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Properties of Graph which satisfy Equations $s \approx t$ where s, t are $(x(yz))z$ Terms of Type (2,0)[1]

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Abstract : Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies a term equation $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$. A class of graph algebras $\mathcal{V}$ is called a graph variety if $\mathcal{V} = \overline{Mod_g\Sigma}$ where Σ is a subset of $T(X) \times T(X)$. A graph variety $\mathcal{V}' = Mod_g\Sigma^{'}$ is called an $(x(yz))z$ graph variety if $\Sigma^{'}$ is a set of $(x(yz))z$ term equations.

In this paper we characterize all graphs which satisfy an equation $s \approx t$ where s, t are $(x(yz))z$ terms.

Keywords : Varieties; $(x(yz))z$ graph varieties; Term; $(x(yz))z$ term; Binary algebra; Graph algebra.
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1 Introduction

Graph algebras have been invented in [1] to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra*

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$\underline{A(G)}$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V, and with two basic operations, namely a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given for $u, v \in V \cup \{\infty\}$ by

$$uv = \begin{cases} u, & \text{if} \quad (u,v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

In [2], Thongmoon and Poomsa-ard characterized all triregular leftmost without loop and reverse arc graph varieties. In [3], Anantpinitwatna and Poomsa-ard characterized all $(x(yz))z$ with loop graph varieties.

We say that a graph variety $\mathcal{V}' = Mod_g\Sigma^{'}$ is called a $(x(yz))z$ graph variety if $\Sigma^{'}$ is a set of $(x(yz))z$ term equations. In this paper we characterize all $(x(yz))z$ graph varieties which $\Sigma^{'}$ is a set of one $(x(yz))z$ term equation.

2 Terms and Graph Varieties

In [4], Pöschel introduced terms for graph algebras, the underlying formal language has to contain a binary operation symbol (juxtaposition) and a symbol for the constant ∞.

Definition 2.1. A term over the alphabet

$$X = \{x_1, x_2, x_3, ...\}$$

is defined inductively as follows:

(i) every variable $x_i, i = 1, 2, 3, ...,$ and ∞ are terms;

(ii) if t_1 and t_2 are terms, then t_1t_2 is a term.

$T(X)$ is the set of all terms which can be obtained from (i) and (ii) in finitely many steps. Terms built up from the two-element set $X_2 = \{x_1, x_2\}$ of variables are thus binary terms. We denote the set of all binary terms by $T(X_2)$. The leftmost variable of a term t is denoted by $L(t)$. A term, in which the symbol ∞ occurs is called a *trivial term*.

Definition 2.2. For each non-trivial term t of type $\tau = (2, 0)$ one can define a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable and } E(t_1t_2) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$$

where $t = t_1t_2$ is a compound term.

$L(t)$ is called the *root* of the graph $G(t)$, and the pair $(G(t), L(t))$ is the *rooted graph* corresponding to t. Formally, we assign the empty graph ϕ to every trivial term t.

Definition 2.3. A non-trivial term t of type $\tau = (2, 0)$ is called an $(x(yz))z$ term if and only if $V(t) = \{x, y, z\}$ and $(x, y), (y, z), (x, z) \in E(t)$. A term equation $s \approx t$ of type $\tau = (2, 0)$ is called $(x(yz))z$ term equation if and only if s, t are $(x(yz))z$ terms.

Definition 2.4. We say that a graph $G = (V, E)$ *satisfies a term equation* $s \approx t$ if the corresponding graph algebra $\underline{A(G)}$ satisfies $s \approx t$ (i.e., we have $s = t$ for every assignment $V(s) \cup V(t) \to V \cup \{\infty\}$), and in this case, we write $G \models s \approx t$. Given a class $\mathcal{G}$ of graphs and a set Σ of term equations (i.e., $\Sigma \subset T(X) \times T(X)$) we introduce the following notation:

$G \models \Sigma$ if $G \models s \approx t$ for all $s \approx t \in \Sigma$, $\mathcal{G} \models s \approx t$ if $G \models s \approx t$ for all $G \in \mathcal{G}$, $\mathcal{G} \models \Sigma$ if $G \models \Sigma$ for all $G \in \mathcal{G}$,

$Id\mathcal{G} = \{s \approx t \mid s, t \in T(X),\ \mathcal{G} \models s \approx t\}$, $Mod_g\Sigma = \{G \mid G$ is a graph and $G \models \Sigma\}$, $\mathcal{V}_g(\mathcal{G}) = Mod_g Id\mathcal{G}$.

$\mathcal{V}_g(\mathcal{G})$ is called the *graph variety generated by* $\mathcal{G}$ and $\mathcal{G}$ is called *graph variety* if $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$. $\mathcal{G}$ is called *equational* if there exists a set Σ' of term equations such that $\mathcal{G} = \mathrm{Mod}_g\Sigma'$. Obviously $\mathcal{V}_g(\mathcal{G}) = \mathcal{G}$ if and only if $\mathcal{G}$ is an equational class.

In [4], Pöschel showed that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. Without difficulties one shows $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

Definition 2.5. Let $G = (V, E)$ and $G^{'} = (V^{'}, E^{'})$ be graphs. A *homomorphism* h from G into G' is a mapping $h : V \to V^{'}$ carrying edges to edges ,that is, for which $(u, v) \in E$ implies $(h(u), h(v)) \in E^{'}$.

In [5], the following proposition was proved:

Proposition 2.6. *Let $G = (V, E)$ be a graph and let $h : X \cup \{\infty\} \longrightarrow V \cup \{\infty\}$ be an evaluation of the variables such that $h(\infty) = \infty$. Consider the canonical extension of h to the set of all terms. Then there holds: if t is a trivial term then $h(t) = \infty$. Otherwise, if $h : G(t) \longrightarrow G$ is a homomorphism of graphs, then $h(t) = h(L(t))$, and if h is not a homomorphism of graphs, then $h(t) = \infty$.*

Further in [6] the following proposition was proved:

Proposition 2.7. *Let $G = (V, E)$ be a graph s and t be non-trivial terms. Then $G \models s \approx t$ if and only if $G \models NF(s) \approx NF(t)$.*

3 $(x(yz))z$ Graph Varieties

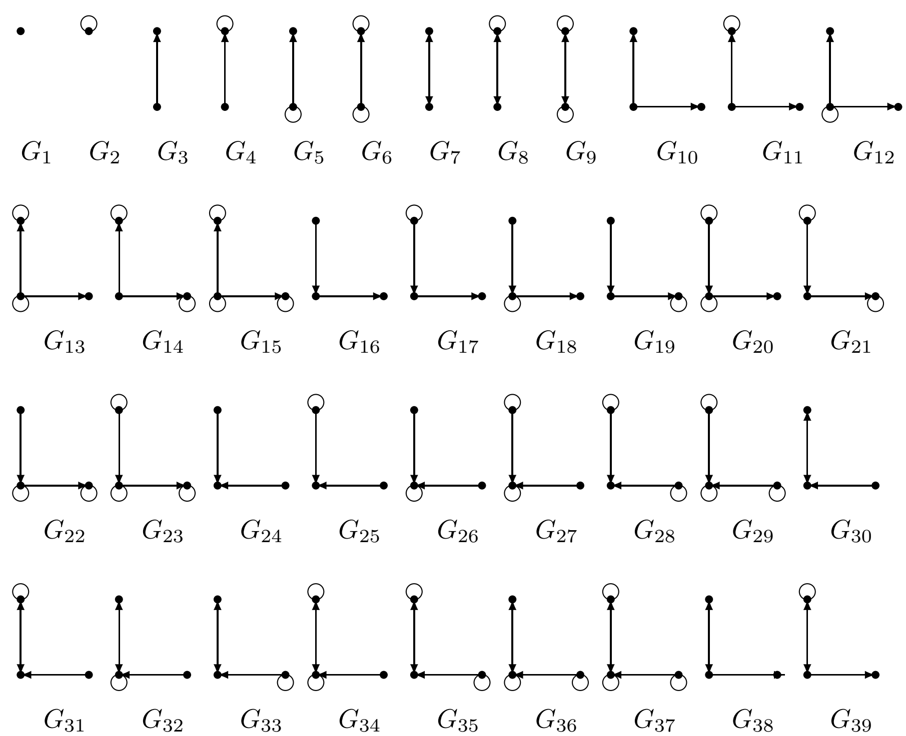
By Proposition 2.7, we see that if $\Sigma \subset T(X) \times T(X)$ and Σ' is the set of term equations $NF(s) \approx NF(t)$ where $s \approx t \in \Sigma$, then $Mod_g\Sigma$ and $Mod_g\Sigma'$ are the same graph variety. Hence, if we want to find all $(x(yz))z$ graph varieties, then it is enough to find all graph varieties $Mod_g\Sigma^{'}$ such that $\Sigma^{'}$ is any subset of $T' \times T'$, where T' is the set of all normal form terms of $(x(yz))z$ terms. Since there are 64 normal form terms of $x(yz))z$ terms (i.e. add loop or reverse arc), there are 4096 $(x(yz))z$ term equations. So, there are 4096 $(x(yz))z$ graph varieties of the form $Mod_g\{s \approx t\}$ but some of them may be the same graph variety (i.e. there are some $(x(yz))z$ term equations $s \approx t$ and $s' \approx t'$ such that $Mod_g\{s \approx t\} = Mod_g\{s' \approx t'\}$). In this study we want to find all different $(x(yz))z$ graph varieties of the form $Mod_g\{s \approx t\}$. Clearly, for each $s \in T'$, $\mathcal{K}_0 = Mod_g\{s \approx s\}$ is the set of all graph algebras.

