_z^2(i,i_2,l) + \gamma^2 O(q^3) \\
&\le \gamma^2 O(q^3).
\end{aligned} \tag{2.29}
$$

Similarly,

$$
\mathcal{R}_{12} = \Big\{\sum_{i\neq k} Es_\gamma[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]\Big\}^2 = \gamma^2 O(q^3). \tag{2.30}
$$

Hence, by (2.28)-(2.30), it implies that

$$
\sum_{A_1} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2 O(q^3). \tag{2.31}
$$

Similar to A_1, we can conclude that

$$
\sum_{A_2} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2 O(q^3), \tag{2.32}
$$

$$\sum_{A_3} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2 O(q^4), \quad (2.33)$$

$$\sum_{A_4} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2 O(q^4), \quad (2.34)$$

$$\sum_{A_5} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2 O(q^4), \quad (2.35)$$

$$\sum_{A_6} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2 O(q^4). \quad (2.36)$$

For the last summation on A_7, we can use the same argument in the proof of Lemma 2.4(3) of Laipaporn and Neammanee(2006) to show that

$$\sum_{A_7} E\big[\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\big] \le \gamma^2 O(q^4). \quad (2.37)$$

From (2.27), (2.31)-(2.37), the lemma is proved. □

3 Proof of Theorem 1.3

Proof. First, we note that

$$P(z \le W \le w) \le P(W \ne \widehat{Y}) + P(z \le \widehat{Y} \le w) \quad (3.1)$$

and by Lemma 2.2(ii),

$$\begin{aligned} P(W \ne \widehat{Y}) &= P\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} \mathbb{I}(|Y(i,j,\rho_\pi(i,j))| > 1+z) \ge 1\Big) \\ &\le E\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} \mathbb{I}(|Y(i,j,\rho_\pi(i,j))| > 1+z)\Big) \\ &= \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} E\mathbb{I}(|Y(i,j,k)| > 1+z) \\ &\le \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \frac{E|Y(i,j,k)|^4}{(1+z)^4} \\ &= \frac{1}{(1+z)^4} O\left(\frac{1}{q^2}\right). \end{aligned} \quad (3.2)$$

Thus it remains to bound the term $P(z \le \widehat{Y} \le w)$.

Let γ be defined as in Lemma 2.5. If $(1+z)\gamma \ge 1$, then by Lemma 2.4(i) and the fact that $\gamma \le O\left(\frac{1}{\sqrt{q}}\right)$, we have

$$\begin{aligned}
P(z \le \widehat{Y} \le w) &= P(1+z \le 1+\widehat{Y}) \\
&\le \frac{E|\widehat{Y}+1|^2}{(1+z)^2} \\
&\le \frac{C}{(1+z)^2} \\
&\le \frac{1}{(1+z)} O\left(\frac{1}{\sqrt{q}}\right).
\end{aligned}$$

Suppose that $(1+z)\gamma < 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 0, & \text{if} \quad t < a-\gamma, \\ (1+t+\gamma)(t-z+\gamma), & \text{if} \quad z-\gamma \le t \le w+\gamma; \\ (1+t+\gamma)(w-z+2\gamma), & \text{if} \quad t > w+\gamma. \end{cases} \tag{3.3}$$

Then f is a non-decreasing function satisfying

$$f'(t) \ge \begin{cases} (1+z), & \text{for} \quad z-\gamma < t < w+\gamma; \\ 0, & \text{otherwise} . \end{cases}$$

We observe that

$$\begin{aligned}
& E\int_{-\infty}^{\infty} f'(\widehat{Y}+t)K(t)dt \\
&\ge (1+z)E\mathbb{I}(z \le \widehat{Y} \le w)\int_{|t|\le\gamma} K(t)dt \\
&= \frac{(q-1)(1+z)}{4}E\mathbb{I}(z \le \widehat{Y} \le w)|\widehat{Y}-\widetilde{Y}|\min(\gamma, |\widehat{Y}-\widetilde{Y}|) \\
&= \frac{(q-1)(1+z)}{4q(q-1)}E\mathbb{I}(z \le \widehat{Y} \le w)U_\gamma \\
&\ge \frac{(1+z)}{4q}E\mathbb{I}(z \le \widehat{Y} \le w)U_\gamma\mathbb{I}(U_\gamma \ge q) \\
&\ge \frac{(1+z)}{4}E\mathbb{I}(z \le \widehat{Y} \le w)\mathbb{I}(U_\gamma \ge q) \\
&= \frac{(1+z)}{4}E\big\{\mathbb{I}(z \le \widehat{Y} \le w) - \mathbb{I}(z \le \widehat{Y} \le w, U_\gamma \le q)\big\} \\
&\ge \frac{(1+z)}{4}\big\{P(z \le \widehat{Y} \le w) - P(U_\gamma \le q)\big\}
\end{aligned}$$

