Thai Journal of Mathematics
Volume 10 (2012) Number 2 : 401–414
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Construction of Vector-valued Multivariate Wavelet Frame Packets

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Abstract : In this paper, the splitting trick coined by Chen [D.-R. Chen, On the splitting trick and wavelet frame packets, SIAM J. Math. Anal. 31 (4) (2000) 726–739] is used to construct vector-valued multivariate wavelet frame packets with an arbitrary dilation matrix A. It is shown that, as long as finitely many splitting steps are applied, the resulting sequence of functions is a frame of $L^2(\mathbb{R}^d)^r$. If the matrix $Q(\xi)$ associated with the splitting is unitary, then the splitting can be applied infinitely many times to prove the existence of frame with the frame bounds as shown in Theorem 3.3.

Keywords : Vector-valued wavelets; Wavelet packet; Wavelet frame packet; Dilation matrix; Splitting trick; Fourier transform.
2010 Mathematics Subject Classification : 42C40; 42C15; 65T60.

1 Introduction

Considerable attention has been given to wavelet packet analysis as an important generalization of wavelet analysis. Wavelet packet functions consist of a rich family of building block functions and are localized in time, but offer more flexibility than wavelets in representing different kinds of signals. The power of wavelet packets lies in the fact that we have much more freedom in selecting which basis functions are to be used to represent the given function. Wavelet packets, due to their nice characteristics have been widely applied to signal processing, coding theory, image compression, fractal theory and solving integral equations and so on.

It is well-known that the classical orthonormal wavelet bases have poor frequency localization. To overcome this disadvantage, Coifman et al. [1] constructed univariate orthogonal wavelet packets. Chui and Li [2] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. The introduction of biorthogonal wavelet packets attributes to Cohen and Daubechies [3]. Shen [4] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Other notable generalizations are the orthogonal version of wavelet packets on a positive half-line $\mathbb{R}^+$ [5], the orthogonal multiwavelet packets [6], non-orthogonal wavelet packets with r-scaling functions [7], the wavelet frame packets [8] on $\mathbb{R}$ for dilation 2 and the orthogonal, biorthogonal and frame packets on $\mathbb{R}^d$ by Long and Chen [9, 10] for the dyadic dilation.

Recently, Sun and Cheng [11] investigated the construction of a class of compactly supported orthogonal vector-valued wavelets. The definition and construction of orthogonal vector-valued wavelet packets are given in a paper by Chen and Chang [12]. Vector-valued wavelets are a class of generalized multiwavelets and multiwavelets can be generated from the component function in vector-valued wavelets (see [13]). Vector-valued wavelets and multiwavelets are different in the following sense. Vector-valued wavelets can be used to decorrelate a vector-valued signal not only in the time domain but also between components for a fixed time where as multiwavelets focuses only on the decorrelation of signals in time domain. Moreover, prefiltering is usually required for discrete multiwavelet transform but not necessary for discrete vector-valued wavelet transforms. The concept of vector-valued wavelet packets was subsequently generalized to vector-valued multivariate wavelet packets by Chen et al. [14] for dilation 2 and for the dilation factor m by Chen et al. [15]. In the same year, Xiao-Feng et al. [16] gave the construction and characterization of all vector-valued multivariate wavelet packets associated with dilation matrix by means of time-frequency analysis, matrix theory and operator theory.

Since frames provide a useful model to obtain signal decompositions in cases where redundancy, robustness, oversampling and irregular sampling play a role. It is, therefore, worthwhile to generalize the construction of vector-valued multivariate wavelet packets to the case of frames. So the main purpose of this paper is to give the construction of vector-valued multivariate wavelet frame packets associated with arbitrary dilation matrix using the splitting trick for frames. When a finitely many splitting steps are used, the resulting sequence of vector-valued functions is a frame of $L^2(\mathbb{R}^d)^r$. Moreover, if the matrix $Q(\xi)$ associated with the splitting is unitary, then the splitting can be applied infinitely many times to prove the existence of frame with frame bounds as shown in Theorem 3.3.

2 Preliminaries and Vector-valued Multivariate Wavelet Packets

Throughout, this paper, we use the following notations. Let $\mathbb{R}$ and $\mathbb{C}$ be all real and complex numbers, respectively. $\mathbb{Z}$ and $\mathbb{N}_0$ denote all integers and all non-negative integers, respectively. $\mathbb{Z}^d$ and $\mathbb{R}^d$ denote the set of all d-tuples of integers and d-tuples of reals, respectively. We denote

$$\mathbb{Z}_+^d = \left\{ z : (z_1, z_2, ..., z_d) \in \mathbb{Z}^d, z_i \geq 0, i = 1, 2, .., d \right\}.$$

Let A denotes a $d \times d$ dilation matrix, whose determinant is $m (m \in \mathbb{Z}, m \geq 2)$. A $d \times d$ matrix A is said to be a dilation matrix for $\mathbb{R}^d$ if

(i) $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$;

(ii) all eigenvalues λ of A satisfy $|\lambda| > 1$.