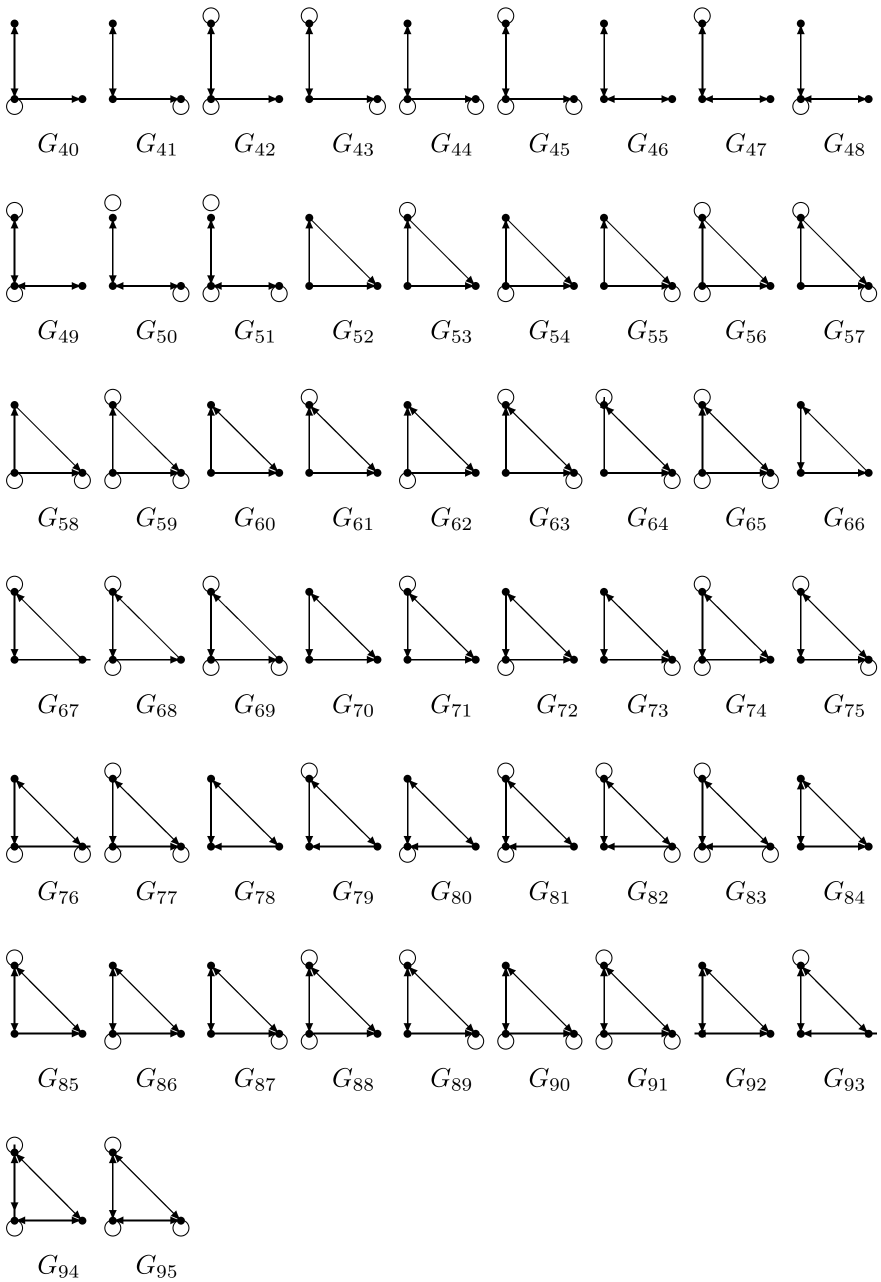
The following proposition was proved in [5].

Proposition 3.1. *Let s and t be non-trivial terms from $T(X)$ with variables $V(s) = V(t) = \{x_0, x_1, ..., x_n\}$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if the graph algebra $\underline{A(G)}$ has the following property:*

A mapping $h : V(s) \longrightarrow V$ is a homomorphism from $G(s)$ into G if and only if it is a homomorphism from $G(t)$ into G.

Proposition 3.1 gives a method to check whether a graph $G = (V, E)$ satisfies the term equation $s \approx t$. The following are all graphs with at most three vertices which satisfy at least one term equation $s \approx t$, $s, t \in T'$ and $s \neq t$.





Next, we will use these graphs to find all different $x(yz))z$ graph varieties and characterize the properties of those graph varieties in the following way:

Since $(x, y), (y, z), (x, z)$ belong to the graph $G(s)$ for every $(x(yz))z$ term s, for any graph $G = (V, E)$ which there are no vertices $a, b, c \in V$ such that

$(a,b),(b,c),(a,c) \in E$, we have the function $h : V(s) \to V$ is not a homomorphism from $G(s)$ into G for all h and for all $(x(yz))z$ terms s. By Proposition 3.1, we get G belongs to every $(x(yz))z$ graph variety. In the same way, for any complete graph $G' = (V', E')$ we have the function $h' : V(s) \to V'$ is a homomorphism from $G(s)$ into G' for all h' and for all $(x(yz))z$ terms s. Hence, G' belongs to every $(x(yz))z$ graph variety. Let $G = (V,E)$ with at most three vertices $a,b,c \in V$ such that $(a,b),(b,c),(a,c) \in E$ but G is not a complete graph and let $s^* = ((xx)(((yx)y)(((zx)y)z)))z$. We will partition the edges of $G(s^*)$ with respect to G in the following way. Let A_G be the set of edges $(u,v) \in E(s^*)$ such that $(h(u),h(v)) \in E$ for all onto functions $h : V(s^*) \to V$ which $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)) \in E$, B_G be the set of edges $(u,v) \in E(s^*)$ such that $(h(u),h(v)) \in E$ for some onto functions $h : V(s^*) \to V$ which $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)) \in E$ and $(h(u),h(v)) \notin E$ for some onto functions $h : V(s^*) \to V$ which $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)) \in E$, C_G be the set of edges $(u,v) \in E(s^*)$ such that $(h(u),h(v)) \notin E$ for all onto functions $h : V(s^*) \to V$ which $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)) \in E$. We see that $(x,y),(y,z),(x,z) \in A_G$ for all G. Then, we have the following lemma.

Lemma 3.2. *Let $G = (V,E)$ with at most three vertices $a,b,c \in V$ such that $(a,b),(b,c),(a,c) \in E$ but G is not a complete graph and $Mod_g\{s \approx t\}$ be an $(x(yz))z$ graph variety. Then, $G \notin Mod_g\{s \approx t\}$ if and only if (i) $E(s)$ contains only element of A_G and $E(t)$ contains some elements of $B_G \cup C_G$ or vise versa or (ii) $E(s)$ contains only element of $A_G \cup B_G$, $E(t)$ contains some elements of $B_G \cup C_G$ and there exists a function $h : V(s) \to V$ such that $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)) \in E$ which is a homomorphism from $G(s)$ into G but it is not a homomorphism from $G(t)$ into G or vise versa*

Proof. Suppose that $G \notin Mod_g\{s \approx t\}$. If $E(s)$ and $E(t)$ contain only element of A_G, then the function $h : V(s) \to V$ which $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)) \in E$ is a homomorphism from both $G(s)$ and $G(t)$ into G. Hence, the function $h' : V(s) \to V$ is a homomorphism from $G(s)$ into G if and only if it is a homomorphism from $G(t)$ into G. By Proposition 3.1, we get $G \in Mod_g\{s \approx t\}$. If both of $E(s)$ and $E(t)$ contain element of C_G, then the function $h : V(s) \to V$ which $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)) \in E$ is not a homomorphism from both $G(s)$ and $G(t)$ into G. Hence, the function $h' : V(s) \to V$ is not a homomorphism from both $G(s)$ and $G(t)$ into G. By Proposition 3.1, we get $G \in Mod_g\{s \approx t\}$. Suppose that $E(s)$ contains only element of $A_G \cup B_G$, $E(t)$ contains some elements of $B_G \cup C_G$ and there exists no a function $h : V(s) \to V$ such that $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)) \in E$ which is a homomorphism from $G(s)$ into G but it is not a homomorphism from $G(t)$ into G. Hence, the function $h' : V(s) \to V$ is a homomorphism from $G(s)$ into G if and only if it is a homomorphism from $G(t)$ into G. By Proposition 3.1, we get $G \in Mod_g\{s \approx t\}$.

