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Some Approximation Properties in Musielak-Orlicz-Sobolev Spaces

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Abstract : Some approximation theorems involving the modular convergence, which improve known density results of interest in the existence theory for strongly nonlinear boundary value problems are presented.

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1 Introduction

Let Ω be a bounded Lipschitz domain in R^n and let φ be a real-valued function defined in $\Omega \times R_+$ and satisfying the following conditions:

a) $\varphi(x,.)$ is an N-function, i.e. convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$, and

$$\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t}=0,\ \ \lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t}=\infty;$$

b) $\varphi(.,t)$ is a Lebesgue measurable function.

A function $\varphi(x,t)$, which satisfies the conditions a) and b) is called a *Musielak-Orlicz function.*

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We define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx$$

where $u : \Omega \mapsto R$ a Lebesgue measurable function. In the following the measurability of a function $u : \Omega \mapsto R$ means the Lebesgue measurability. The set

$$K_\varphi(\Omega) = \{u : \Omega \to R \text{ mesurable } | \varrho_{\varphi,\Omega}(u) < +\infty\}$$

is called the *generalized Orlicz class*. The Musielak-Orlicz space (called also the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivelently:

$$L_\varphi(\Omega) = \left\{u : \Omega \to R \text{ mesurable } | \varrho_{\varphi,\Omega}\left(\frac{|u(x)|}{\lambda}\right) < +\infty, \text{ for some } \lambda > 0\right\}.$$

Let $\psi(x, s) = \sup_{t\geq 0}\{st - \varphi(x, t)\}$, for any $x \in \Omega$ and $s \in R^+$, that is, ψ be the Musielak-Orlicz function complementary to $\varphi(x, t)$ in the sense of Young with respect to the variable s. In the space $L_\varphi(\Omega)$ we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\left\{\lambda > 0 | \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1\right\},$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_\psi \leq 1} \int_\Omega |u(x)v(x)|dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [1].

We say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. The space $L_\varphi(\Omega)$ is isomorph to the dual of $E_\psi(\Omega)$. For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall|\alpha| \leq m \; D^\alpha u \in L_\varphi(\Omega)\}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1|+|\alpha_2|+\cdots+|\alpha_n|$ and $D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$ denote the distributional derivatives of u. The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.

Let

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \varrho_{\varphi,\Omega}(D^\alpha u) \text{ and } ||u||^m_{\varphi,\Omega} = \inf\left\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

for any $u \in W^m L_\varphi(\Omega)$. These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $< W^m L_\varphi(\Omega), ||u||^m_{\varphi,\Omega} >$ is a Banach space if φ satisfies the following condition [1]:

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c. \tag{1.1}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \prod L_\varphi$; this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

We denote by $D(\Omega)$ the space of infinitely smooth functions with compact support in Ω and by $D(\overline{\Omega})$ the restriction of $D(R^n)$ on Ω. Let $W_0^m L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$. We say that a sequence of functions u_n belong to $W^m L_\varphi(\Omega)$ (respectively to $W_0^m L_\varphi(\Omega)$) is modular convergent to $u \in W^m L_\varphi(\Omega)$ (respectively $\in W_0^m L_\varphi(\Omega)$) if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

For two complementary Musielak-Orlicz functions φ and ψ the following inequality is called the young inequality [1]:

$$t.s \leq \varphi(x,t) + \psi(x,s) \text{ for } t,s \geq 0,\ x \in \Omega. \tag{1.2}$$

This inequality implies the inequality

$$|||u|||_{\varphi,\Omega} \leq \varrho_{\varphi,\Omega}(u) + 1. \tag{1.3}$$

In $L_\varphi(\Omega)$ we have the following relations between the norm and the modular :

$$||u||_{\varphi,\Omega} \leq \varrho_{\varphi,\Omega}(u) \text{ if } ||u||_{\varphi,\Omega} > 1 \tag{1.4}$$

$$||u||_{\varphi,\Omega} \geq \varrho_{\varphi,\Omega}(u) \text{ if } ||u||_{\varphi,\Omega} \leq 1. \tag{1.5}$$

For two complementary Musielak-Orlicz functions φ and ψ, if $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$ we have the Hölder inequality [1]:

$$\left|\int_\Omega u(x)v(x)\ dx\right| \leq ||u||_{\varphi,\Omega}|||v|||_{\psi,\Omega}. \tag{1.6}$$

In this paper we assume that there exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ we have:

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\frac{A}{\log\left(\frac{1}{|x-y|}\right)}} \tag{1.7}$$

for all $t \geq 1$. For some Musielak-Orlicz functions which verify (1.7) see examples in the end of this chapter. In this paper we study the problem of density of smooth functions in $W^m L_\varphi(\Omega)$ and $W_0^m L_\varphi(\Omega)$ for the modular convergence, under the assumption (1.7). Our result generalizes that of the authors in [2] in the case of Musielak-Orlicz-Sobolev spaces, that of Gossez in [3] in the case of classical Orlicz spaces and those of Zhikov [4, 5] and Samko [6] in the case of variable exponent Sobolev spaces.