Property (i) implies that A has integer entries and hence $|detA|$ is greater than 1. Let $B = A^t$, the transpose of A and $m = |detA| = |detB|$. It is known that there exists m-elements $\rho_0, \rho_1, \rho_2, ..., \rho_{m-1}$ in $\mathbb{Z}_+^d$, by the finite group theory such that

$$\mathbb{Z}^d = \bigcup_{\rho \in \Omega_0} (\rho + A\mathbb{Z}^d); \quad (\rho_1 + A\mathbb{Z}^d) \cap (\rho_1 + A\mathbb{Z}^d) = \emptyset$$

where $\Omega_0 = \{\rho_0, \rho_1, \rho_2, ..., \rho_{m-1}\}$ denotes the set of all different representative elements in the quotient group $\mathbb{Z}^d / A\mathbb{Z}^d$ and ρ_1, ρ_2 denote two arbitrary elements in Ω_0. Set $\rho_0 = \{0\}$, where $\{0\}$ is the null of the set $\mathbb{Z}_+^d$. Let $\Omega = \Omega_0 - \{0\}$ and Ω, Ω_0 be two index sets. Let $L^2(\mathbb{R}^d)^r$ denotes the set of all $r \times 1$ vector-valued functions $\Phi(x)$, i.e.,

$$L^2(\mathbb{R}^d)^r = \left\{ \Phi(x) = \big(\phi_1(x), \phi_2(x), ..., \phi_r(x)\big)^t : \phi_i(x) \in L^2(\mathbb{R}^d), i = 1, 2, ..., r \right\}.$$

For $\Phi(x) \in L^2(\mathbb{R}^d)^r$, $\|\Phi\|$ denotes the norm of vector-valued function $\|\Phi(x)\|$ as follows:

$$\|\Phi\| = \left(\sum_{i=1}^{r} \int_{\mathbb{R}^d} |\phi_i(x)|^2 dx \right)^{1/2}. \tag{2.1}$$

For $\Phi \in L^2(\mathbb{R}^d)^r$, its integration and Fourier transform are defined, respectively, as follows:

$$\int_{\mathbb{R}^d} \Phi(x) dx = \left(\int_{\mathbb{R}^d} \phi_1(x) dx, \int_{\mathbb{R}^d} \phi_2(x) dx, ..., \int_{\mathbb{R}^d} \phi_r(x) dx \right), \tag{2.2}$$

$$\hat{\Phi}(\xi) = \int_{\mathbb{R}^d} \Phi(x) e^{-i\langle x, \xi \rangle} dx, \tag{2.3}$$

where $\langle x, \xi \rangle$ denotes the inner product of real vector x and ξ. Moreover, for any two vector-valued functions $\Phi, \Psi \in L^2(\mathbb{R}^d)^r, \langle \Phi, \Psi \rangle$ denote their symbol as inner product, i.e.,

$$\langle \Phi, \Psi \rangle = \int_{\mathbb{R}^d} \Phi(x)\Psi(x)^* \, dx, \tag{2.4}$$

where the superscript $*$ means the transpose and complex conjugate.

We now recall the notion of higher dimensional vector-valued multiresolution analysis and orthogonal vector-valued wavelets of $L^2(\mathbb{R}^d)^r$.

Definition 2.1 ([12]). A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)^r$ is called a vector-valued multiresolution analysis (MRA) of $L^2(\mathbb{R}^d)^r$ associated with a dilation matrix A if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)^r$,

(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, where $\{0\}$ is r-dimensional zero vector,

(iv) $F \in V_j$ if and only if $F(A.) \in V_{j+1}$ for all $j \in \mathbb{Z}$,

(v) there exists a vector-valued function Φ in V_0, called the scaling function, such that the system of vector-valued functions $\left\{\Phi(x-k) : k \in \mathbb{Z}^d\right\}$ forms an orthonormal basis for V_0.

Since $\Phi(x) \in V_0 \subset V_1$, there exists finitely supported $r \times r$ matrix constant sequence $\{P_k\}_{k\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)^{r\times r}$, which has finite non-zero terms, such that

$$\Phi(x) = \sum_{k\in\mathbb{Z}^d} P_k \Phi(Ax-k) \tag{2.5}$$

Equation (2.5) is called a *refinement equation* and $\Phi(x)$ is a vector-valued scaling function. Taking the Fourier transform of (2.5), we get

$$\hat{\Phi}(\xi) = P(\xi)\hat{\Phi}(\xi), \quad \xi \in \mathbb{R}^d \tag{2.6}$$

where

$$P(\xi) = \frac{1}{m} \sum_{k\in\mathbb{Z}^d} P_k e^{-i\langle k,\xi\rangle}, \quad \xi \in \mathbb{R}^d \tag{2.7}$$

is a $2\pi\mathbb{Z}^d$-periodic function, called the symbol of $\Phi(x)$.