Conversely, suppose s and t satisfying (i) or (ii). Suppose that $E(s)$ contains only element of A_G and $E(t)$ contains some elements of $B_G \cup C_G$. Let $(u,v) \in B_G \cup C_G$ and $(u,v) \in E(t)$. We have there exists a function $h : V(t) \to V$ which $(h(x),h(y)),(h(y),h(z)),(h(x),h(z)) \in E$ such that $(h(u),h(v)) \notin E$. Hence, h

is not a homomorphism $G(t)$ into G. By assumption, we get $(h(u'), h(v')) \in E$ for all $(u', v') \in E(s)$. Hence, h is a homomorphism from $G(s)$ into G. By Proposition 3.1, we get $G \notin Mod_g\{s \approx t\}$. Suppose that $E(s)$ contains only element of $A_G \cup B_G$, $E(t)$ contains some elements of $B_G \cup C_G$ and there exists a function $h : V(s) \to V$ such that $(h(x), h(y)), (h(y), h(z)), (h(x), h(z)) \in E$ which is a homomorphism from $G(s)$ into G but it is not a homomorphism from $G(t)$ into G. By Proposition 3.1, we get $G \notin Mod_g\{s \approx t\}$. □

From Lemma 3.1, we have some remarks.

Remark 3.3. *Let $\mathcal{K} = Mod_g\{s \approx t\}$. Then, we have*

(i) $G_4 \in \mathcal{K}$ *if and only if* $E(s) \subseteq A_{G_4}$, $E(t) \subseteq A_{G_4}$ *or* $E(s) \cap C_{G_4} \neq \phi$, $E(t) \cap C_{G_4} \neq \phi$,

(ii) $G_5 \in \mathcal{K}$ *if and only if* $E(s) \subseteq A_{G_5}$, $E(t) \subseteq A_{G_5}$ *or* $E(s) \cap C_{G_5} \neq \phi$, $E(t) \cap C_{G_5} \neq \phi$,

(iii) $G_6 \in \mathcal{K}$ *if and only if* $E(s) \cap (B_{G_6} \cup C_{G_6}) = E(t) \cap (B_{G_6} \cup C_{G_6})$ *or both of* $E(s)$ *and* $E(t)$ *contain either* (z, x) *or* $(y, x), (z, y)$,

(iv) $G_8 \in \mathcal{K}$ *if and only if* $E(s) \cap B_{G_8} = E(t) \cap B_{G_8}$.

Consider the graph at most two vertices, G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9. We see that the graphs G_1, G_2, G_3, G_7, G_9 belong to every $(x(yz))z$ graph variety. For convenience to classify the $(x(yz))z$ graph varieties, we will partition the set of all $(x(yz))z$ graph varieties in to at most sixteen sets which generated by G_4, G_5, G_6 and G_8 i.e. the set of graph varieties which do not contain all of G_4, G_5, G_6 and G_8, the set of graph varieties which contain only G_4, the set of graph varieties which contain only G_5, the set of graph varieties which contain only G_6, the set of graph varieties which contain only G_8, the set of graph varieties which contain only G_4 and G_5, and so on until the set of graph varieties which contain all of G_4, G_5, G_6 and G_8. We will denote these classes by $\mathcal{G}_i,\ i = 1, 2, 3, ..., 16$ respectively. By Lemma 3.1 and Remark 3.1, we have $\mathcal{G}_{11}, \mathcal{G}_{14}, \mathcal{G}_{15}$ are empty sets, since if G_6, G_8 belong to graph variety $\mathcal{K}$, then G_4, G_5 belong to graph variety $\mathcal{K}$.

Next we will use Lemma 3.1 to classify graph varieties in each $\mathcal{G}_i,\ i = 1, 2, 3, ...,$ 16. In this case we need the A_G, B_G and C_G of any graph which consider. We see that $(x, y), (y, z), (x, z) \in A_G$ for every G. We collect these properties of graphs which we need to consider as the following:

$$A_{G_4} = \{(y, y), (z, y), (z, z)\}, B_{G_4} = \phi, C_{G_4} = \{(x, x), (y, x), (z, x)\}.$$
$$A_{G_5} = \{(x, x), (y, x), (y, y)\}, B_{G_5} = \phi, C_{G_5} = \{(z, x), (z, y), (z, z)\}.$$
$$A_{G_6} = \{(x, x), (y, y), (z, z)\}, B_{G_6} = \{(y, x), (z, y)\}, C_{G_6} = \{(z, x)\}.$$
$$A_{G_8} = \{(y, x), (z, x), (z, y)\}, B_{G_8} = \{(x, x), (y, y), (z, z)\}, C_{G_8} = \phi.$$
$$A_{G_{52}} = \phi, B_{G_{52}} = \phi, C_{G_{52}} = \{(x, x), (y, y), (z, z), (y, x), (z, x), (z, y)\}.$$
$$A_{G_{60}} = \{(z, y)\}, B_{G_{60}} = \phi, C_{G_{60}} = \{(x, x), (y, y), (z, z), (y, x), (z, x)\}.$$
$$A_{G_{70}} = \{(z, x)\}, B_{G_{70}} = \phi, C_{G_{70}} = \{(x, x), (y, y), (z, z), (y, x), (z, y)\}.$$
$$A_{G_{78}} = \{(y, x)\}, B_{G_{78}} = \phi, C_{G_{78}} = \{(x, x), (y, y), (z, z), (z, x), (z, y)\}.$$

$A_{G_{84}} = \phi, B_{G_{84}} = \{(y,x),(z,x),(z,y)\}, C_{G_{84}} = \{(x,x),(y,y),(z,z)\}.$
$A_{G_{92}} = \{(y,x),(z,x),(z,y)\}, B_{G_{92}} = \phi, C_{G_{92}} = \{(x,x),(y,y),(z,z)\}.$

Since $\mathcal{G}_1$ is the set of all graph varieties which do not contain all of G_4, G_5, G_6, G_8, we see that each element of $\mathcal{G}_1$ contain at most these graphs G_1, G_2, G_3, G_7, G_9, G_{10}, G_{16}, G_{24}, G_{30}, G_{38}, G_{46}, G_{51}, G_{52}, G_{60}, G_{66}, G_{70}, G_{78}, G_{84}, G_{92}, G_{95}. We have G_1, G_2, G_3, G_7, G_9, G_{10}, G_{16}, G_{24}, G_{30}, G_{38}, G_{46}, G_{51}, G_{66}, G_{95} belong to all graph varieties in $\mathcal{G}_1$. Hence, the graph varieties in $\mathcal{G}_1$ generated by G_{52}, G_{60}, G_{70}, G_{78}, G_{84}, G_{92} are given as the following theorem:

Theorem 3.4. *There are only seven graph varieties in $\mathcal{G}_1$.*

Proof. Since elements of $\mathcal{G}_1$ generated by G_{52}, G_{60}, G_{70}, G_{78}, G_{84}, G_{92}, we see that $\mathcal{G}_1$ has at most sixty four graph varieties. From the properties of G_4, G_5, G_6, G_8, G_{52}, G_{60}, G_{70}, G_{78}, G_{84}, G_{92}, by Lemma 3.1 and the properties of $\mathcal{G}_1$, we have the following:

Consider for G_{52}, by Lemma 3.1 we see that $G_{52} \notin \mathcal{K} = Mod_g\{s \approx t\}$ if $s = (x(yz))z$, $E(t) \cap C_{G_4} \neq \phi$, $E(t) \cap C_{G_5} \neq \phi$, $E(t) \cap (B_{G_6} \cup C_{G_6}) \neq \phi$, $E(t) \cap B_{G_8} \neq \phi$ and $E(t) \cap C_{G_{52}} \neq \phi$. Since $E(t) \cap B_{G_8} \neq \phi$, we have $\mathcal{K}$ does not contain all of G_{52} $G_{60}, G_{70}, G_{78}, G_{84}, G_{92}$. Hence, the graph variety in $\mathcal{G}_1$ which does not contain all of G_{52}, G_{60}, G_{70}, G_{78}, G_{84}, G_{92} is $\mathcal{K}_1 = Mod_g\{(x(yz))z \approx ((xx)(y(zx)))z\}$.