Similar results have been provided by Hudzik in [7] and [8] by assuming the following condition:

$$\int M(x,|f_\varepsilon(x)|)dx \leq K \int M(x,|f(x)|)dx \tag{1.8}$$

for all functions $f \in L_M(R^n)$,where f_ε is a regularized function of f. In our paper we don't assume any condition of this type. For others approximations results in Musielak-Orlicz-Sobolev spaces and some their applications to nonlinear partial differential equations see [9]. And for nonlinear equations in classical Orlicz spaces see [10–13, 16, 17] and references within.

2 Main Results

Let $K(x)$ be a measurable function with support in the ball $B_R = B(0,R)$ and let

$$K_\varepsilon(x) = \frac{1}{\varepsilon^n} K\left(\frac{x}{\varepsilon}\right).$$

We consider the family of operators

$$K_\varepsilon f(x) = \kappa_\varepsilon^{-1} \int_\Omega K_\varepsilon(x-y) f(\kappa_\varepsilon y)\, dy. \tag{2.1}$$

Theorem 2.1. *Let $K(x) \in L^\infty(B_R)$ and let φ and ψ be two complementary Musielak-Orlicz functions such that φ satisfies the conditions (1.1), (1.7) and*

$$\text{if } D \subset \Omega \text{ is a bounded measurable set, then } \int_D \varphi(x,1)dx < \infty \tag{2.2}$$

and ψ satisfies the following condition:

$$\psi(x,1) \leq C \ \text{ a.e. in } \Omega. \tag{2.3}$$

Then the operators K_ε are uniformly bounded from $L_\varphi(\Omega)$ into $L_\varphi(\Omega)$, namely

$$||K_\varepsilon f||_{\varphi,\Omega} \leq C||f||_{\varphi,\Omega} \ \forall f \in L_\varphi(\Omega), \tag{2.4}$$

where $C > 0$ does not depend on ε.

Remark 2.2. *For any Musielak-Orlicz function φ we can replace it by a Musielak-Orlicz function $\overline{\varphi}$ which is globally equivalent to φ such that $\overline{\varphi}(x,1) + \overline{\psi}(x,1) = 1$, where $\overline{\psi}$ is the Musielak-Orlicz function complementary to $\overline{\varphi}$ (see [14], §2.4). Hence by (1.1) we may assume without loss of generality that the condition (2.3) is always satisfied.*

Theorem 2.3. *Let φ and $K(x)$ satisfy the assumptions of Theorem 1 and*

$$\int_{B_R} K(y)\ dy = 1. \tag{2.5}$$

Then (2.1) is an identity approximation in $L_\varphi(\Omega)$, that is,

$$\exists\lambda > 0 : \lim_{\varepsilon\to 0} \varrho_{\varphi,\Omega}\left(\frac{K_\varepsilon f - f}{\lambda}\right) = 0,\ f \in L_\varphi(\Omega). \tag{2.6}$$

Corollary 2.4. *Under the assumptions of Theorem 2.1, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology.*

Theorem 2.5. *Let φ be a Musielak-Orlicz function which satisfies the assumptions of Theorem 2.1 and let $f \in W_0^m L_\varphi(\Omega)$. Then there exist $\lambda > 0$ and a sequence $f_n \in \mathcal{D}(\Omega)$ such that for $|\alpha| \le m$,*

$$\int_\Omega \varphi\left(x, \left(\frac{D^\alpha f_n - D^\alpha f}{\lambda}\right)\right) \to 0 \quad \text{as } n \to \infty.$$

Theorem 2.6. *Let φ be a Musielak-Orlicz function which satisfies the assumptions of Theorem 2.1 and let $f \in W^m L_\varphi(\Omega)$. Then there exist $\lambda > 0$ and a sequence $f_n \in \mathcal{D}(\overline{\Omega})$ such that for $|\alpha| \le m$,*

$$\int_\Omega \varphi\left(x, \left(\frac{D^\alpha f_n - D^\alpha f}{\lambda}\right)\right) \to 0 \quad \text{as } n \to \infty.$$