Let $W_j, j \in \mathbb{Z}$ be the orthogonal complementary subspaces of V_j in V_{j+1}, i.e., $W_j = V_{j+1} \ominus V_j, j \in \mathbb{Z}$. These subspaces inherit the scaling property of $\{V_j\}$, namely

$$f \in W_j \ \text{ if and only if } \ f(A.) \in W_{j+1}. \tag{2.8}$$

Moreover, they are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(\mathbb{R}^d)^r = \bigoplus_{j\in\mathbb{Z}} W_j = V_0 \oplus \left(\bigoplus_{j\geq 0} W_j \right). \tag{2.9}$$

A set of vector-valued functions $\Psi_\sigma, \sigma \in \Omega$ in $L^2(\mathbb{R}^d)^r$ is said to be a set of basic vector-valued wavelets associated with the vector-valued MRA if the collection $\left\{\Psi_\sigma(x-k) : \sigma \in \Omega,\ k \in \mathbb{Z}^d\right\}$ forms an orthonormal basis for W_0. In view of (2.8) and (2.9), it is clear that if $\{\Psi_1, \Psi_2, ..., \Psi_{m-1}\}$ is a basic set of vector-valued wavelets, then

$$\left\{m^{j/2}\Psi_\sigma(A^j x - k) : j \in \mathbb{Z},\ k \in \mathbb{Z}^d, \sigma \in \Omega\right\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)^r$ (see [11, 12]). Hence, Eq. (2.9) becomes

$$L^2(\mathbb{R}^d)^r = \bigoplus_{j\in\mathbb{Z}} \left(\bigoplus_{\sigma\in\Omega} W_j^\sigma\right). \tag{2.10}$$

Since $\Psi_\sigma(x) \in W_0 \subset V_1, \sigma \in \Omega$, there exist $m-1$ finite supported constant $r \times r$ matrix sequences $\{P_k^\sigma\}_{k\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)^{r\times r}$ such that

$$\Psi_\sigma(x) = \sum_{k\in\mathbb{Z}^d} P_k^\sigma m^{1/2}\Phi(Ax-k), \quad \sigma \in \Omega. \tag{2.11}$$

In order to define the vector-valued wavelet packets, we set

$$G_0(x) = \Phi(x),\ G_\sigma(x) = \Psi_\sigma(x),\ P_k = Q_k,\ P_k^\sigma = Q_k^\sigma,\ \sigma \in \Omega, k \in \mathbb{Z}^d.$$

Then, the Eqs. (2.5) and (2.11) can be jointly written as follows:

$$G_\sigma(x) = \sum_{k\in\mathbb{Z}^d} Q_k^\sigma m^{1/2} G_0(Ax-k), \quad \sigma \in \Omega_0. \tag{2.12}$$

The Fourier transform of (2.12) yields

$$\hat{G}_\sigma\left(A\xi\right) = Q^\sigma(\xi)\,\hat{G}_0(\xi), \tag{2.13}$$

where

$$Q^\sigma(\xi) = \sum_{k\in\mathbb{Z}^d} Q_k^\sigma\, e^{-i\langle k,\xi\rangle}, \qquad \sigma \in \Omega_0. \tag{2.14}$$

The functions $Q^\sigma, \sigma \in \Omega_0$, are in $L^2(\mathbb{R}^d)^r$ such that

$$Q(\xi) = \left(Q^\sigma\left(B^{-1}(\xi + 2\pi\rho)\right)\right)_{\sigma,\rho\in\Omega_0} \tag{2.15}$$

is a unitary matrix for $a.e.\ \xi \in [0, 2\pi]^d$ (see [12, 13]).

For any $\alpha \in \mathbb{Z}_+^d$, the basic vector-valued multivariate wavelet packets associated with the orthogonal vector-valued scaling function G_0 are defined recursively by

$$G_\alpha(x) = G_{A\mu+\sigma}(x) = \sum_{k\in\mathbb{Z}^d} m^{1/2} Q_k^\sigma\, G_\mu(Ax-k), \quad \sigma \in \Omega_0, \mu \in \mathbb{Z}^+. \tag{2.16}$$

Let D be a dilation operator $(Df)(x) = f(Ax)$, where $f \in L^2(\mathbb{R}^d)^r$ and denote a set $DU = \{Df : f \in U\}$, where $U \subset L^2(\mathbb{R}^d)^r$. For any $\mu \in \mathbb{Z}_+^d$, define

$$U_\mu = \left\{ f(x) : f(x) = \sum_{k \in \mathbb{Z}^d} S_k G_\mu(x-k), \ \{S_k\} \in l^2(\mathbb{Z}^d)^{r\times r} \right\}$$

where the family $\left\{G_\mu(x) : \mu \in \mathbb{Z}_+^d\right\}$ are the vector-valued wavelet packets with respect to orthogonal vector-valued scaling function $G_0(x) = \Phi(x)$. Then, we observe that $U_0 = V_0$ and $U_\sigma = W_0^\sigma, \sigma \in \Omega$.