For G_{60}, we see that $G_{60} \notin \mathcal{K} = Mod_g\{s \approx t\}$ different from $\mathcal{K}_1$ if $s = (x(y(zy)))z$, $E(t) \cap C_{G_4} \neq \phi$, $E(t) \subseteq A_{G_5}$, $E(t) \cap B_{G_8} \neq \phi$ and $E(t) \cap C_{G_{60}} \neq \phi$ which there is one graph variety. The graph variety in $\mathcal{G}_1$ which does not contain only G_{60}, G_{84}, G_{92} is $\mathcal{K}_2 = Mod_g\{(x(y(zy)))z \approx ((xx)(yz))z\}$.

For G_{70}, we see that $G_{70} \notin \mathcal{K} = Mod_g\{s \approx t\}$ different from $\mathcal{K}_1$, $\mathcal{K}_2$ if $s = (x(y(zx)))z$, $E(t) \subseteq A_{G_4}$, $E(t) \subseteq A_{G_5}$, $E(t) \cap B_{G_8} \neq \phi$ and $E(t) \cap C_{G_{70}} \neq \phi$ which there is one graph variety. The graph variety in $\mathcal{G}_1$ which does not contain only G_{70}, G_{84}, G_{92} is $\mathcal{K}_3 = Mod_g\{(x(y(zx)))z \approx (x((yy)z))z\}$.

For G_{78}, we see that $G_{78} \notin \mathcal{K} = Mod_g\{s \approx t\}$ different from $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ if $s = (x((yx)z))z$, $E(t) \subseteq A_{G_4}$, $E(t) \cap C_{G_5} \neq \phi$, $E(t) \cap B_{G_8} \neq \phi$ and $E(t) \cap C_{G_{78}} \neq \phi$ which there is one graph variety. The graph variety in $\mathcal{G}_1$ which does not contain only G_{78}, G_{84}, G_{92} is $\mathcal{K}_4 = Mod_g\{(x((yx)z))z \approx (x(y(zz)))z\}$.

For G_{84}, we see that $G_{84} \notin \mathcal{K} = Mod_g\{s \approx t\}$ different from $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ if $s = (x((yx)(zy)))z$ or $s = (x((yx)(zx)))z$ or $s = (x(y((zx)y)))z$, $E(t) \subseteq A_{G_4}$, $E(t) \subseteq A_{G_5}$ and $E(t) \cap C_{G_{84}} \neq \phi$ which there is one graph variety. The graph variety in $\mathcal{G}_1$ which does not contain only G_{84}, G_{92} is $\mathcal{K}_5 = Mod_g\{(x((yx)(zy)))z \approx (x((yy)z))z\}$.

For G_{92}, we see that $G_{92} \notin \mathcal{K} = Mod_g\{s \approx t\}$ different from $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_5$ if $s = (x((yx)((zx)y)))z)$, $E(t) \subseteq A_{G_4}$, $E(t) \subseteq A_{G_5}$ and $E(t) \cap C_{G_{92}} \neq \phi$ which there is one graph variety. The graph variety in $\mathcal{G}_1$ which does not contain only G_{92} is $\mathcal{K}_6 = Mod_g\{(x((yx)((zx)y)))z \approx (x((yy)z))z\}$.

The graph variety which contains all G_{52}, G_{60}, G_{70}, G_{78}, G_{84}, G_{92} is $\mathcal{K}_7 = Mod_g\{((xx)((yx)z))z \approx (x((yy)(zy)))z\}$. By the properties of G_4, G_5, G_6, G_8, G_{52}, G_{60}, G_{70}, G_{78}, G_{84}, G_{92}, by Lemma 3.1 and the properties of $\mathcal{G}_1$, we have there are no other graph varieties in $\mathcal{G}_1$. Hence, there are only seven graph varieties in $\mathcal{G}_1$. □

Next, we will use the Proposition 3.1 to characterize the properties of the graphs in each graph variety in $\mathcal{G}_1$.

Theorem 3.5. *Let $G = (V, E)$ be a graph and $\mathcal{K}_1 = Mod_g\{(x(yz))z \approx ((xx)(y(zx)))z\}$. Then, $G \in \mathcal{K}$ if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(a, a), (c, a) \in E$.*

Proof. Let $G = (V, E)$ be a graph. Suppose that $G \in \mathcal{K}_1$ and for any $a, b, c \in V$, $(a, b), (b, c), (a, c) \in E$. Let $s = (x(yz))z$, $t = ((xx)(y(zx)))z$ and let $h : V(s) \to V$ be a function such that $h(x) = a$, $h(y) = b$ and $h(z) = c$. We see that h is a homomorphism from $G(s)$ into G. By Proposition 3.1, we have h is a homomorphism from $G(t)$ into G. Since $(x, x) \in E(t)$ and $(z, x) \in E(t)$, we have $(h(x), h(x)) = (a, a) \in E$ and $(h(z), h(x)) = (c, a) \in E$.

Conversely, suppose that $G = (V, E)$ is a graph which has property that, for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(a, a), (c, a) \in E$. Let $s = (x(yz))z$, $t = ((xx)(y(zx)))z$ and let $h : V(s) \to V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G. Since $(x, y), (y, z), (x, z) \in E(s)$, we have $(h(x), h(y)), (h(y), h(z)), (h(x), h(z)) \in E$. By assumption, we get $(h(x), h(x)), (h(z), h(x)) \in E$. Hence, h is a homomorphism from $G(t))$ into G. Clearly, if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Then, by Proposition 3.1 we get $\underline{A(G)}$ satisfies $s \approx t$. □

Theorem 3.6. *Let $G = (V, E)$ be a graph and $\mathcal{K}_2 = Mod_g\{(x(y(zy)))z \approx ((xx)(yz))z\}$. Then, $G \in \mathcal{K}_2$ if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(c, b) \in E$ if and only if $(a, a) \in E$.*

Proof. Let $G = (V, E)$ be a graph. Suppose that $G \in \mathcal{K}_2$ and for any $a, b, c \in V$ suppose that $(a, b), (b, c), (a, c), (c, b) \in E$. Let $s = (x(y(zy)))z$, $t = ((xx)(yz))z\}$ and let $h : V(s) \to V$ be a function such that $h(x) = a$, $h(y) = b$ and $h(z) = c$. We see that h is a homomorphism from $G(s)$ into G. By Proposition 3.1, we have h is a homomorphism from $G(t)$ into G. Since $(x, x) \in E(t)$, we have $(h(x), h(x)) = (a, a) \in E$. For any $a, b, c \in V$ suppose that $(a, b), (b, c), (a, c), (a, a) \in E$. Let $s = (x(y(zy)))z$, $t = ((xx)(yz))z\}$ and let $h : V(s) \to V$ be a function such that $h(x) = a$, $h(y) = b$ and $h(z) = c$. We see that h is a homomorphism from $G(t)$ into G. By Proposition 3.1, we have h is a homomorphism from $G(s)$ into G. Since $(z, y) \in E(t)$, we have $(h(z), h(y)) = (c, b) \in E$.

Conversely, suppose that $G = (V, E)$ is a graph which has property that, for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(c, b) \in E$ if and only if $(a, a) \in E$. Let $s = (x(y(zy)))z$, $t = ((xx)(yz))z$ and let $h : V(s) \to V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G. Since $(x, y), (y, z), (x, z), (z, y) \in E(s)$, we have $(h(x), h(y)), (h(y), h(z)), (h(x), h(z)), (h(z), h(y)) \in E$. By assumption, we get $(h(x), h(x)) \in E$. Hence, h is a homomorphism from $G(t))$ into G. In the same way, we can prove that if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Then, by Proposition 3.1 we get $\underline{A(G)}$ satisfies $s \approx t$. $\square$

Theorem 3.7. *Let $G = (V, E)$ be a graph and $\mathcal{K}_3 = Mod_g\{(x(y(zx)))z \approx (x((yy)z))z\}$. Then $G \in \mathcal{K}_3$ if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(c, a) \in E$ if and only if $(b, b) \in E$.*

Proof. The proof is similar to the proof of Theorem 3.6. $\square$

Theorem 3.8. *Let $G = (V, E)$ be a graph and $\mathcal{K}_4 = Mod_g\{(x(x((yx)z))z \approx (x(y(zz)))z\}$. Then $G \in \mathcal{K}_4$ if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(b, a) \in E$ if and only if $(c, c) \in E$.*

Proof. The proof is similar to the proof of Theorem 3.6. $\square$

Theorem 3.9. *Let $G = (V, E)$ be a graph and $\mathcal{K}_5 = Mod_g\{(x((yx)(zy)))z \approx (x((yy)z))z\}$. Then $G \in \mathcal{K}_5$ if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(b, a), (c, b) \in E$ if and only if $(b, b) \in E$.*

Proof. Let $G = (V, E)$ be a graph. Suppose that $G \in \mathcal{K}_5$. For any $a, b, c \in V$, suppose that $(a, b), (b, c), (a, c), (b, a), (c, b) \in E$. Let $s = (x((yx)(zy)))z$, $t = (x((yy)z))z$ and let $h : V(s) \to V$ be a function such that $h(x) = a$, $h(y) = b$ and $h(z) = c$. We see that h is a homomorphism from $G(s)$ into G. By Proposition 3.1, we have h is a homomorphism from $G(t)$ into G. Since $(y, y) \in E(t)$, we have $(h(y), h(y)) = (b, b) \in E$. In the same way, we can prove that if $(a, b), (b, c), (a, c), (b, b) \in E$, then $(b, a), (c, b) \in E$.