Example 2.7. *Let $p : \Omega \mapsto [1, \infty)$ be a measurable function such that there exist a constant $c > 0$ such that for all points $x, y \in \Omega$ with $|x - y| < \frac{1}{2}$, we have the inequality*

$$|p(x) - p(y)| \le \frac{c}{\log\left(\frac{1}{|x-y|}\right)}.$$

Then the following Musielak-Orlicz functions satisfy the conditions of Theorem 2.1:

(1) $\varphi(x,t) = t^{p(x)}$ *such that* $\sup_{x\in\Omega} p(x) < \infty$*;*

(2) $\varphi(x,t) = t^{p(x)} \log(1 + t)$*;*

(3) $\varphi(x,t) = t(\log(t + 1))^{p(x)}$*;*

(4) $\varphi(x,t) = (e^t)^{p(x)} - 1$.

3 Proofs

Proof of Theorem 2.1. We can assume that Ω is a starlike domain relative to some ball $B_{r_0} = \{|x| < r_0\}$. This means that the segment joining a point in Ω

with a point in B_{r_0} is contained in Ω. Setting $t = r_0^{-1}$ for the starlike domain we can write

$$(1 - t\varepsilon)\Omega + \varepsilon B \subset \Omega, \tag{3.1}$$

where $B = \{|x| < 1\}$ is the unit ball. It is not difficult to pass from a starlike domain to a Lipschitz domain because of the localization property.

We extend $f \in L_\varphi(\Omega)$ by zero to R^n, so we have

$$K_\varepsilon f(x) = \kappa_\varepsilon^{-1} \int_\Omega K_\varepsilon(x - y) f(\kappa_\varepsilon y)\ dy,$$

where $\kappa_\varepsilon^{-1} = 1 - t\varepsilon$. Then (3.1) implies that $K_\varepsilon f \in \mathcal{D}(\Omega)$. We also assume that

$$||f||_{\varphi,\Omega} \leq 1. \tag{3.2}$$

It suffices to show that

$$\varrho_{\varphi,\Omega}(K_\varepsilon f) = \int_\Omega \varphi(x, |K_\varepsilon f(x)|) dx \leq c \tag{3.3}$$

for some ε such that $0 < \varepsilon \leq \varepsilon^0 \leq 1$ and $c > 0$ independent of f. Let

$$\Omega = \cup_{k=1}^N \omega^k$$

be any partition of Ω into small parts ω^k comparable with the given ε:

$$diam\ \omega^k \leq \varepsilon,\ k = 1, 2, 3, ..., N = N(\varepsilon).$$

We represent the integral in (3.3) as

$$\varrho_{\varphi,\Omega}(K_\varepsilon f) = \sum_{k=1}^N \int_{\omega^k} \varphi\left(x, \left|\kappa_\varepsilon^{-1} \int_\Omega K_\varepsilon(x - y) f(\kappa_\varepsilon y)\ dy\right|\right) dx. \tag{3.4}$$

We put

$$\varphi_k(t) = \inf\{\varphi(x, t), x \in \Omega^k\} \leq \inf\{\varphi(x, t), x \in \omega^k\} \tag{3.5}$$

where some larger partition $\Omega^k \supset \omega^k$ comparable with ε will be chosen later:

$$diam\ \Omega^k \leq m\varepsilon, m > 1. \tag{3.6}$$

Hence

$$\varrho_{\varphi,\Omega}(K_\varepsilon f) = \sum_{k=1}^N \int_{\omega^k} A_k(x, \varepsilon)\ \varphi_k\left(\left|\kappa_\varepsilon^{-1} \int_\Omega K_\varepsilon(x - y) f(\kappa_\varepsilon y)\ dy\right|\right)\ dx, \tag{3.7}$$

where

$$A_k(x, \varepsilon) := \frac{\varphi(x, |\kappa_\varepsilon^{-1} \int_\Omega K_\varepsilon(x - y) f(\kappa_\varepsilon y)\ dy|)}{\varphi_k(|\kappa_\varepsilon^{-1} \int_\Omega K_\varepsilon(x - y) f(\kappa_\varepsilon y)\ dy|)}.$$