For $\mu \in \mathbb{Z}_+^d$, the set DU_μ can be orthogonally decomposed into subspaces $U_{A\mu+\sigma}, \sigma \in \Omega_0$, i.e.,

$$DU_\mu = \bigoplus_{\sigma \in \Omega_0} U_{A\mu+\sigma}. \tag{2.17}$$

Let

$$\Delta_j = \left\{\alpha : (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{Z}_+^d, \, m^{j-1} \le \alpha_\ell \le m^j - 1, j \in \mathbb{Z}, 1 \le \ell \le d\right\}.$$

For any $a \in \mathbb{N}_0$, set $\tilde{E}_a = \sum_{\ell=0}^{a} A^\ell \Omega_0, E_a = \tilde{E}_{a+1} - \tilde{E}_a = A\Omega_0$. Now, in view of (2.17), we have

$$DU_0 = U_0 \bigoplus_{\sigma \in \Omega_0} U_\sigma. \tag{2.18}$$

Since $U_0 = V_0$ and $W_0 = \bigoplus_{\sigma\in\Omega} W_j^\sigma = \bigoplus_{\sigma\in\Omega} U_\sigma$, hence $DU_0 = V_0 \oplus W_0$. Therefore, by the repeated applications of (2.17) and (2.18), we obtain

$$D^j U_0 = D^{j-1} U_0 \bigoplus_{\alpha \in E_a} U_\alpha. \tag{2.19}$$

But $V_{j+1} = V_j \oplus W_j$, thus it follows that

$$D^j U_0 = D^{j-1} U_0 \oplus D^{j-1} W_0.$$

Therefore, from (2.18), we have

$$L^2(\mathbb{R}^d)^r = V_0 \left(\bigoplus_{j\ge 0} D^j W_0 \right) = U_0 \bigoplus \left(\bigoplus_{j>0} \left(\bigoplus_{\alpha \in E_a} U_\alpha \right)\right) = \bigoplus_{\alpha \in \mathbb{Z}_+^d} U_\alpha. \tag{2.20}$$

Let $\Xi = \{(a, j) : a \in \mathbb{N}_0, j \in \mathbb{Z}\}$ be a set which satisfies for any $n \in \mathbb{Z}$, there exists a unique pair of number $(a, j) \in \Xi$, such that $n = a + j$. It was shown in [13] that the collection of vector-valued functions

$$\mathcal{F} = \left\{ m^{j/2} G_\alpha (A^j . - k) : \alpha \in E_a, \ (a, j) \in \Xi, k \in \mathbb{Z}^d \right\}$$

forms an orthonormal basis of $L^2(\mathbb{R}^d)^r$.

3 Splitting Trick and Vector-valued Wavelet Frame Packets

Let $\mathbb{H}$ be a Hilbert space. A sequence $\{x_k : k \in \mathbb{Z}\}$ of $\mathbb{H}$ is said to be a frame for $\mathbb{H}$ if there exist constants C_1 and $C_2,\ 0 < C_1 \le C_2 < \infty$ such that

$$C_1 \|x\|^2 \le \sum_{k\in\mathbb{Z}} \left|\langle x, x_k\rangle\right|^2 \le C_2 \|x\|^2 \tag{3.1}$$

for all $x \in \mathbb{H}$. The largest C_1 and the smallest C_2 for which (3.1) holds are called frame bounds. A frame is a tight frame if C_1 and C_2 can be chosen so that $C_1 = C_2$, and is a normalized tight frame if $C_1 = C_2 = 1$.

Let $\Phi = \left\{\Phi_{\sigma_1}, \Phi_{\sigma_2}, ..., \Phi_{\sigma_{m-1}}\right\}$ be a family of $m-1$ vector-valued functions in $L^2(\mathbb{R}^d)^r$ such that $\left\{\Phi_\sigma(x-k) : \sigma \in \Omega_0, k \in \mathbb{Z}^d\right\}$ is a frame for its closed linear span $S(\Phi)$. Suppose that the $m-1$ vector-valued functions $\left\{\Psi_{\sigma_1}, \Psi_{\sigma_2}, ..., \Psi_{\sigma_{m-1}}\right\}$ are in $S(\Phi)$, so that each Ψ_σ, is a linear combination of $\Phi_\sigma(x-k) : \sigma \in \Omega_0, k \in \mathbb{Z}^d$. Now, it is natural to ask: whether $\left\{\Psi_\sigma(x-k) : \sigma \in \Omega_0, k \in \mathbb{Z}^d\right\}$ is also a frame for $S(\Phi)$. If $\Psi_\sigma, \sigma \in \Omega_0$ are in $S(\Phi_\sigma)$, then there exists constant $r \times r$ matrix sequences $\{H^\sigma_{\mu,k}\}_{\mu\in\Omega_0, k\in\mathbb{Z}^d}$ in $l^2(\mathbb{Z}^d)^{r\times r}$ such that

$$\Psi_\sigma(x) = \sum_{\mu\in\Omega_0} \sum_{k\in\mathbb{Z}^d} H^\sigma_{\mu,k}\, \Phi_\mu(x-k), \quad \sigma \in \Omega_0. \tag{3.2}$$

Taking Fourier transform, we get

$$\begin{aligned}\hat{\Psi}_\sigma(\xi) &= \sum_{\mu\in\Omega_0} \sum_{k\in\mathbb{Z}^d} H^\sigma_{\mu,k}\, \hat{\Phi}_\mu(\xi)\, e^{-i\langle k,\xi\rangle} \\ &= \sum_{\mu\in\Omega_0} H^\sigma_\mu\, \hat{\Phi}_\mu(\xi), \quad \sigma \in \Omega_0\end{aligned}$$

where $H^\sigma_\mu(\xi) = \sum_{k\in\mathbb{Z}^d} H^\sigma_{\mu,k}\, e^{-i\langle k,\xi\rangle}$. Let $H(\xi) = \left(H^\sigma_\mu(\xi)\right)_{\sigma,\mu\in\Omega_0}$.