Conversely, suppose that $G = (V, E)$ be a graph which has property that, for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(b, a), (c, b) \in E$ if and only if $(b, b) \in E$. Let $s = (x((yx)(zy)))z$, $t = (x((yy)z))z$ and let $h : V(s) \to V$ be a function. Suppose that h is a homomorphism from $G(s)$ into G. Since $(x, y), (y, z), (x, z), (y, x), (z, y) \in E(s)$, we have $(h(x), h(y)), (h(y), h(z)), (h(x), h(z)), (h(y), h(x)), (h(z), h(y)) \in E$. By assumption, we get $(h(y), h(y)) \in E$. Hence, h is a homomorphism from $G(t))$ into G. In the same way, we can prove that if h is a homomorphism from $G(t)$ into G, then it is a homomorphism from $G(s)$ into G. Then, by Proposition 3.1 we get $\underline{A(G)}$ satisfies $s \approx t$. $\square$

Theorem 3.10. *Let $G = (V, E)$ be a graph and $\mathcal{K}_6 = Mod_g\{(x((yx)((zx)y)))z \approx (x((yy)z))z\}$. Then $G \in \mathcal{K}_6$ if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(b, a), (c, a), (c, b) \in E$ if and only if $(b, b) \in E$.*

Proof. The proof is similar to the proof of Theorem 3.9. $\square$

Theorem 3.11. *Let $G = (V, E)$ be a graph and $\mathcal{K}_7 = Mod_g\{((xx)((yx)z))z \approx (x((yy)(zy)))z\}$. Then $G \in \mathcal{K}_7$ if and only if for any $a, b, c \in V$ if $(a, b), (b, c), (a, c) \in E$, then $(a, a), (b, a) \in E$ if and only if $(b, b), (c, b) \in E$.*

Proof. The proof is similar to that of Theorem 3.9. $\square$

Consider the same as $\mathcal{G}_1$, we have the graph varieties in $\mathcal{G}_2$ are generated by G_{52}, G_{55}, G_{60}, G_{70}, G_{78}, G_{80}, G_{84}, G_{92}. The graph varieties in $\mathcal{G}_3$ are generated by G_{52}, G_{54}, G_{60}, G_{62}, G_{70}, G_{78}, G_{82}, G_{84}, G_{92}. The graph varieties in $\mathcal{G}_4$ are generated by G_{52}, G_{59}, G_{60}, G_{65}, G_{70}, G_{77}, G_{78}, G_{83}, G_{84}, G_{91}, G_{92}. The graph varieties in $\mathcal{G}_5$ are generated by G_{52}, G_{60}, G_{70}, G_{78}, G_{84}, G_{85}, G_{92}, G_{93}, G_{94}. The graph varieties in $\mathcal{G}_6$ are generated by G_{53}, G_{54}, G_{55}, G_{60}, G_{62}, G_{64}, G_{70}, G_{72}, G_{75}, G_{78}, G_{80}, G_{82}, G_{84}, G_{92}. The graph varieties in $\mathcal{G}_7$ are generated by G_{52}, G_{55}, G_{57}, G_{59}, G_{60}, G_{64}, G_{65}, G_{70}, G_{77}, G_{78}, G_{80}, G_{83}, G_{84}, G_{91}, G_{92}. The graph varieties in $\mathcal{G}_8$ are generated by G_{52}, G_{55}, G_{60}, G_{61}, G_{64}, G_{70}, G_{73}, G_{78}, G_{80}, G_{84}, G_{85}, G_{87}, G_{89}, G_{92}, G_{93}, G_{94}. The graph varieties in $\mathcal{G}_9$ are generated by G_{52}, G_{54}, G_{59}, G_{60}, G_{62}, G_{65}, G_{70}, G_{77}, G_{78}, G_{82}, G_{83}, G_{84}, G_{91}, G_{92}. The graph varieties in $\mathcal{G}_{10}$ are generated by G_{52}, G_{54}, G_{60}, G_{62}, G_{70}, G_{71}, G_{78}, G_{79}, G_{82}, G_{84}, G_{85}, G_{86}, G_{88}, G_{92}, G_{93}, G_{94}. The graph varieties in $\mathcal{G}_{12}$ are generated by G_{52}, G_{53}, G_{54}, G_{55}, G_{56}, G_{57}, G_{58}, G_{59}, G_{60}, G_{62}, G_{64}, G_{65}, G_{70}, G_{72}, G_{75}, G_{77}, G_{78}, G_{80}, G_{82}, G_{83}, G_{84}, G_{91}, G_{92}. The graph varieties in $\mathcal{G}_{13}$ are generated by G_{52}, G_{53}, G_{54}, G_{55}, G_{60}, G_{61}, G_{62}, G_{64}, G_{70}, G_{71}, G_{72}, G_{73}, G_{75}, G_{78}, G_{79}, G_{80}, G_{82}, G_{84}, G_{85} G_{86}, G_{87}, G_{88}, G_{89}, G_{92}, G_{93}, G_{94}. The graph varieties in $\mathcal{G}_{16}$ are generated by G_{52}, G_{53}, G_{54}, G_{55}, G_{56}, G_{57}, G_{58}, G_{59}, G_{60}, G_{61}, G_{62}, G_{63}, G_{64}, G_{65},, G_{70}, G_{71}, G_{72}, G_{73}, G_{74}, G_{75}, G_{76}, G_{77}, G_{78}, G_{79}, G_{80}, G_{81}, G_{82}, G_{83}, G_{84}, G_{85}, G_{86}, G_{87}, G_{88}, G_{89}, G_{90}, G_{91}, G_{92}, G_{93}, G_{94}. By the same method that use in $\mathcal{G}_1$, we get all graph varieties in other classes and the properties of graphs as the following table:

Table. Other $(x(yz))z$ graph varieties and the properties of graphs.

Graph variety	Properties of graphs, for any a, $b, c \in V$ if $(a,b),(b,c),(a,c) \in E$,
$\mathcal{K}_8 = Mod_g\{(x((yx)z))z \approx (x((yy)(zy)))z\}$	then $(b,a) \in E$ iff $(b,b),(c,b) \in E$.
$\mathcal{K}_9 = Mod_g\{(x(y(zy)))z \approx (x((yy)z))z\}$	then $(c,b) \in E$ iff $(b,b) \in E$.
$\mathcal{K}_{10} = Mod_g\{(x(y(zx)))z \approx ((xx)(yz))z\}$	then $(c,a) \in E$ iff $(a,a) \in E$.
$\mathcal{K}_{11} = Mod_g\{(x((yx)z))z \approx ((xx)(y(zx)))z\}$	then $(b,a) \in E$ iff $(a,a),(c,a) \in E$.
$\mathcal{K}_{12} = Mod_g\{(x((yx)(zz)))z \approx ((xx)(yz))z\}$	then $(b,a),(c,c) \in E$ iff $(a,a) \in E$.
$\mathcal{K}_{13} = Mod_g\{(x((yx)(zy)))z \approx ((xx)(yz))z\}$	$(b,a),(c,b) \in E$ iff $(a,a) \in E$.
$\mathcal{K}_{14} = Mod_g\{(x((yx)((zx)y)))z \approx ((xx)(yz))z\}$	$(b,a),(c,a),(c,b) \in E$ iff $(a,a) \in E$.
$\mathcal{K}_{15} = Mod_g\{((xx)(y(zy)))z \approx (x(((yx)y)z))z\}$	$(a,a),(c,b) \in E$ iff $(b,a),(b,b) \in E$.
$\mathcal{K}_{16} = Mod_g\{(x(yz))z \approx ((xx)((yx)z))z\}$	then $(a,a),(b,a) \in E$.

Table. (continue).

