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A Fixed Point Theorem in Generalized Menger Spaces

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Abstract : The present authors had introduced generalized Menger space as a generalization of certain probabilistic metric spaces. In this paper we establish a basic fixed point theorem in this types of spaces. Our result is supported by an example.

Keywords : Generalized Menger space; Fixed point; Contraction mapping; t-norm.
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1 Introduction

The concept of metric space has been generalized in several directions. One such generalization was introduced in the work of Branciari [1] where instead of the triangle inequality a quadrangular inequality was assumed. The definition of generalized metric space introduced by Branciari [1] is the following.

Definition 1.1 ([1])**.** Let X be a non-empty set and $d : X^2 \to R^+$ be a mapping such that for all $x, y \in X$ for all points $\xi, \eta \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0 \Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x,y) \leq d(x,\xi) + d(\xi,\eta) + d(\eta,y)$.

then we will say that (X,d) is a *generalized metric space*.

Branciari [1] had also given an example to show the existence of generalized metric space which are not metric spaces. Also in the same work Banach contraction mapping theorem in generalized metric space was established. Further fixed point studies of functions defined on generalized metric space were done in works like [2–4].

There is a long history of the studies in probabilistic extensions of the contraction mapping principle. Sehgal and Bharucha-Reid [5] introduced q-contraction in probabilistic metric spaces. Probabilistic metric spaces are probabilistic generalizations of metric spaces. Several aspects of this structure are described in [6]. Fixed point theory in probabilistic metric spaces is an active branch of research. A comprehensive survey of research in this line is given in [7]. Some other recent works are noted in [8–15].

In the same spirit of Branciari [1], a generalization of Menger space has been introduced by the present authors where the probabilistic triangular inequality has been replaced by a quadrangular inequality. In particular the generalized metric space (Definition 1.1) is a special case of the generalized Menger space. Here we establish the probabilistic q-contraction principle in the generalized Menger space. Finally we cite an example to which our theorem is applicable. In particular generalized Menger space described in this example is not a Menger space which establishes the fact that this generalization is an effective generalization. Also the contraction mapping principle in generalized metric space proved by Branciari [1] follows as a special case of our theorem.

Definition 1.2 ([16])**.** A mapping $T : \prod_{i=1}^{n}[0,1] \to [0,1]$ is called a *n-th order t-norm* if the following conditions are satisfied:

(i) $T(0,0,...,0) = 0$, $T(a,1,1,...,1) = a$ for all $a \in [0,1]$;

(ii) $T(a_1,a_2,a_3,...,a_n) = T(a_2,a_1,a_3,...,a_n) = T(a_2,a_3,a_1,...,a_n) = \cdots = T(a_2,a_3,a_4,...,a_n,a_1)$;

(iii) $a_i \geq b_i$, $i = 1,2,3,...,n$ implies $T(a_1,a_2,a_3,...,a_n) \geq T(b_1,b_2,b_3,...,b_n)$;

(iv) $T(T(a_1,a_2,a_3,...,a_n),b_2,b_3,...,b_n) = T(a_1,T(a_2,a_3,...,a_n,b_2),b_3,...,b_n)$
$= T(a_1,a_2,T(a_3,a_4,...,a_n,b_2,b_3),b_4,...,b_n)$
$= \cdots = T(a_1,a_2,...,a_{n-1},T(a_n,b_2,b_3,...,b_n))$.

Definition 1.3 ([17, Generalized Menger Space (g.M.s.)])**.** Let S be a non-empty set. Then (S,F,T) is said to be a *generalized Menger space* (briefly g.M.s.) if for all $x,y \in S$ and all distinct points $z,w \in S$ each of them different from x and y , the following conditions are satisfied.

(i) $F_{x,y}(0) = 0$;

(ii) $F_{x,y}(t) = 1$ for all $t > 0$ iff $x = y$;

(iii) $F_{x,y}(t) = F_{y,x}(t)$ for all $t > 0$ and for all $x, y \in S$;

(iv) $F_{x,y}(t) \geq T(F_{x,z}(t_1), F_{z,w}(t_2), F_{w,y}(t_3))$ where $t_1 + t_2 + t_3 = t$ and T is a 3-rd order t-norm.