Now, we state a lemma which is the generalization of Lemma 3.1 in [8].

Lemma 3.1. *Let Φ_σ, Ψ_σ for $\sigma, \mu \in \Omega_0$ and $H(\xi)$ be as above. Suppose that there exist constants C_1 and $C_2,\ 0 < C_1 \le C_2 < \infty$ such that*

$$C_1 I_r \le H^*(\xi) H(\xi) \le C_2 I_r \quad \textit{for a.e. } \xi \in \mathbb{R}^d.$$

Then, for all $f \in L^2(\mathbb{R}^d)^r$, we have

$$\begin{aligned}C_1 \sum_{\mu\in\Omega_0} \sum_{k\in\mathbb{Z}^d} \left|\langle f, \Phi_\mu(x-k)\rangle\right|^2 &\le \sum_{\sigma\in\Omega_0} \sum_{k\in\mathbb{Z}^d} \left|\langle f, \Psi_\sigma(x-k)\rangle\right|^2 \\ &\le C_2 \sum_{\mu\in\Omega_0} \sum_{k\in\mathbb{Z}^d} \left|\langle f, \Phi_\mu(x-k)\rangle\right|^2. \end{aligned} \tag{3.3}$$

Now, we apply the splitting trick to vector-valued wavelet frames. Let $\Phi \in L^2(\mathbb{R}^d)^r,\ V_0 = \overline{\operatorname{span}}\left\{\Phi(x-k) : k \in \mathbb{Z}^d\right\}$ and $\left\{\Phi(x-k) : k \in \mathbb{Z}^d\right\}$ be a frame for V_0 with frame bounds C_1 and C_2. For $\sigma \in \Omega_0$, suppose that there exists matrix sequences $\{Q_k^\sigma\}_{k\in\mathbb{Z}^d}$ with $\sum_{k\in\mathbb{Z}^d} |Q_k^\sigma|^2 < \infty$. We define G_σ as in (2.12), that is,

$$G_\sigma(x) = \sum_{k\in\mathbb{Z}^d} Q_k^\sigma\, m^{1/2}\Phi(Ax-k). \tag{3.4}$$

Let $Q(\xi)$ be the matrix defined in (2.15) and $Q^\sigma(\xi)$ be the functions as in (2.14). Assume that there exist constants C_1 and $C_2,\ 0 < C_1 \leq C_2 < \infty$ such that

$$C_1 I_r \leq Q^*(\xi)Q(\xi) \leq C_2 I_r \ \text{ for a.e. } \xi \in [0, 2\pi]^d. \tag{3.5}$$

Furthermore, the vector-valued functions $G_\sigma(x)$ has the equivalent expression

$$\begin{aligned} G_\sigma(x) &= \sum_{k\in\mathbb{Z}^d} Q_k^\sigma\, m^{1/2} G_0(Ax-k) \\ &= \sum_{\mu\in\Omega_0}\sum_{k'\in\mathbb{Z}^d} R_{\mu,k'}^\sigma\, \Phi^{(\mu)}(x-k') \end{aligned}$$

where

$$R_{\mu,k'}^\sigma = m^{1/2} Q_k^\sigma,\ \text{when } k = mk' - \mu, k' \in \mathbb{Z}^d, \mu \in \Omega_0$$

and

$$\Phi^{(\mu)}(x) = m^{1/2}\Phi(Ax-k'),\ \mu \in \Omega_0. \tag{3.6}$$

Taking Fourier transform, we obtain

$$\begin{aligned} \hat{G}_\sigma\,(\xi) &= \sum_{\mu\in\Omega_0}\sum_{k'\in\mathbb{Z}^d} R_{\mu,k'}^\sigma\ (\Phi^{(\mu)})^\wedge(\xi)\, e^{-i\langle k,\xi\rangle} \\ &= \sum_{\mu\in\Omega_0} H_\mu^\sigma(\xi)\, (\Phi^{(\mu)})^\wedge(\xi), \quad \sigma \in \Omega, \end{aligned}$$

where $H_\mu^\sigma(\xi) = \sum_{k'\in\mathbb{Z}^d} R_{\mu,k'}^\sigma\, e^{-i\langle k,\xi\rangle}$.

Let $H(\xi) = \left(H_\mu^\sigma(\xi)\right)_{\sigma,\mu\in\Omega}$. Then it is easy to verify that $Q^*(\xi)Q(\xi)$ and $H^*(\xi)H(\xi)$ are similar matrices (see [10]). Let $\lambda(\xi)$ and $\Lambda(\xi)$ be the minimal and maximal eigenvalues of the positive definite matrix $Q^*(\xi)Q(\xi)$, respectively and let $\lambda = \inf_\xi \lambda(\xi)$ and $\Lambda = \sup_\xi \Lambda(\xi)$. Suppose $0 < \lambda \leq \Lambda < \infty$. Then, by Eq. (3.6), we have

$$\lambda I_r \leq Q^*(\xi)Q(\xi) \leq \Lambda I_r \ \text{ for a.e. } \xi \in [0, 2\pi]^d.$$

This is equivalent to say that

$$\lambda I_r \leq H^*(\xi)H(\xi) \leq \Lambda I_r \ \text{ for a.e. } \xi \in [0, 2\pi]^d.$$