Graph variety	Properties of graphs, for any a, $b, c \in V$ if $(a,b),(b,c),(a,c) \in E$,
$\mathcal{K}_{17} = Mod_g\{(x(y(zy)))z \approx (x(y((zx)z)))z\}$	then $(c,b) \in E$ iff $(c,a),(c,c) \in E$.
$\mathcal{K}_{18} = Mod_g\{((xx)(y(zy)))z \approx (x(y(zz)))z\}$	then $(a,a),(c,b) \in E$ iff $(c,c) \in E$.
$\mathcal{K}_{19} = Mod_g\{(x(y(zx)))z \approx (x(y(zz)))z\}$	then $(c,a) \in E$ iff $(c,c) \in E$.
$\mathcal{K}_{20} = Mod_g\{(x((yx)z))z \approx (x((yy)z))z\}$	then $(b,a) \in E$ iff $(b,b) \in E$.
$\mathcal{K}_{21} = Mod_g\{(x((yx)(zy)))z \approx (x(y(zz)))z\}$	then $(b,a),(c,b) \in E$ iff $(c,c) \in E$.
$\mathcal{K}_{22} = Mod_g\{(x((yx)((zx)y)))z \approx (x(y(zz)))z\}$	then $(b,a),(c,a),(c,b) \in E$ iff $(c,c) \in E$.
$\mathcal{K}_{23} = Mod_g\{(x((yx)(zz)))z \approx (x((yy)(zx)))z\}$	then $(b,a),(c,c) \in E$ iff $(b,b),(c,a) \in E$.
$\mathcal{K}_{24} = Mod_g\{(x(yz))z \approx ((xx)(y(zz)))z\}$	then $(a,a),(c,c) \in E$.
$\mathcal{K}_{25} = Mod_g\{((xx)(yz))z \approx (x((yy)(zz)))z\}$	then $(a,a) \in E$ iff $(b,b),(c,c) \in E$.
$\mathcal{K}_{26} = Mod_g\{(x(yz))z \approx (x(y(zx)))z\}$	then $(c,a) \in E$.
$\mathcal{K}_{27} = Mod_g\{(x(yz))z \approx (x((yx)(zy)))z\}$	then $(b,a),(c,b) \in E$.
$\mathcal{K}_{28} = Mod_g\{(x(y(zy)))z \approx (x((yx)z))z\}$	then $(c,b) \in E$ iff $(b,a) \in E$.
$\mathcal{K}_{29} = Mod_g\{(x((yy)z))z \approx (x((yy)(zx)))z\}$	and $(b,b) \in E$, then $(c,a) \in E$.
$\mathcal{K}_{30} = Mod_g\{((xx)(yz))z \approx (x((yx)z))z\}$	then $(a,a) \in E$ iff $(b,a) \in E$.
$\mathcal{K}_{31} = Mod_g\{((xx)(yz))z \approx (x(((yx)y)z))z\}$	then $(a,a) \in E$ iff $(b,a),(b,b) \in E$.
$\mathcal{K}_{32} = Mod_g\{(x(y(zz))z \approx (x(y(zy)))z\}$	then $(c,c) \in E$ iff $(c,b) \in E$.
$\mathcal{K}_{33} = Mod_g\{(x(y(zz))z \approx (x((yy)(zy)))z\}$	then $(c,c) \in E$ iff $(b,b),(c,b) \in E$.
$\mathcal{K}_{34} = Mod_g\{(x(y(zy))z \approx (x((yy)(zz)))z\}$	then $(c,b) \in E$ iff $(b,b),(c,c) \in E$.
$\mathcal{K}_{35} = Mod_g\{((xx)(y(zy)))z \approx (x(y(zx))z\}$	then $(a,a),(c,b) \in E$ iff $(c,a) \in E$.
$\mathcal{K}_{36} = Mod_g\{((xx)(y(zy))z \approx (x((yy)(zx)))z\}$	then $(a,a),(c,b) \in E$ iff $(b,b),(c,a) \in E$.

Table. (continue).

Graph variety	Properties of graphs, for any a, $b, c \in V$ if $(a,b),(b,c),(a,c) \in E$,
$\mathcal{K}_{37} = Mod_g\{((xx)(y(zy))z \approx (x((yx)(zz)))z\}$	then $(a,a),(c,b) \in E$ iff $(b,a),(c,c) \in E$.
$\mathcal{K}_{38} = Mod_g\{((xx)(y(zy))z \approx (x(y((zx)z)))z\}$	then $(a,a),(c,b) \in E$ iff $(c,a),(c,c) \in E$.
$\mathcal{K}_{39} = Mod_g\{((xx)(y(zy))z \approx (x((yx)(zx)))z\}$	then $(a,a),(c,b) \in E$ iff $(b,a),(c,a) \in E$.
$\mathcal{K}_{40} = Mod_g\{((xx)(y(zy))z \approx (x((yx)((zx)y)))z\}$	and $(c,b) \in E$, then $(a,a) \in E$ iff $(b,a),(c,a) \in E$.
$\mathcal{K}_{41} = Mod_g\{(x(y(zx))z \approx (x((yx)(zz)))z\}$	then $(c,a) \in E$ iff $(b,a),(c,c) \in E$.
$\mathcal{K}_{42} = Mod_g\{(x(y(zx))z\} \approx ((xx)(y(zy)))z\}$	then $(c,a) \in E$ iff $(a,a),(c,b) \in E$.
$\mathcal{K}_{43} = Mod_g\{(x(y(zx)))z\} \approx ((xx)(y(zz)))z\}$	then $(c,a) \in E$ iff $(a,a),(c,c) \in E$.
$\mathcal{K}_{44} = Mod_g\{(x((yy)(zx)))z \approx (x((yx)(zz)))z\}$	then $(b,b),(c,a) \in E$ iff $(b,a),(c,c) \in E$.
$\mathcal{K}_{45} = Mod_g\{(x((yy)(zx)))z \approx ((xx)(y(zz)))z\}$	then $(b,b),(c,a) \in E$ iff $(a,a),(c,c) \in E$.
$\mathcal{K}_{46} = Mod_g\{((xx)(y(zz)))z \approx (x((yx)(zy)))z\}$	then $(a,a),(c,c) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{47} = Mod_g\{((xx)(y(zz)))z \approx (x((yx)(zz))z\}$	and $(c,c) \in E$, then $(a,a) \in E$ iff $(b,a) \in E$.
$\mathcal{K}_{48} = Mod_g\{(x(y((zx)z)))z \approx ((xx)(y(zy))z\}$	then $(c,a) \in E$ iff $(a,a),(c,b) \in E$.
$\mathcal{K}_{49} = Mod_g\{(x((yx)z))z \approx ((xx)((yy)z))z\}$	then $(b,a) \in E$ iff $(a,a),(b,b) \in E$.
$\mathcal{K}_{50} = Mod_g\{(x((yx)(zz)))z \approx (x((yx)(zy)))z\}$	and $(b,a) \in E$, then $(c,c) \in E$ iff $(c,b) \in E$.
$\mathcal{K}_{51} = Mod_g\{(x((yx)(zz)))z \approx ((xx)(y((zx)y)))z\}$	then $(b,a),(c,c) \in E$ iff $(a,a),(c,a),(c,b) \in E$.
$\mathcal{K}_{52} = Mod_g\{(x((yx)(zz)))z \approx (x((yx)((zx)y)))z\}$	and $(b,a) \in E$, then $(c,c) \in E$ iff $(c,a),(c,b) \in E$.
$\mathcal{K}_{53} = Mod_g\{(x((yx)(zy))z \approx ((xx)((yy)(zz)))z\}$	then $(b,a),(c,b) \in E$ iff $(a,a),(b,b),(c,c) \in E$.
$\mathcal{K}_{54} = Mod_g\{(x((yx)((zx)y))z \approx ((xx)((yy)(zz)))z\}$	then $(b,a),(c,a),(c,b) \in E$ iff $(a,a),(b,b),(c,c) \in E$.
$\mathcal{K}_{55} = Mod_g\{(x(((yx)y)(zx)))z \approx ((xx)(y((zy)z)))z\}$	then $(b,a),(b,b),(c,a) \in E$ iff $(a,a),(c,b),(c,c) \in E$.
$\mathcal{K}_{56} = Mod_g\{(x(yz))z \approx (x(y(zz)))z\}$	then $(c,c) \in E$.

Table. (continue).