Definition 1.4. Let (S, F, T) be a generalized Menger space. A sequence $\{x_n\} \subset S$ is said to *converge* to some point $x \in S$ if given $\epsilon > 0$, $\lambda > 0$ we can find a positive integer $N_{\epsilon,\lambda}$ such that for all $n > N_{\epsilon,\lambda}$,

$$F_{x_n,\, x}(\epsilon) > 1 - \lambda.$$

Definition 1.5. A sequence $\{x_n\}$ is said to be a *Cauchy sequence* in S if given $\epsilon > 0$, $\lambda > 0$ there exists a positive integer $N_{\epsilon,\lambda}$ such that

$$F_{x_n,\, x_m}(\epsilon) > 1 - \lambda \text{ for all } m, n > N_{\epsilon,\lambda}.$$

Definition 1.6. A generalized Menger space (S, F, T) is said to be *complete* if every Cauchy sequence is convergent in it.

Definition 1.7. Let (S, F, T) be a generalized Menger space. A mapping $f : S \to S$ is called a *q-contraction* if there exists $q \in (0, 1)$ such that for every $x, y \in S$ and $t > 0$ the following implication holds:

$$F_{fx,\, fy}(t) \geq F_{x,\, y}\left(\frac{t}{q}\right). \tag{1.1}$$

2 Main Results

Theorem 2.1. *Let $(S, F, \min)$ be a complete generalized Menger space, $q \in (0, 1)$ and the mapping $f : S \to S$ be a q-contraction, then f has a unique fixed point.*

Proof. Let $x \in S$. We consider the sequence $\{f^n x\}_{n \epsilon N}$. Now we have for $t > 0$,

$$F_{f^{n+1}x, f^n x}(t) \geq F_{f^n x, f^{n-1}x}\left(\frac{t}{q}\right) \geq F_{f^{n-1}x, f^{n-2}x}\left(\frac{t}{q^2}\right) \geq \cdots \geq F_{fx,x}\left(\frac{t}{q^n}\right) \to 1 \tag{2.1}$$

as $n \to \infty$. If $f^n x = f^m x$ for some $m, n \in N (m > n)$, where N is the set of natural numbers, then we have $f^p x = x$ for $p = m - n$. Therefore for $t > 0$,

$$F_{x,fx}(t) = F_{f^p x, f^{p+1}x}(t) \geq F_{x,fx}\left(\frac{t}{q^p}\right) \geq F_{x,fx}\left(\frac{t}{q^{2p}}\right) \geq \cdots \geq F_{x,fx}\left(\frac{t}{q^{np}}\right) \to 1 \tag{2.2}$$

as $n \to \infty$, that is, $x = fx$.

So, we assume $f^n x \neq f^m x$ for all distinct $m, n \in N$. We claim that $\{f^n x\}$ is a Cauchy sequence. If possible, let $\{f^n x\}$ be not a Cauchy sequence. Then there exist $\varepsilon > 0$ and $\lambda > 0$ for which we can find subsequences $\{f^{m(k)}x\}$ and $\{f^{n(k)}x\}$ of $\{f^n x\}$ with $n(k) > m(k) > k$ for all positive integer k such that

$$F_{f^{m(k)}x, f^{n(k)}x}(\varepsilon) \leq 1 - \lambda. \tag{2.3}$$

We take $n(k)$ corresponding to $m(k)$ to be the smallest integer satisfying (2.3) so that

$$F_{f^{m(k)},f^r x}(\varepsilon) > 1-\lambda \text{ fot all } r < n(k). \tag{2.4}$$

Equivalently, the construction is finding a point $f^{n(k)}x$ in the sequence with $n(k) > m(k)$ which will fall outside the (ε-λ)-neighborhood of $f^{m(k)}x = \{z : F_{f^{m(k)}x,\,z}(\varepsilon) \geq 1-\lambda\}$, but the points in the sequence preceding the point $f^{n(k)}x$, that is, the points $f^{n(k)-1}x, f^{n(k)-2}x, ...$ will fall inside the set. This is guaranteed by the fact the sequence is assumed not to be a Cauchy sequence.