Then by Lemma 3.1, for all $g \in L^2(\mathbb{R}^d)^r$, we have

$$\begin{aligned}\lambda \sum_{\mu\in\Omega_0}\sum_{k\in\mathbb{Z}^d} \left|\langle g, \Phi^{(\mu)}(x-k)\rangle\right|^2 &\le \sum_{\mu\in\Omega_0}\sum_{k\in\mathbb{Z}^d} \left|\langle g, G_\mu(x-k)\rangle\right|^2 \\ &\le \Lambda \sum_{\mu\in\Omega_0}\sum_{k\in\mathbb{Z}^d} \left|\langle g, \Phi^{(\mu)}(x-k)\rangle\right|^2, \end{aligned} \tag{3.7}$$

where $\Phi^{(\mu)}$ is defined in (3.6). Since

$$\sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{1/2}\Phi(Ax-k)\rangle\right|^2 = \sum_{\mu\in\Omega_0}\sum_{k\in\mathbb{Z}^d} \left|\langle g, \Phi^{(\mu)}(x-k)\rangle\right|^2,$$

which follows from (3.6), inequality (3.7) can be written as

$$\begin{aligned}\lambda \sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{1/2}\Phi(Ax-k)\rangle\right|^2 &\le \sum_{\mu\in\Omega_0}\sum_{k\in\mathbb{Z}^d} \left|\langle g, G_\mu(x-k)\rangle\right|^2 \\ &\le \Lambda \sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{1/2}\Phi(Ax-k)\rangle\right|^2. \end{aligned} \tag{3.8}$$

This is the *splitting trick for frames*. We now apply this trick to the vector-valued functions G_μ, for each $\mu \in \Omega_0$ to obtain

$$\begin{aligned}\lambda \sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{1/2}G_\mu(Ax-k)\rangle\right|^2 &\le \sum_{\sigma\in\Omega_0}\sum_{k\in\mathbb{Z}^d} \left|\langle g, G_{\sigma,\mu}(x-k)\rangle\right|^2 \\ &\le \Lambda \sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{1/2}G_\mu(Ax-k)\rangle\right|^2 \end{aligned} \tag{3.9}$$

where $G_{\sigma,\mu},\ \sigma,\mu \in \Omega_0$ are defined as in (3.4) (G_μ now replaces Φ):

$$G_{\sigma,\mu}(x) = \sum_{k\in\mathbb{Z}^d} Q_k^\sigma\, m^{1/2} G_\mu(Ax-k), \quad \mu \in \Omega_0. \tag{3.10}$$

Summing (3.9) over $\mu \in \Omega_0$, we have

$$\begin{aligned}\lambda \sum_{\mu\in\Omega_0}\sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{1/2}G_\mu(Ax-k)\rangle\right|^2 &\le \sum_{\sigma\in\Omega_0}\sum_{\mu\in\Omega_0}\sum_{k\in\mathbb{Z}^d} \left|\langle g, G_{\sigma,\mu}(x-k)\rangle\right|^2 \\ &\le \Lambda \sum_{\mu\in\Omega_0}\sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{1/2}G_\mu(Ax-k)\rangle\right|^2. \end{aligned}$$

Using (3.8), we obtain

$$\begin{aligned}\lambda^2 \sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{2/2}\Phi(A^2x-k)\rangle\right|^2 &\le \sum_{\sigma\in\Omega_0}\sum_{\mu\in\Omega_0}\sum_{k\in\mathbb{Z}^d} \left|\langle g, G_{\sigma,\mu}(x-k)\rangle\right|^2 \\ &\le \Lambda^2 \sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{2/2}\Phi(A^2x-k)\rangle\right|^2. \end{aligned} \tag{3.11}$$

We now define the vector-valued wavelet frame packets $G_\alpha, \alpha \in \mathbb{Z}^d$, similar to the orthonormal case (see Eq. (2.16)). Thus, in order to ensure that G_α are in $L^2(\mathbb{R}^d)^r$, it is sufficient to assume that all the entries in the matrix $Q(\xi)$, defined in (2.15), are bounded functions. Comparing (3.10) and (2.16), we see that

$$\{G_{\mu,\sigma} : \sigma, \mu \in \Omega_0\} = \{G_{A\mu+\sigma} : \sigma, \mu \in \Omega_0\} = \{G_\alpha : \alpha \in \Delta_2\}.$$

So (3.11) can be written as

$$\begin{aligned}\lambda^2 \sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{2/2}\Phi(A^2x-k)\rangle\right|^2 &\le \sum_{\alpha\in\Delta_2}\sum_{k\in\mathbb{Z}^d} \left|\langle g, G_\alpha(x-k)\rangle\right|^2 \\ &\le \Lambda^2 \sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{2/2}\Phi(A^2x-k)\rangle\right|^2.\end{aligned}$$

By induction, we get for each $j \ge 1$,

$$\begin{aligned}\lambda^j \sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{j/2}\Phi(A^jx-k)\rangle\right|^2 &\le \sum_{\alpha\in\Delta_j}\sum_{k\in\mathbb{Z}^d} \left|\langle g, G_\alpha(x-k)\rangle\right|^2 \\ &\le \Lambda^j \sum_{k\in\mathbb{Z}^d} \left|\langle g, m^{j/2}\Phi(A^jx-k)\rangle\right|^2. \qquad (3.12)\end{aligned}$$

The vector-valued functions $\left\{G_\alpha : \alpha \in \mathbb{Z}^d_+\right\}$ will be called the *vector-valued multivariate wavelet frame packets* and we summarize the above discussion in the following theorem.

