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Notes on the Spectrum of Lower Triangular Double-Band Matrices

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Abstract : In the paper by Srivastava and Kumar [P.D. Srivastava, S. Kumar, Thai J. Math. 8 (2) (2010) 221–233], the authors have introduced the lower triangular double-band matrix Δ_v as an operator on the sequence space l_1 and studied the spectrum and fine spectrum of this operator over l_1. The operator Δ_v on l_1 is defined by $\Delta_v x = (v_k x_k - v_{k-1} x_{k-1})_{k=0}^{\infty}$ with $x_{-1} = 0$, where $x = (x_k) \in l_1$ and (v_k) is either constant or strictly decreasing sequence of positive real numbers satisfying certain conditions. In this paper we give notes on the point spectrum and the residual spectrum of the operator Δ_v over the space l_1 in the case when (v_k) is a strictly decreasing sequence of positive real numbers.

Keywords : Spectrum of an operator; Generalized difference operator; Sequence spaces.
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1 Introduction, Preliminaries and Notation

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx, \ x \in X\}.$$

By $B(X)$, we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^*

of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

We shall need some basic concepts in spectral theory which are given as follows (see [1, pp. 370-371]). Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T we associate the operator

$$T_\lambda = T - \lambda I, \tag{1.1}$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse, which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1}, \tag{1.2}$$

and call it the *resolvent operator* of T. A *regular value* λ of T is a complex number such that

(R1) T_λ^{-1} exists,

(R2) T_λ^{-1} is bounded,

(R3) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent set* of T, denoted by $\rho(T, X)$, is the set of all regular values λ of T. Its complement $\sigma(T, X) = \mathbb{C}\backslash\rho(T, X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point* (*discrete*) *spectrum* $\sigma_p(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T, X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists and satisfies (R3) but not (R2), that is, T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T, X)$ is the set of all $\lambda \in \mathbb{C}$ such that T_λ^{-1} exists (and may be bounded or not) but does not satisfy (R3), that is, the domain of T_λ^{-1} is not dense in X.

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called a *sequence space*. We write l_1 for the space of all absolutely summable sequences, i.e.,

$$l_1 = \left\{ x = (x_k) : \sum_{k=0}^{\infty} |x_k| < \infty \right\}.$$

Also, we write l_∞ for the space of all bounded sequences. It is well-known that the dual space l_1^* of l_1 is isomorphic to the space l_∞.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N} = \{0, 1, 2, ...\}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \tag{1.3}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By (λ, μ), we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda, \mu)$ if and only if the series on the right side of (1.3) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. We use the convention that any term with negative subscript is equal to naught.

Now, we may give:

Lemma 1.1 ([2, p. 59])**.** *T has a dense range if and only if T^* is one to one.*

The generalized difference operator Δ_v on the space l_1 has been introduced by Srivastava and Kumar [3]. The operator Δ_v on the space l_1 is defined by

$$\Delta_v x = \Delta_v(x_k) = (v_k x_k - v_{k-1} x_{k-1})_{k=0}^{\infty} \text{ with } x_{-1} = 0,$$

where $x = (x_k) \in l_1$ and (v_k) is either constant or strictly decreasing sequence of positive real numbers satisfying

$$\lim_{k \to \infty} v_k = L > 0 \text{ and} \tag{1.4}$$

$$\sup_k (v_k) \leq 2L. \tag{1.5}$$

The operator Δ_v can be represented by the matrix

$$\Delta_v = \begin{pmatrix} v_0 & 0 & 0 & \cdots \\ -v_0 & v_1 & 0 & \cdots \\ 0 & -v_1 & v_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Srivastava and Kumar [3] studied the spectrum and fine spectrum of the operator Δ_v over the sequence space l_1. We summarize the main results concerning the spectrum of the generalized difference operator Δ_v on the space l_1 as stated in [3].

Theorem 1.2 ([3, Theorem 3.1]). *The operator $\Delta_v : l_1 \to l_1$ is a bounded linear operator and*

$$\|\Delta_v\|_{l_1} = 2 \sup_k (v_k).$$

Theorem 1.3 ([3, Theorem 3.2]). *The spectrum of the operator Δ_v on l_1 is given by*

$$\sigma(\Delta_v, l_1) = \left\{ \lambda \in \mathbb{C} : \left| 1 - \frac{\lambda}{L} \right| \leq 1 \right\}.$$

