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Some Multiple Integral Transforms of Multidimensional Polynomial and I-function

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Abstract : The aim of this paper is to study multiple Mellin and Laplace transforms involving multidimensional analogues of a general class of polynomials together with multivariable I-function. In this regard, we have proved five theorems, corollaries have also been recorded. Similar results obtain by other authors follows as special cases of our findings.

Keywords : Generalized H-function; General class of polynomial; Integral transform; Multiple transforms; Transforms of special functions.
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1 Introduction

Recently, the multivariable I-function which is a generalization of the multivariable H-function [1], was introduced and studied by Prasad [2] and Prasad and Yadav [3]. Moreover Prasad and Singh [4] studied the Mellin and Laplace transform of the multivariable I-function. They observed that the Mellin transform of the I-function of r variables reduces to the I-function of $(r-1)$ variables. Saxena and Singh [5] discussed the derivatives of the multivariable I-function. In the recent paper of Chaurasia and Kumar[6], Singh and Yadav [7] studied some fractional integrals of a general polynomials, $\bar{H}$-functions and multivariable I-functions.

Since the multivariable I-function is of a general nature, its multiple Mellin and Laplace transforms reduce to many simpler special functions as particular cases.

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The main objective of this paper is to study multiple Mellin and Laplace transforms of the multidimensional analogue of general class of polynomials together with the multivariable I-functions.

2 Preliminaries

In this section, we shall first recall some definitions and fundamental facts of integral transforms and special functions.

2.1 Basic Definition of Integral Transform

Definition 2.1. The multidimensional Mellin transform is defined for the function $\varphi(t_1,\ldots,t_r)$ as follows [8]

$$\begin{aligned}(\mathfrak{M}\varphi)(s) :&= \int_{(R)^n_{+\cdots+}} t^{s-1}\varphi(t)dt \\ &= \int_0^\infty \cdots \int_0^\infty t_1^{s_1-1}\cdots t_r^{s_r-1}\varphi(t_1,\cdots,t_r)dt_1\cdots dt_r, \qquad (2.1)\end{aligned}$$

where $t_1,\ldots,t_r > 0$.

The multiple Mellin convolution for the two function g and h will be defined and represented as follows

$$\begin{aligned}&(g*\cdots*h)(u_1,\ldots,u_r) \\ &\quad = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} t_i^{-1} g\left(\frac{u_1}{t_1},\ldots,\frac{s_r}{t_r}\right) h(t_1,\ldots,t_r)dt_1\cdots dt_r, \qquad (2.2)\end{aligned}$$

provided that the integral in (2.2) converges absolutely.

Definition 2.2. The multidimensional Laplace transform is defined for the function $f(x_1,\ldots,x_r)$ as follows [8]

$$\begin{aligned}(\mathfrak{L}f)(s) :&= \mathfrak{L}\{f(x_1,\ldots,x_r);s_1,\ldots,s_r\} \\ &= \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^{r} s_i x_i\right) f(x_1,\ldots,x_r)dx_1\cdots dx_r, \qquad (2.3)\end{aligned}$$

where $\Re(s_i) > 0$, $i = 1,\ldots,r$.

2.2 Basic Definition of Special Function

Definition 2.3. The *multivariable I-function* is represented [2] as

$$I[z_1,\ldots,z_r] := I^{\{0,n_i\}_{2,r}:\{(m^{(i)},n^{(i)})\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\{(p^{(i)},q^{(i)})\}^{1,r}}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\,\middle|\, \begin{matrix} \mathcal{A}:\mathcal{B} \\ \mathcal{C}:\mathcal{D} \end{matrix}\right]$$
$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\zeta_1,\ldots,\zeta_r)\prod_{i=1}^{r}\left\{\phi_i(\zeta_i)z_i^{\zeta_i}\right\}d\zeta_1\cdots d\zeta_r \tag{2.4}$$