Theorem 3.2. *Let $\Phi \in L^2(\mathbb{R}^d)^r$ such that $\left\{\Phi(x-k) : k \in \mathbb{Z}^d\right\}$ is a frame for its closed linear span V_0 with frame bounds C_1 and C_2. Let $Q(\xi)$, λ and Λ be as above. Assume that the entries of $Q(\xi)$ are bounded measurable functions such that $0 < \lambda \le \Lambda < \infty$. Let $\left\{G_\alpha : \alpha \in \mathbb{Z}^d_+\right\}$ be the vector-valued wavelet frame packets and let $V_j = \left\{f : f(A^{-j}.) \in V_0\right\}$. Then, for all $j \ge 0$, the system of vector-valued functions*

$$\left\{G_\alpha(x-k) : \alpha \in \mathbb{Z}^d_+, k \in \mathbb{Z}^d\right\}$$

is a frame of V_j with frame bounds $\lambda^j C_1$ and $\Lambda^j C_2$.

Proof. Since $\left\{\Phi(x-k) : k \in \mathbb{Z}^d\right\}$ is a frame of V_0 with frame bounds C_1 and C_2, it is clear that for all j, $\left\{m^{j/2}\Phi(A^jx-k) : k \in \mathbb{Z}^d\right\}$ is a frame of V_j with same frame bounds. So from (3.12), we have

$$\lambda^j C_1\|g\|^2 \le \sum_{\alpha\in\Delta_j}\sum_{k\in\mathbb{Z}^d} \left|\langle g, G_\alpha(x-k)\rangle\right|^2 \le \Lambda^j C_2\|g\|^2, \quad \forall\ g \in V_j. \qquad (3.13)$$

$\square$

Let $V_0 = \overline{\text{span}}\left\{\Phi(x-k) : k \in \mathbb{Z}^d\right\}$, $V_j = \left\{f : f(A^{-j}.) \in V_0\right\}$ and $V_j \subset V_{j+1}$. Let $W = \cup V_j$. Then, it is easy to check that W is invariant under translations by $A^{-j}k$ and these elements are dense in $\mathbb{R}^d$. Therefore, $\overline{W}$ is a closed translation invariant subspace of $L^2(\mathbb{R}^d)^r$. Hence, $\overline{W} = L^2_E(\mathbb{R}^d)^r$ for some $E \subset \mathbb{R}^d$ (see [17]), where

$$L^2_E(\mathbb{R}^d)^r = \left\{ f \in L^2(\mathbb{R}^d)^r : \text{supp } \hat{f} \subset E \right\}.$$

Since the basic vector-valued wavelet packets form an orthonormal basis for $L^2(\mathbb{R}^d)^r = \overline{\cup V_j}$, an analogous result holds for the vector-valued wavelet frame packets if the matrix $Q(\xi)$ defined in (2.15), is unitary. Hence, when $Q(\xi)$ is unitary for *a.e.* ξ, then the splitting trick can be operated for infinitely many times as shown by the following theorem.

Theorem 3.3. *Let $\left\{\Phi(x-k) : k \in \mathbb{Z}^d\right\}$ be a frame for its closed linear span V_0, with frame bounds C_1 and C_2 and let $V_0 \subset V_1$, where $V_j = \left\{f : f(A^{-j}.) \in V_0\right\}$. Assume that $Q(\xi)$ is unitary for a.e. ξ. Then $\left\{G_\alpha(x-k) : \alpha \in \mathbb{Z}^d_+,\ k \in \mathbb{Z}^d\right\}$ is a frame for the spaces $\overline{\cup_{j\geq 0} V_j}$ with same frame bounds.*

More generally, let $\Xi = \{(a,j) \in \mathbb{N}_0 \times \mathbb{Z}\}$ be such that $\bigcup_{(a,j)\in\Xi} \Delta_j$ is a partition of $\mathbb{N}_0$. Then, the collection of functions

$$\mathcal{F}_\Xi = \left\{ m^{j/2} G_\alpha(A^j x - k) : (a,j) \in \Xi,\ k \in \mathbb{Z}^d \right\}$$

is a frame for $\overline{\cup_{j\geq 0} V_j}$ with same bounds C_1 and C_2.