Theorem 1.4 ([3, Theorem 3.3]). *The point spectrum of the operator Δ_v on l_1 is given by*

$$\sigma_p(\Delta_v, l_1) = \begin{cases} \varnothing, \ if \ (v_k) \ is \ a \ constant \ sequence. \\ \{v_0, v_1, v_2, ...\}, \ if \ (v_k) \ is \ a \ strictly \ decreasing \ sequence. \end{cases}$$

Theorem 1.5 ([3, Theorem 4.1]). *The point spectrum of the operator Δ_v^* on l_1^* is given by*

$$\sigma_p(\Delta_v^*, l_1^*) = \left\{ \lambda \in \mathbb{C} : \left| 1 - \frac{\lambda}{L} \right| \leq 1 \right\}.$$

Theorem 1.6 ([3, Theorem 4.2]). *The residual spectrum of the operator Δ_v on l_1 is given by*

$$\sigma_r(\Delta_v, l_1) = \begin{cases} \left\{ \lambda \in \mathbb{C} : \left| 1 - \frac{\lambda}{L} \right| \leq 1 \right\}, \ if \ (v_k) \ is \ a \ constant \ sequence. \\ \left\{ \lambda \in \mathbb{C} : \left| 1 - \frac{\lambda}{L} \right| \leq 1 \right\} \setminus \{v_0, v_1, v_2, ...\}, \ if \ (v_k) \ is \ a \ strictly \\ decreasing \ sequence. \end{cases}$$

Theorem 1.7 ([3, Theorem 4.3]). *The continuous spectrum of the operator Δ_v on l_1 is $\sigma_c(\Delta_v, \ l_1) = \varnothing$.*

However, Theorems 1.4 and 1.6 which appear on pages 227 and 230 of [3], are incorrect in the case when the sequence (v_k) is a strictly decreasing sequence of positive real numbers. This will be shown by a counterexample in Section 2. Next, we provide the corrected results in Section 3.

2 A Counterexample

Consider the sequence (v_k), where $v_k = \frac{k+3}{2k+5}$. Clearly, (v_k) is a strictly decreasing sequence of positive real numbers satisfying the conditions (1.4) and (1.5); indeed

$$\lim_{k \to \infty} v_k = L = \frac{1}{2}, \ \sup_k (v_k) = \frac{3}{5} \leq 1 = 2L.$$

We can prove that $v_0 = \frac{3}{5} \notin \sigma_p(\Delta_v, l_1)$. Indeed, suppose for contrary that there exists $x = (x_k) \neq \theta$ in l_1 such that $\Delta_v x = v_0 x$. Then

$$(v_0 - v_0)x_0 = 0 \quad \text{and} \quad -v_k x_k + (v_{k+1} - v_0)x_{k+1} = 0,$$

for all $k \in \mathbb{N}$. If $x_0 = 0$, then $x_k = 0$, for all $k \geq 1$, and so we have a contradiction since $x \neq \theta$. Also, if $x_0 \neq 0$ then

$$\lim_{k\to\infty} \left| \frac{x_{k+1}}{x_k} \right| = \left| \frac{L}{L - v_o} \right| = 5 > 1,$$

and so we have a contradiction since $x \in l_1$. Then $v_0 \notin \sigma_p(\Delta_v, l_1)$.

The operator $\Delta_v - v_0 I$ on l_1 is defined by

$$(\Delta_v - v_0 I)x = (0, -v_0 x_0 + (v_1 - v_0)x_1, -v_1 x_1 + (v_2 - v_0)x_2, ...), \quad (2.1)$$

where $x = (x_k) \in l_1$. The operator $(\Delta_v - v_0 I)^{-1}$ exists since $v_0 \notin \sigma_p(\Delta_v, l_1)$. But $(\Delta_v - v_0 I)^{-1}$ does not satisfy (R3). Indeed, consider the sequence $y = (1, 0, 0, ...)$ in l_1 and let y be the center of a small ball, say, of radius $\frac{1}{3}$. Clearly, by (2.1), this ball does not intersect the range of the operator $\Delta_v - v_0 I$. Then, the operator $\Delta_v - v_0 I$ does not have a dense range in l_1. Hence, by definition, $v_0 \in \sigma_r(\Delta_v, l_1)$. Thus, we have the following assertions

$$\sigma_p(\Delta_v, l_1) \neq \{v_0, v_1, v_2, ...\} \text{ and}$$

$$\sigma_r(\Delta_v, l_1) \neq \left\{ \lambda \in \mathbb{C} : \left| 1 - \frac{\lambda}{L} \right| \leq 1 \right\} \setminus \{v_0, v_1, v_2, ...\}.$$

This proves that Theorems 1.4 and 1.6 are incorrect in the case when the sequence (v_k) is a strictly decreasing sequence of positive real numbers.