Graph variety	Properties of graphs, for any a, $b, c \in V$ if $(a,b),(b,c),(a,c) \in E$,
$\mathcal{K}_{57} = Mod_g\{(x(yz))z \approx (x((yy)(zz)))z\}$	then $(b,b),(c,c) \in E$.
$\mathcal{K}_{58} = Mod_g\{(x(y(zz)))z \approx (x((yy)z))z\}$	then $(c,c) \in E$ iff $(b,b) \in E$.
$\mathcal{K}_{59} = Mod_g\{(x((yx)z)))z \approx (x((yx)(zz)))z\}$	and $(b,a) \in E$, then $(c,c) \in E$.
$\mathcal{K}_{60} = Mod_g\{(x((yx)z)))z \approx (x(((yx)y)(zz)))z\}$	and $(b,a) \in E$, then $(b,b),(c,c) \in E$.
$\mathcal{K}_{61} = Mod_g\{(x((yx)(zz))))z \approx ((xx)((yx)z))z\}$	and $(b,a) \in E$, then $(c,c) \in E$ iff $(a,a) \in E$.
$\mathcal{K}_{62} = Mod_g\{(x(yz))z \approx (x(y(zy)))z\}$	then $(c,b) \in E$.
$\mathcal{K}_{63} = Mod_g\{(x(y(zx))z \approx (x((yx)z))z\}$	then $(c,a) \in E$ iff $(b,a) \in E$.
$\mathcal{K}_{64} = Mod_g\{(x((yx)z)z \approx (x((yx)(zy)))z\}$	and $(b,a) \in E$, then $(c,b) \in E$.
$\mathcal{K}_{65} = Mod_g\{((xx)((yx)z)z \approx ((xx)(y(zy)))z\}$	and $(a,a) \in E$, then $(b,a) \in E$ iff $(c,b) \in E$.
$\mathcal{K}_{66} = Mod_g\{(x((yy)(zx))))z \approx (x(((yx)y)z)))z\}$	and $(b,b) \in E$, then $(c,a) \in E$ iff $(b,a) \in E$.
$\mathcal{K}_{67} = Mod_g\{(x((yy)(zx))))z \approx (x(((yx)y)z)))z\}$	and $(b,b) \in E$, then $(c,a) \in E$ iff $(b,a) \in E$.
$\mathcal{K}_{68} = Mod_g\{((xx)((yx)(zz))))z \approx ((xx)(y((zy)z))))z\}$	and $(a,a) \in E$, then $(b,a),(c,c) \in E$ iff $(c,b) \in E$.
$\mathcal{K}_{69} = Mod_g\{(x(yz))z \approx ((xx)(yz))z\}$	then $(a,a) \in E$.
$\mathcal{K}_{70} = Mod_g\{(x(yz))z \approx ((xx)((yy)z))z\}$	then $(a,a),(b,b) \in E$.
$\mathcal{K}_{71} = Mod_g\{((xx)(yz))z \approx (x((yy)z))z\}$	then $(a,a) \in E$ iff $(b,b) \in E$.
$\mathcal{K}_{72} = Mod_g\{(x(y(zy))z \approx ((xx)(y(zy)))z\}$	and $(c,b) \in E$, then $(a,a) \in E$.
$\mathcal{K}_{73} = Mod_g\{(x(y(zy)))z \approx ((xx)(y((zy)z)))z\}$	and $(c,b) \in E$, then $(a,a),(c,c) \in E$.
$\mathcal{K}_{74} = Mod_g\{((xx)(y(zy)))z \approx (x(y((zy)z)))z\}$	and $(c,b) \in E$, then $(a,a) \in E$ iff $(c,c) \in E$.
$\mathcal{K}_{75} = Mod_g\{(x(yz))z \approx (x((yx)z))z\}$	then $(b,a) \in E$.
$\mathcal{K}_{76} = Mod_g\{(x(y(zy))z \approx (x(y(zx)))z\}$	then $(c,b) \in E$ iff $(c,a) \in E$.

Table. (continue).

Graph variety	Properties of graphs, for any a, $b, c \in V$ if $(a,b),(b,c),(a,c) \in E$,
$\mathcal{K}_{77} = Mod_g\{(x(y(zy)))z \approx (x((yx)(zy))z\}$	and $(c,b) \in E$, then $(b,a) \in E$.
$\mathcal{K}_{78} = Mod_g\{(x(y(zz)))z \approx (x((yx)(zz))z\}$	and $(c,c) \in E$, then $(b,a) \in E$.
$\mathcal{K}_{79} = Mod_g\{(x((yy)z))z \approx (x((yy)(zy))z\}$	and $(b,b) \in E$, then $(c,b) \in E$.
$\mathcal{K}_{80} = Mod_g\{(x(yz))z \approx (x((yy)z))z\}$	then $(b,b) \in E$.
$\mathcal{K}_{81} = Mod_g\{((xx)(yz))z \approx ((xx)((yy)z))z\}$	and $(a,a) \in E$, then $(b,b) \in E$.
$\mathcal{K}_{82} = Mod_g\{(x(y(zz)))z \approx (x((yy)(zz)))z\}$	and $(c,c) \in E$, then $(b,b) \in E$.
$\mathcal{K}_{83} = Mod_g\{((xx)(y(zz)))z \approx ((xx)((yy)(zz)))z\}$	and $(a,a),(c,c) \in E$, then $(b,b) \in E$.
$\mathcal{K}_{84} = Mod_g\{(x(y(zy)))z \approx (x((yy)(zy)))z\}$	and $(c,b) \in E$, then $(b,b) \in E$.
$\mathcal{K}_{85} = Mod_g\{((xx)(y(zy)))z \approx ((xx)((yy)(zy)))z\}$	and $(a,a),(c,b) \in E$, then $(b,b) \in E$.
$\mathcal{K}_{86} = Mod_g\{(x(y(zx)))z \approx ((xx)(y(zx)))z\}$	and $(c,a) \in E$, then $(a,a) \in E$.
$\mathcal{K}_{87} = Mod_g\{(x(y(zx)))z \approx (x((yy)(zx)))z\}$	and $(c,a) \in E$, then $(b,b) \in E$.
$\mathcal{K}_{88} = Mod_g\{(x(y(zx)))z \approx ((xx)((yy)(zx)))z\}$	and $(c,a) \in E$, then $(a,a),(b,b) \in E$.
$\mathcal{K}_{89} = Mod_g\{(x(y(zx)))z \approx (x((yx)(zy)))z\}$	then $(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{90} = Mod_g\{(x((yy)(zx)))z \approx ((xx)(y(zx)))z\}$	and $(c,a) \in E$, then $(b,b) \in E$ iff $(a,a) \in E$.
$\mathcal{K}_{91} = Mod_g\{(x((yy)(zx)))z \approx ((xx)((yy)(zx)))z\}$	and $(b,b),(c,a) \in E$, then $(a,a) \in E$.
$\mathcal{K}_{92} = Mod_g\{(x((yy)(zx)))z \approx ((xx)((yx)(zy)))z\}$	then $(b,b),(c,a) \in E$ iff $(a,a),(b,a),(c,b) \in E$.
$\mathcal{K}_{93} = Mod_g\{(x((yy)(zx)))z \approx (x((yx)(zy)))z\}$	then $(b,b),(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{94} = Mod_g\{((xx)(y(zx)))z \approx ((xx)((yy)(zx)))z\}$	and $(a,a),(c,a) \in E$, then $(b,b) \in E$.
$\mathcal{K}_{95} = Mod_g\{((xx)(y(zx)))z \approx (x((yx)(zy)))z\}$	then $(a,a),(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{96} = Mod_g\{((xx)(y(zx)))z \approx (x(((yx)y)(zy)))z\}$	then $(a,a),(c,a) \in E$ iff $(b,a),(b,b),(c,b) \in E$.

Table. (continue).