We shall prove the uniform estimate

$$A_k(x,\varepsilon) \le c,\ x \in \omega^k, \tag{3.8}$$

where $c > 0$ does not depend on $x \in \omega^k,\ k$ and $\varepsilon \in (0, \varepsilon^0)$ with some $\varepsilon^0 > 0$. By (1.6) we have

$$\begin{aligned}\alpha(x,\varepsilon) := \left|\kappa_\varepsilon^{-1}\int_\Omega K_\varepsilon(x-y) f(\kappa_\varepsilon y)\ dy\right| &\le \frac{(1+t)M}{\varepsilon^n}\int_\Omega |\chi_{B_{\varepsilon R}}(y) f(\kappa_\varepsilon y)| dy \\ &\le \frac{(1+t)M}{\varepsilon^n} ||f||_\varphi\ |||\chi_{B_{\varepsilon R}}|||_\psi,\end{aligned}$$

where $M = \sup_{B_R} |K(y)|$. By (1.3) and condition (2.3) we obtain

$$|||\chi_{B_{\varepsilon R}}|||_\psi \le c_2 |B_{\varepsilon R}| + 1 \le c_2 + 1 \tag{3.9}$$

for $0 < \varepsilon \le |B(0,1)|^{-\frac{1}{n}} := \varepsilon_1^0$. Hence

$$\alpha(x,\varepsilon) \le \frac{c_1}{\varepsilon^n}. \tag{3.10}$$

We observe now that by (1.7) and (3.5) we have

$$\frac{\varphi(x,t)}{\varphi_k(t)} = \frac{\varphi(x,t)}{\varphi(\xi_k,t)} \le t^{\frac{A}{\log\left(\frac{1}{|x-\xi_k|}\right)}}, \tag{3.11}$$

where $x \in \omega^k,\ \xi_k \in \Omega^k$. Evidently $|x - \xi_k| \le diam\ \Omega^k \le m\varepsilon$. Therefore,

$$\begin{aligned}A_k(x,\varepsilon) = \frac{\varphi(x,\alpha(x,\varepsilon))}{\varphi(\xi_k,\alpha(x,\varepsilon))} &\le (\alpha(x,\varepsilon))^{\frac{A}{\log\left(\frac{1}{m\varepsilon}\right)}} \\ &\le (c_1\varepsilon^{-n})^{\frac{A}{\log\left(\frac{1}{m\varepsilon}\right)}} \le (c_1)^{\frac{A}{\log\left(\frac{1}{m}\right)}} (\varepsilon^{-n})^{\frac{A}{\log\left(\frac{1}{m\varepsilon}\right)}}\end{aligned} \tag{3.12}$$

under the assumption that $0 < \varepsilon \le \frac{1}{2m} := \varepsilon_2^0$. Then from (3.12)

$$A_k(x,\varepsilon) \le c_4 := c_3 e^{2nA},\ c_3 = (c_1)^{\frac{A}{\log\left(\frac{1}{m}\right)}} \tag{3.13}$$

for $x \in \omega^k$ and

$$0 < \varepsilon \le \frac{1}{m^2} := \varepsilon_3^0. \tag{3.14}$$

Therefore, we have the uniform estimate (3.8) with $c = c_3 e^{2nA}$ and $0 < \varepsilon \le \varepsilon^0,\ \varepsilon^0 = \min_{1\le k\le 3} \varepsilon_k^0,\ \varepsilon_k^0$ being given above. Using estimate (3.8) we obtain from (3.7)

$$\varrho_{\varphi,\Omega}(K_\varepsilon f) = c\sum_{k=1}^N \int_{\omega^k} \varphi_k\left(\left|\int_\Omega K_\varepsilon(x-y) f(\kappa_\varepsilon y)\ dy\right|\right) dx. \tag{3.15}$$

So by the Jensen integral inequality we obtain

$$\begin{aligned}\varrho_{\varphi,\Omega}(K_\varepsilon f) &\leq \sum_{k=1}^{N} \int_{|y|<\varepsilon R} |K_\varepsilon(y)| dy \int_{\omega^k} \varphi_k(f(\kappa_\varepsilon(x-y)))dx \\ &= c \sum_{k=1}^{N} \int_{|y|<R} |K(y)| dy \int_{x+\varepsilon\kappa_\varepsilon^{-1} y \in \omega^k} \varphi_k(f(x))dx. \end{aligned} \tag{3.16}$$

Obviously, the domain of the integration in x in the last integral is embedded into the domain

$$\bigcup_{y \in B_{\varepsilon R}} \{x : x + \kappa_\varepsilon^{-1} y \in \omega^k\} \tag{3.17}$$

which does not depend on y. Now, we choose in (3.5) the sets Ω^k which were not determined until now, as the sets (3.17). Then, evidently, $\Omega^k \supset \omega^k$, and it is easily seen that

$$diam\ \Omega^k \leq (1+2R)\varepsilon, \tag{3.18}$$

so the requirement (3.6) is satisfied with $m = 1 + 2R$.