Now we have for $k > 2$,

$$\begin{aligned} 1-\lambda &\geq F_{f^{m(k)}x,f^{n(k)}x}(\varepsilon) \geq F_{f^{m(k)-1}x,f^{n(k)-1}x}\left(\frac{\varepsilon}{q}\right) \geq F_{f^{m(k)-2}x,f^{n(k)-2}x}\left(\frac{\varepsilon}{q^2}\right) \\ &\geq \min\{F_{f^{m(k)-2}x,\,f^{m(k)-1}x}(\eta_1), F_{f^{m(k)-1}x,\,f^{m(k)}x}(\eta_2), F_{f^{m(k)}x,\,f^{n(k)-2}x}(\varepsilon)\} \end{aligned} \tag{2.5}$$

where η_1 and η_2 are positive numbers such that $\eta_1+\eta_2+\varepsilon = \frac{\varepsilon}{q^2}$. This is possible since $0 < q < 1$.

We now take $\lambda_1 > 0$ such that $\lambda_1 < \lambda$. By (2.1) k is chosen so that

$$F_{f^{m(k)-2}x,\,f^{m(k)-1}x}(\eta_1) > 1-\lambda_1 \tag{2.6}$$

and

$$F_{f^{m(k)-1}x,\,f^{m(k)}x}(\eta_2) > 1-\lambda_1. \tag{2.7}$$

Using (2.6) , (2.7) and (2.4) we have from (2.5)

$$\begin{aligned} 1-\lambda &\geq \min\{F_{f^{m(k)-2}x,\,f^{m(k)-1}x}(\eta_1), F_{f^{m(k)-1}x,\,f^{m(k)}x}(\eta_2), F_{f^{m(k)}x,\,f^{n(k)-2}x}(\varepsilon)\} \\ &> \min\{1-\lambda_1, 1-\lambda_1, 1-\lambda\} = 1-\lambda, \end{aligned}$$

which is a contradiction. Hence $\{f^n x\}$ is a Cauchy sequence in $(S, F, \min)$. As $(S, F, \min)$ is a complete g.M.s., we have $\{f^n x\}$ is convergent in S. Let

$$\lim_{n\to\infty} f^n x = z. \tag{2.8}$$

We now show that $fz = z$. For all $t > 0$, $t_1, t_2, t_3 > 0$ and $t_1+t_2+t_3 = t$,

$$\begin{aligned} F_{z,fz}(t) &\geq \min\{F_{z,f^{n-1}x}(t_1), F_{f^{n-1}x,f^n x}(t_2), F_{f^n x,fz}(t_3)\} \\ &\geq \min\left\{F_{z,f^{n-1}x}(t_1), F_{f^{n-1}x,f^n x}(t_2), F_{f^{n-1}x,z}\left(\frac{t_3}{q}\right)\right\}. \end{aligned}$$

Making $n \to \infty$ and using (2.1) and (2.8) we have from above $F_{z,fz}(t) \geq \min\{1,1,1\} = 1$. Hence $fz = z$.

For uniqueness, let z and u be two fixed points. Therefore for all $t > 0$,

$$F_{z,u}(t) = F_{fz,fu}(t) \geq F_{z,u}\left(\frac{t}{q}\right) \geq \cdots \geq F_{z,u}\left(\frac{t}{q^n}\right) \to 1$$

as $n \to \infty$. Therefore $z = u$. □

We now show that Definition 1.1 is special case of Definition 1.3. Let (S,d) be a generalized metric space. This spaces can be treated as a generalized Menger space if we put $F_{x,\,y}(t) = H(t - d(x,y))$, where $x, y \in S$, H is defined as

$$H(s) = \begin{cases} 1, & \text{if } s > 0, \\ 0, & \text{if } s \le 0, \end{cases}$$

and T is defined as $T(a,b) = \min\{a,b\}$.

Conditions (i) and (ii) of Definition 1.1 trivially follows from conditions (ii) and (iii) of Definition 1.3, respectively. We now show that condition (iii) of Definition 1.1 follows from conditions (iv) of Definition 1.3. Let $x, y \in S$ and z, w be two distinct points in S different from x and y. If possible let

$$d(x,y) \le d(x,z) + d(z,w) + d(w,y) \tag{2.9}$$

be true and the following equation is false.