Graph variety	Properties of graphs, for any a, $b, c \in V$ if $(a,b),(b,c),(a,c) \in E$,
$\mathcal{K}_{97} = Mod_g\{((xx)((yy)(zx)))z \approx (x((yx)(zy)))z\}$	then $(a,a),(b,b),(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{98} = Mod_g\{((xx)((yy)(zx)))z \approx (x((yx)((zy)z)))z\}$	then $(a,a),(b,b),(c,a) \in E$ iff $(b,a),(c,b),(c,c) \in E$.
$\mathcal{K}_{99} = Mod_g\{(x((yx)z))z \approx ((xx)((yx)z))z\}$	and $(b,a) \in E$, then $(a,a) \in E$.
$\mathcal{K}_{100} = Mod_g\{(x((yx)(zz)))z \approx ((xx)((yx)(zz)))z\}$	and $(b,a),(c,c) \in E$, then $(a,a) \in E$.
$\mathcal{K}_{101} = Mod_g\{(x((yx)(zy)))z \approx ((xx)((yx)(zy)))z\}$	and $(b,a),(c,b) \in E$, then $(a,a) \in E$.
$\mathcal{K}_{102} = Mod_g\{(x((yx)(zy)))z \approx ((xx)(y(zx)))z\}$	then $(b,a),(c,b) \in E$ iff $(a,a),(c,a) \in E$.
$\mathcal{K}_{103} = Mod_g\{((xx)((yx)(zx)))z \approx (x(((yx)y)((zy)z))z\}$	and $(b,a) \in E$, then $(a,a),(c,a)$ $\in E$ iff $(b,b),(c,b),(c,c) \in E$.
$\mathcal{K}_{104} = Mod_g\{(x(yx)((zx)y)))z \approx (xx)((yx)((zx)y))z\}$	and $(b,a),(c,a),(c,b) \in E$, then $(a,a) \in E$.
$\mathcal{K}_{105} = Mod_g\{((xx)(yz))z \approx ((xx)((yx)z))z\}$	and $(a,a) \in E$, then $(b,a) \in E$.
$\mathcal{K}_{106} = Mod_g\{(x(y(zz)))z \approx (x(y((zy)z)))z\}$	and $(c,c) \in E$, then $(c,b) \in E$.
$\mathcal{K}_{107} = Mod_g\{((xx)(y(zy)))z \approx ((xx)(y(zx)))z\}$	and $(a,a) \in E$, then $(c,b) \in E$ iff $(c,a) \in E$.
$\mathcal{K}_{108} = Mod_g\{((xx)(y(zy)))z \approx ((xx)((yx)(zx)))z\}$	and $(a,a) \in E$, then $(c,b) \in E$ iff $(b,a),(c,a) \in E$.
$\mathcal{K}_{109} = Mod_g\{((xx)(y(zy)))z \approx ((xx)((yx)(zy)))z\}$	and $(a,a),(c,b) \in E$, then $(b,a) \in E$.
$\mathcal{K}_{110} = Mod_g\{(x(y((zx)z)))z \approx (x((yx)(zz)))z\}$	then $(c,c) \in E$, then $(c,a) \in E$ iff $(b,a) \in E$.
$\mathcal{K}_{111} = Mod_g\{((xx)(y((zx)z)))z \approx ((xx)((yx)(zz)))z\}$	and $(a,a),(c,c) \in E$, then $(c,a) \in E$ iff $(b,a) \in E$.
$\mathcal{K}_{112} = Mod_g\{((xx)(y((zx)z)))z \approx ((xx)(y((zy)z)))z\}$	and $(a,a),(c,c) \in E$, then $(c,a) \in E$ iff $(c,b) \in E$.
$\mathcal{K}_{113} = Mod_g\{(x((yx)(zz)))z \approx (x(y(((zx)y)z)))z\}$	and $(c,c) \in E$, then $(b,a) \in E$ iff $(c,a),(c,b) \in E$.
$\mathcal{K}_{114} = Mod_g\{(x((yx)(zz)))z \approx (x((yx)((zx)z)))z\}$	and $(b,a),(c,c) \in E$, then $(c,a) \in E$.
$\mathcal{K}_{115} = Mod_g\{((xx)((yy)(zx)))z \approx ((xx)((yy)(zy)))z\}$	and $(a,a),(b,b) \in E$, then $(c,a) \in E$ iff $(c,b) \in E$.
$\mathcal{K}_{116} = Mod_g\{((xx)(y((zy)z)))z \approx ((xx)((yx)(zz)))z\}$	and $(a,a),(c,c) \in E$, then $(c,b) \in E$ iff $(b,a) \in E$.

Table. (continue).

Graph variety	Properties of graphs, for any a, $b, c \in V$ if $(a,b),(b,c),(a,c) \in E$,
$\mathcal{K}_{117} = Mod_g\{((xx)(y((zx)z)))z \approx ((xx)(y((zy)z)))z\}$	and $(a,a),(c,c) \in E$, then $(c,a) \in E$ iff $(c,b) \in E$.
$\mathcal{K}_{118} = Mod_g\{(x(y(zx)))z \approx (x((yx)(zy)))z\}$	then $(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{119} = Mod_g\{((xx)(y(zx)))z \approx ((xx)((yx)(zy)))z\}$	and $(a,a) \in E$, then $(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{120} = Mod_g\{(x((yy)(zx)))z \approx (x(((yx)y)(zy)))z\}$	and $(b,b) \in E$, then $(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{121} = Mod_g\{(x(y((zx)z)))z \approx (x((yx)((zy)z)))z\}$	and $(c,c) \in E$, then $(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{122} = Mod_g\{((xx)((yy)(zx))z \approx ((xx)(((yx)y)(zy))z\}$	and $(a,a),(b,b) \in E$, then $(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{123} = Mod_g\{((xx)(y((zx)z)))z \approx ((xx)((yx)((zy)z)))z\}$	and $(a,a),(c,c) \in E$, then $(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{124} = Mod_g\{(x((yy)((zx)z)))z \approx (x(((yx)y)((zy)z)))z\}$	and $(b,b),(c,c) \in E$, then $(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{125} = Mod_g\{((xx)((yy)((zx)z)))z \approx ((xx)((yx)y)((zy)z)))z\}$	and $(a,a),(b,b),(c,c) \in E$, then $(c,a) \in E$ iff $(b,a),(c,b) \in E$.
$\mathcal{K}_{126} = Mod_g\{(x((yx)(zy)))z \approx (x(y((zx)y)))z\}$	and $(c,b) \in E$, then $(b,a) \in E$ iff $(c,a) \in E$.
$\mathcal{K}_{127} = Mod_g\{(x(((yx)y)(zy)))z \approx (x(((yx)y)(zx))z\}$	and $(b,a),(b,b) \in E$, then $(c,b) \in E$ iff $(c,a) \in E$.
$\mathcal{K}_{128} = Mod_g\{(x((yx)((zy)z))z \approx (x(y(((zx)y)z)))z\}$	and $(c,b),(c,c) \in E$, then $(b,a) \in E$ iff $(c,a) \in E$.
$\mathcal{K}_{129} = Mod_g\{(x((yx)((zy)z)))z \approx (x(y(((zx)y)z)))z\}$	and $(c,b),(c,c) \in E$, then $(b,a) \in E$ iff $(c,a) \in E$.
$\mathcal{K}_{130} = Mod_g\{((xx)((yx)(zy)))z \approx ((xx)((yx)(zx)))z\}$	and $(a,a),(b,a) \in E$, then $(c,b) \in E$ iff $(c,a) \in E$.
$\mathcal{K}_{131} = Mod_g\{((xx)((yx)(zy)))z \approx ((xx)(y((zx)y)))z\}$	and $(a,a),(c,b) \in E$, then $(b,a) \in E$ iff $(c,a) \in E$
$\mathcal{K}_{132} = Mod_g\{(x(((yx)y)(zx)))z \approx (x(((yx)y)(zy)))z\}$	and $(b,a),(b,b) \in E$, then $(c,a) \in E$ iff $(c,b) \in E$.
$\mathcal{K}_{133} = Mod_g\{(x((yy)((zx)y)))z \approx (x(((yx)y)(zy)))z\}$	and $(b,b),(c,b) \in E$, then $(c,a) \in E$ iff $(b,a) \in E$.
$\mathcal{K}_{134} = Mod_g\{(x((yx)((zx)z)))z \approx (x((yx)((zy)z)))z\}$	and $(b,a),(c,c) \in E$, then $(c,a) \in E$ iff $(c,b) \in E$.
$\mathcal{K}_{135} = Mod_g\{((xx)(((yx)y)(zy)))z \approx ((xx)(((yx)y)(zx)))z\}$	and $(a,a),(b,a),(b,b) \in E$, then $(c,b) \in E$ iff $(c,a) \in E$.
$\mathcal{K}_{136} = Mod_g\{((xx)(y(((zx)y)z)))z \approx ((xx)((yx)((zx)z)))z\}$	and $(a,a),(c,a),(c,c) \in E$, then $(c,b) \in E$ iff $(b,a) \in E$.

Table. (continue).

Graph variety	Properties of graphs, for any a, $b, c \in V$ if $(a, b), (b, c), (a, c) \in E$,
$\mathcal{K}_{137} = Mod_g\{((xx)(y(((zx)y)z)))z \approx ((xx)((yx)((zy)z)))z\}$	and $(a, a), (c, b), (c, c) \in E$, then $(c, a) \in E$ iff $(b, a) \in E$.
$\mathcal{K}_{138} = Mod_g\{((xx)((yx)((zx)z)))z \approx ((xx)((yx)((zy)z)))z\}$	and $(a, a), (b, a), (c, c) \in E$, then $(c, a) \in E$ iff $(c, b) \in E$.
$\mathcal{K}_{139} = Mod_g\{((xx)((yx)((zx)z)))z \approx ((xx)(y(((zx)y)z)))z\}$	and $(a, a), (c, a), (c, c) \in E$, then $(b, a) \in E$ iff $(c, b) \in E$.
$\mathcal{K}_{140} = Mod_g\{((xx)((yy)((zx)y)))z \approx ((xx)(((yx)y)(zy)))z\}$	and $(a, a), (b, b), (c, b) \in E$, then $(c, a) \in E$ iff $(b, a) \in E$.
$\mathcal{K}_{141} = Mod_g\{((xx)((yy)(((zx)y)z)))z \approx ((xx)(((yx)y)((zx)z)))z\}$	and $(a, a), (b, b), (c, a), (c, c) \in E$, then $(c, b) \in E$ iff $(b, a) \in E$.

Let $\mathcal{K}_0 = Mod_g\{(x(yz))z \approx (x(yz))z\}$. We see that there are 142 $(x(yz))z$ graph varieties of the form $Mod_g\{s \approx t\}$.

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