From (3.17) we have

$$\begin{aligned}\varrho_{\varphi,\Omega}(K_\varepsilon f) &\leq c \sum_{k=1}^{N} \int_{|y|<R} |K(y)| dy \int_{\Omega^k} \varphi_k(f(x))dx \\ &\leq c \int_{|y|<R} |K(y)| dy \sum_{k=1}^{N} \int_{\Omega^k} \varphi_k(f(x))dx. \end{aligned} \tag{3.19}$$

Therefore,

$$\varrho_{\varphi,\Omega}(K_\varepsilon f) \leq c_5 \int_\Omega \tilde{\varphi}(x, f(x))\ dx, \tag{3.20}$$

where $\tilde{\varphi}(x,t) = \max_i \varphi_i(t)$, the maximum being taken with respect to all the sets Ω_k. Evidently, $\tilde{\varphi}(x,t) \leq \varphi(x,t) \ \ \forall x \in \Omega$. Then from (3.20) and (3.2) we arrive to the final estimate

$$\varrho_{\varphi,\Omega}(K_\varepsilon f) \leq c_5 \int_\Omega \varphi(x, f(x))\ dx \leq c_5. \tag{3.21}$$

□

Proof of Theorem 2.3. To prove (2.6), we use Theorem 2.1, which provides the uniform boundedness of the operators K_ε from $L_\varphi(\Omega)$ into $L_\varphi(\Omega)$. Then by the Banach-Steinhaus theorem it suffices to verify that (2.6) holds for some dense set in $L_\varphi(\Omega)$. So, it is sufficient to prove (2.6) for the characteristic functions $\chi_E(x)$ of all bounded measurable sets $E \subset \Omega$ [1]. We have

$$K_\varepsilon(\chi_E)(x) - \chi_E(x) = \int_{B_R} k(y)[\kappa_\varepsilon^{-1}\chi_E(x - \varepsilon\kappa_\varepsilon y) - \chi_E(x)]dy,$$

whence for $\lambda > 0$,

$$\begin{aligned}\varrho_{\varphi,\Omega}\left(\frac{K_\varepsilon(\chi_E) - \chi_E}{\lambda}\right) &= \int_\Omega \varphi\left(x, \frac{1}{\lambda}\int_{B_R} k(y)[\kappa_\varepsilon^{-1}\chi_E(x - \varepsilon\kappa_\varepsilon y) - \chi_E(x)]dy\right)dx \\ &\le \int_{B_R} k(y)\left(\int_\Omega \varphi\left(x, \frac{1}{\lambda}[\kappa_\varepsilon^{-1}\chi_E(x - \varepsilon\kappa_\varepsilon y) - \chi_E(x)]\right)dx\right)dy\end{aligned}$$

by the Fubini theorem and the Jensen inequality. Hence by condition (2.2) and the Lebesgue dominated convergence theorem we obtain (2.6) for some $\lambda > 0$. □

Proof of Corollary 2.4. The proof is immediate from the Theorem 2.3. □

Proof of Theorem 2.5. Let $f(x) \in W_0^m L_\varphi(\Omega)$ and let us extend f by zero to R^n and apply the same smoothing procedure as above, obtaining

$$\overline{\varrho}_{\varphi,\Omega}\left(\frac{f - K_\varepsilon f}{\lambda}\right) = \sum_{|j| \le m} \varrho_{\varphi,\Omega}\left(\frac{D^j f - K_\varepsilon(D^j f)}{\lambda}\right).$$

Then it suffices to apply Theorem 2.3. □

Proof of Theorem 2.6. It is essentially the same as that above except that one takes $\kappa_\varepsilon = 1 - t\varepsilon$ and we put

$$v_\varepsilon(x) = \begin{cases} \kappa_\varepsilon^{-1} u(\kappa_\varepsilon x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in R^n \setminus \Omega. \end{cases}$$

This definition has meaning since (3.1) implies that $\kappa_\varepsilon\Omega \subset \Omega$. Hence $v_\varepsilon * K_\varepsilon \in \mathcal{D}(R^n)$ and we also put

$$u_\varepsilon(x) = \begin{cases} v_\varepsilon * K_\varepsilon(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in R^n \setminus \Omega. \end{cases}$$

Therefore $u_\varepsilon \in \mathcal{D}(\overline{\Omega})$ and the remaining arguments remain the same. □

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