Proof. Since $Q(\xi)$ is unitary, $\lambda = \Lambda = 1$ so that the inequalities in (3.12) are equalities, and from (3.13) we have

$$C_1\|g\|^2 \leq \sum_{\alpha\in\Delta_j} \sum_{k\in\mathbb{Z}^d} \left|\langle g, G_\alpha(x-k)\rangle\right|^2 \leq C_2\|g\|^2, \quad \forall\ g \in V_j. \tag{3.14}$$

Now, let $h \in \overline{\cup_{j\geq 0} V_j}$, then there exists $h_j \in V_j$ such that $h_j \to h$ as $j \to \infty$. We now, fix j, then for $j < j'$, we have from equation (3.14)

$$\sum_{\alpha\in\Delta_j} \sum_{k\in\mathbb{Z}^d} \left|\langle h_{j'}, G_\alpha(x-k)\rangle\right|^2 \leq C_2\|h_{j'}\|^2.$$

Letting $j' \to \infty$ first and then $j \to \infty$, we have for all $h \in \overline{\cup_{j\geq 0} V_j}$

$$\sum_{\alpha\in\mathbb{Z}^d_+} \sum_{k\in\mathbb{Z}^d} \left|\langle h, G_\alpha(x-k)\rangle\right|^2 \leq C_2\|h\|^2. \tag{3.15}$$

To get the reverse inequality, we again use (3.14)

$$\begin{aligned} C_1\|h_j\|^2 &\leq \sum_{\alpha\in\Delta_j} \sum_{k\in\mathbb{Z}^d} \left|\langle h_j, G_\alpha(x-k)\rangle\right|^2 \\ &= \sum_{\alpha\in\Delta_j} \sum_{k\in\mathbb{Z}^d} \left|\langle h_j - h, G_\alpha(x-k)\rangle + \langle h, G_\alpha(x-k)\rangle\right|^2. \end{aligned}$$

Therefore,

$$C_1^{1/2}\|h_j\| \le \left(\sum_{\alpha\in\Delta_j}\sum_{k\in\mathbb{Z}^d}\left|\langle h_j-h, G_\alpha(x-k)\rangle\right|^2\right)^{\frac{1}{2}} + \left(\sum_{\alpha\in\Delta_j}\sum_{k\in\mathbb{Z}^d}\left|\langle h, G_\alpha(x-k)\rangle\right|^2\right)^{\frac{1}{2}}$$
$$\le C_2^{1/2}\|h_j-h\| + \left(\sum_{\alpha\in\Delta_j}\sum_{k\in\mathbb{Z}^d}\left|\langle h, G_\alpha(x-k)\rangle\right|^2\right)^{\frac{1}{2}}. \quad \text{(by (3.15))}$$

Letting $j\to\infty$, we get

$$C_1\|h\|^2 \le \sum_{\alpha\in\mathbb{Z}_+^d}\sum_{k\in\mathbb{Z}^d}\left|\langle h, G_\alpha(x-k)\rangle\right|^2, \quad \text{for all } h\in\overline{\cup V_j}.$$

Hence the first part is proved. Since $Q(\xi)$ is unitary, we have $\lambda=\Lambda=1$ and hence there must be an equality in (3.9). Therefore,

$$\sum_{k\in\mathbb{Z}^d}\left|\langle g, m^{1/2}G_\alpha(Ax-k)\rangle\right|^2 = \sum_{\sigma\in\Omega_0}\sum_{k\in\mathbb{Z}^d}\left|\langle g, G_{A\mu+\sigma}(x-k)\rangle\right|^2.$$

Using this result, we get

$$\sum_{k\in\mathbb{Z}^d}\left|\langle g, m^{2/2}G_\alpha(A^2x-k)\rangle\right|^2 = \sum_{\sigma\in\Omega_0}\sum_{\tau\in\Omega_0}\sum_{k\in\mathbb{Z}^d}\left|\langle g, G_{A(A\mu+\sigma)+\tau}(x-k)\rangle\right|^2$$
$$= \sum_{\sigma\in\Delta_2}\sum_{k\in\mathbb{Z}^d}\left|\langle g, G_\sigma(x-k)\rangle\right|^2.$$

Similarly, we obtain

$$\sum_{k\in\mathbb{Z}^d}\left|\langle g, m^{j/2}G_\alpha(A^jx-k)\rangle\right|^2 = \sum_{\sigma\in\Delta_j}\sum_{k\in\mathbb{Z}^d}\left|\langle g, G_\sigma(x-k)\rangle\right|^2. \tag{3.16}$$

From the first part of the theorem, we have

$$C_1\|g\|^2 \le \sum_{\alpha\in\mathbb{Z}_+^d}\sum_{k\in\mathbb{Z}^d}\left|\langle g, G_\alpha(x-k)\rangle\right|^2 \le C_2\|g\|^2, \quad \forall\ g\in\overline{\cup V_j}.$$

But the set Ξ is such that $\bigcup_{(a,j)\in\Xi}\Delta_j=\mathbb{N}_0$. Therefore,

$$C_1\|g\|^2 \le \sum_{(a,j)\in\Xi}\sum_{\sigma\in\Delta_j}\sum_{k\in\mathbb{Z}^d}\left|\langle g, G_\sigma(x-k)\rangle\right|^2 \le C_2\|g\|^2.$$

Using (3.16), we get

$$C_1\|g\|^2 \le \sum_{(a,j)\in\Xi}\sum_{k\in\mathbb{Z}^d}\left|\langle g, m^{j/2}G_\alpha(A^jx-k)\rangle\right|^2 \le C_2\|g\|^2$$

for all $g\in\overline{\cup V_j}$. This completes the proof of the theorem complelety. $\square$

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(Received 26 April 2011)
(Accepted 4 November 2011)