and by the same argument of Lemma 2.3(ii),

$$E\widehat{Y}f(\widehat{Y}) = E\int_{-\infty}^{\infty} f'(\widehat{Y}+t)K(t)dt + \widetilde{\Delta}f(\widehat{Y}). \tag{3.4}$$

By this fact, Lemma 2.4(i) and Lemma 2.5, we have

$$\begin{aligned}
P(z \le \widehat{Y} \le w) &\le \frac{4}{(1+z)}E\widehat{Y}f(\widehat{Y}) - \frac{4}{(1+z)}\widetilde{\Delta}f(\widehat{Y}) + P(U_\gamma \le q) \\
&\le \frac{4}{(1+z)}(w-z+2\gamma)E|\widehat{Y}||1+\gamma+\widehat{Y}| \\
&+ \frac{4}{(1+z)}|\widetilde{\Delta}f(\widehat{Y})| + P(EU_\gamma - U_\gamma \ge 2q) \\
&\le \frac{4}{(1+z)}(w-z+2\gamma)\Big\{E|\widehat{Y}| + E|\widehat{Y}|^2\Big\} \\
&+ \frac{4}{(1+z)}|\widetilde{\Delta}f(\widehat{Y})| + \frac{C}{q^2}E(U_\gamma - EU_\gamma)^2 \\
&\le \frac{C}{(1+z)}(w-z) + \frac{4}{(1+z)}|\widetilde{\Delta}f(\widehat{Y})| + \frac{C}{(1+z)}O\left(\frac{1}{\sqrt{q}}\right).
\end{aligned} \tag{3.5}$$

Note that, by (2.8), Lemma 2.2(i) and Lemma 2.4(i),

$$\begin{aligned}
|\widetilde{\Delta}f(\widehat{Y})| &= \Big|\frac{1}{q}Ef(\widehat{Y})\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\Big| \\
&\le \frac{1}{q}(w-z+2\gamma)E\Big|(1+\gamma+\widehat{Y})\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\Big| \\
&\le (w-z+2\gamma)(1+\gamma)\frac{1}{q}\sum_{k=0}^{q-1}E\Big|\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,k)\Big| \\
&+ \frac{1}{q}(w-z+2\gamma)E\Big|\widehat{Y}\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y(i,j,k) - \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y_z(i,j,k)\Big)\Big| \\
&\le (w-z+2\gamma)(1+\gamma)E\Big|\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j))\Big| \\
&+ \frac{1}{q}(w-z+2\gamma)E\Big|\widehat{Y}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y(i,j,k)\Big| \\
&+ \frac{1}{q}(w-z+2\gamma)E\Big|\widehat{Y}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y_z(i,j,k)\Big|
\end{aligned}$$

$$\leq (w-z+2\gamma)(1+\gamma)+\frac{1}{q}(w-z+2\gamma)\Big\{\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY^2(i,j,k)$$
$$+E\Big(\sum_{i_1,j_1,k_1}\sum_{\substack{i_2,j_2,k_2\\(i_1,j_1,k_1)\neq(i_2,j_2,k_2)}}Y(i_1,j_1,k_1)Y(i_2,j_2,k_2)\Big)\Big\}^{\frac{1}{2}}$$
$$+\frac{1}{q}(w-z+2\gamma)\{q^3\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY_z^2(i,j,k)\}^{\frac{1}{2}}$$
$$\leq (w-z+2\gamma)(1+\gamma)+\frac{1}{q}(w-z+2\gamma)\Big\{O(q)+\sum_{i,j,k}\tilde{\mu}^2(i,j,k)\Big\}^{\frac{1}{2}}$$
$$+\frac{(w-z+2\gamma)}{q(1+z)^2}\{q^3(q^{3-4})\}^{\frac{1}{2}}$$
$$\leq (w-z+2\gamma)O(1).$$

From this fact and (3.5), the theorem is proved. □

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