3 Corrected Results for the Point Spectrum and the Residual Spectrum of the Operator Δ_v on the Space l_1

In this section, we introduce corrected versions of Theorems 1.4 and 1.6 in the case when the sequence (v_k) is a strictly decreasing sequence of positive real numbers satisfying the conditions (1.4) and (1.5).

Theorem 3.1. $\sigma_p(\Delta_v, l_1) = \varnothing$.

Proof. Consider the equation $\Delta_v x = \lambda x$ for $x \neq \theta = (0, 0, 0, ...)$ in l_1. Then

$$(v_0 - \lambda)x_0 = 0 \quad \text{and} \quad - v_k x_k + (v_{k+1} - \lambda)x_{k+1} = 0, \quad \text{for all } k \in \mathbb{N}.$$

Hence, for all $\lambda \notin \{v_k : k \in \mathbb{N}\}$, we have $x_k = 0$, for all $k \in \mathbb{N}$. So, $\lambda \notin \sigma_p(\Delta_v, l_1)$. This shows that $\sigma_p(\Delta_v, l_1) \subseteq \{v_k : k \in \mathbb{N}\}$. Now, let $\lambda = v_0$. If $x_0 = 0$, then $x_k = 0$ for all $k \geq 1$ which contradicts the assumption that $x \neq \theta$. Also, if $x_0 \neq 0$ then

$$x_k = \frac{v_{k-1}}{v_k - v_0} x_{k-1} \neq 0,$$

for all $k \geq 1$, and hence we can take $x = (x_k) \neq \theta$ which satisfies $\Delta_v x = v_0 x$, but

$$|x_k| = \left| \frac{v_0\, v_1\, ...\, v_{k-1}}{(v_1 - v_0)(v_2 - v_0)\, ...\, (v_k - v_0)} \right| |x_0| > |x_0|\,,$$

for all $k \geq 1$. This contradicts the assumption that $x \in l_1$, and so $v_0 \notin \sigma_p(\Delta_v, l_1)$.

Similarly, we can prove that $v_k \notin \sigma_p(\Delta_v, l_1)$ for all $k \geq 1$. Thus $\sigma_p(\Delta_v, l_1) = \varnothing$. This completes the proof. □

Now, we establish the corrected result for the residual spectrum of the operator Δ_v over l_1.

Theorem 3.2. $\sigma_r(\Delta_v, l_1) = \left\{\lambda \in \mathbb{C} : \left|1 - \frac{\lambda}{L}\right| \leq 1\right\}$.

Proof. For $\left|1 - \frac{\lambda}{L}\right| \leq 1$, the operator $\Delta_v - \lambda I$ is one to one and hence has inverse. But $\Delta_v^* - \lambda I$ is not one to one by Theorem 1.5. Now, Lemma 1.1 yields the fact that $\overline{R(\Delta_v - \lambda I)} \neq l_1$ and this completes the proof. □

Combining Theorems 1.2, 1.3, 1.5, 1.7, 3.1 and 3.2, we can have the following main theorem:

Theorem 3.3.

1. *The operator* $\Delta_v : l_1 \to l_1$ *is a bounded linear operator and*

$$\|\Delta_v\|_{l_1} = 2 \sup_k (v_k) = 2v_0.$$

2. $\sigma(\Delta_v, l_1) = \left\{\lambda \in \mathbb{C} : \left|1 - \frac{\lambda}{L}\right| \leq 1\right\}$.
3. $\sigma_p(\Delta_v, l_1) = \varnothing$.
4. $\sigma_p(\Delta_v^*, l_1^*) = \left\{\lambda \in \mathbb{C} : \left|1 - \frac{\lambda}{L}\right| \leq 1\right\}$.

5. $\sigma_r(\Delta_v, l_1) = \left\{\lambda \in \mathbb{C} : \left|1 - \frac{\lambda}{L}\right| \leq 1\right\}.$

6. $\sigma_c(\Delta_v, l_1) = \varnothing.$

Remark 3.4. *Note That, if* (v_k) *is a constant sequence, say* $v_k = L \neq 0$ *for all* $k \in \mathbb{N}$*, then the operator* Δ_v *is reduced to the operator* $B(r, s)$ *with* $r = L$*,* $s = -L$ *and the results for the spectrum and fine spectrum of the operator* Δ_v *on* l_1 *follow immediately from the corresponding results in [4].*

Remark 3.5. *A modification of the operator* Δ_v *over the sequence spaces* c *and* l_p*, where* $1 < p < \infty$ *introduced and studied in [5].*

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