$$F_{x,y}(t) \ge T(F_{x,z}(t_1), F_{z,w}(t_2), F_{w,y}(t_3)). \tag{2.10}$$

where $t = t_1 + t_2 + t_3$. We assumed that $F_{x,\,y}(t) = H(t - d(x,y))$. Inequality (2.10) will be false if $F_{x,\,y}(t) = 0$ and $F_{x,\,z}(t_1) = 1$, $F_{z,\,w}(t_2) = 1$, $F_{w,\,y}(t_3) = 1$. Now

$$F_{x,\,y}(t) = 0 \text{ implies that } t - d(x,y) \le 0, \text{ that is, } t \le d(x,y), \tag{2.11}$$

$$F_{x,\,z}(t_1) = 1 \text{ implies that } t_1 - d(x,y) > 0, \text{ that is, } t_1 > d(x,z). \tag{2.12}$$

Similarly we must have

$$t_2 > d(z,w) \tag{2.13}$$

and

$$t_3 > d(w,y). \tag{2.14}$$

Therefore

$$d(x,y) \ge t = t_1 + t_2 + t_3 > d(x,z) + d(z,w) + d(w,y) \tag{2.15}$$

which contradicts inequality (2.9).

Thus we can say that condition (iii) of Definition 1.1 follows from conditions (iv) of Definition 1.3. By similar arguments we can establish that a sequence in a generalized metric space is a Cauchy sequence if and only if it is a Cauchy sequence in the corresponding generalized Menger space. Then it follows that whenever a generalized metric space is complete the corresponding generalized Menger space is also complete. We also show by a similar argument that the inequality

$$F_{fx,\,fy}(t) \ge F_{x,\,y}\left(\frac{t}{c}\right) \tag{2.16}$$

in the generalized Menger space corresponding generalized metric space (S,d) implies the inequality

$$d(fx, fy) \le c d(x,y) \tag{2.17}$$

in that generalized metric space. Taking into account all the points discussed above we have as a corollary the result of Branciari [1].

Corollary 2.2. *Let (S,d) be a complete generalized metric space, $c \in [0,1)$ and $f : S \to S$ a mapping such that for each $x,y \in S$ one has*

$$d(fx,fy) \leq cd(x,y)$$

then

(i) there exists a point $a \in S$ such that for each $x \in S$ one has $\lim_{n\to\infty} f^n x = a$;

(ii) $fa = a$ and for each $e \in S$ such that $fe = e$ one has $e = a$;

(iii) for all $n \in N$ one has

$$d(f^n x, a) \leq \frac{c^n}{1-c} \max\{d(x,fx), d(x,f^2x)\}.$$

Example 2.3. *Let $S = \{x_1,x_2,x_3,x_4\}$, $T(a,b) = \min\{a,b\}$ and $F_{x,y}(t)$ be defined as*

$$F_{x_1,x_2}(t) = F_{x_2,x_1}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.70, & \text{if } 0 < t < 6, \\ 1, & \text{if } t \geq 6. \end{cases}$$

$$F_{x_1,x_3}(t) = F_{x_3,x_1}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.90, & \text{if } 0 < t \leq 3, \\ 1, & \text{if } t > 3. \end{cases}$$

$$F_{x_1,x_4}(t) = F_{x_4,x_1}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.80, & \text{if } 0 < t \leq 4, \\ 1, & \text{if } t > 4. \end{cases}$$

$$F_{x_2,x_3}(t) = F_{x_3,x_2}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.95, & \text{if } 0 < t \leq 3, \\ 1, & \text{if } t > 3. \end{cases}$$

$$F_{x_2,x_4}(t) = F_{x_4,x_2}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.80, & \text{if } 0 < t \leq 4, \\ 1, & \text{if } t > 4. \end{cases}$$

$$F_{x_3,x_4}(t) = F_{x_4,x_3}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.70, & \text{if } 0 < t < 6, \\ 1, & \text{if } t \geq 6. \end{cases}$$

Then $(S,F,\min)$ is a complete generalized Menger space. Let $f : S \to S$ be given by $fx_1 = fx_2 = fx_3 = x_3$ and $fx_4 = x_1$. The mapping f is q-contractive if we take $q = 0.80$. Also f satisfies all the conditions of Theorem 2.1 and x_3 is the unique fixed point of f. This example is not a Menger space as we have seen

$$F_{x_3,x_4}(5) \not\geq \min\{F_{x_3,x_2}(1), F_{x_2,x_4}(4)\}.$$

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