where $\omega=\sqrt{-1}$,

$$\psi(\zeta_1,\ldots,\zeta_r) = \frac{\prod_{k=2}^{r}\left[\prod_{j=1}^{n_k}\Gamma(1-a_{kj}+\sum_{i=1}^{k}\alpha_{kj}^{(i)}\zeta_i)\right]}{\prod_{k=2}^{r}\left[\prod_{j=n_k+1}^{p_k}\Gamma(a_{kj}-\sum_{i=1}^{k}\alpha_{kj}^{(i)}\zeta_i)\right]} \times \frac{1}{\prod_{k=2}^{r}\left[\prod_{j=1}^{q_k}\Gamma(1-b_{kj}+\sum_{i=1}^{k}\beta_{kj}^{(i)}\zeta_i)\right]}, \tag{2.5}$$

$$\phi_i(\zeta_i) = \frac{\left[\prod_{k=1}^{m^{(i)}}\Gamma(b_k^{(i)}-\beta_k^{(i)}\zeta_i)\right]\left[\prod_{j=1}^{n^{(i)}}\Gamma(1-a_j^{(i)}+\alpha_j^{(i)}\zeta_i)\right]}{\left[\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma(a_j^{(i)}-\alpha_j^{(i)}\zeta_i)\right]\left[\prod_{k=m^{(i)}+1}^{q^{(i)}}\Gamma(1-b_k^{(i)}+\beta_k^{(i)}\zeta_i)\right]}, \tag{2.6}$$

$\forall\, i \in \{1,\ldots,r\}$. Also,

$$\begin{aligned}
&\{0,n_i\}_{2,r} && := 0,n_2:\cdots:0,n_r,\\
&\{p_i,q_i\}_{2,r} && := p_2,q_2:\cdots:p_r,q_r,\\
&\left\{\left(m^{(i)},n^{(i)}\right)\right\}^{1,r} && := (m^{(1)},n^{(1)}):\cdots:(m^{(r)},n^{(r)}),\\
&\left\{\left(p^{(i)},q^{(i)}\right)\right\}^{1,r} && := (p^{(1)},q^{(1)}):\cdots:(p^{(r)},q^{(r)}),\\
\mathcal{A} &:=: \left\{\left(a_{ij};\alpha_{ij}^{(1)},\ldots,\alpha_{ij}^{(i)}\right)_{1,p_i}^{2,r}\right\} && := (a_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)})_{1,p_2},(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)})_{1,p_3},\\
& && \quad \ldots,(a_{rj};\alpha_{rj}^{(1)},\ldots,\alpha_{rj}^{(r)})_{1,p_r},\\
\mathcal{B} &:=: \left\{\left(a_j^{(i)},\alpha_j^{(i)}\right)_{1,p^{(i)}}^{1,r}\right\} && := (a_j^{(1)},\alpha_j^{(1)})_{1,p^{(1)}},\ldots,(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}},\\
\mathcal{C} &:=: \left\{\left(b_{ij};\beta_{ij}^{(1)},\ldots,\beta_{ij}^{(i)}\right)_{1,q_i}^{2,r}\right\} && := (b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)})_{1,q_2},(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)})_{1,q_3},\\
& && \quad \ldots,(b_{rj};\beta_{rj}^{(1)},\ldots,\beta_{rj}^{(r)}))_{1,q_r},\\
\mathcal{D} &:=: \left\{\left(b_j^{(i)},\beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r}\right\} && := (b_j^{(1)},\beta_j^{(1)})_{1,q^{(1)}},\ldots,(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}}
\end{aligned}$$

such that $n_i, p_i, q_i, m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)}$ are non-negative integers and all $a_{ij}, b_{ij}, \alpha_{ij}$, $\beta_{ij}, a_j^{(i)}, b_j^{(i)}, \alpha_j^{(i)}, \beta_j^{(i)}$ are complex numbers and the empty product denotes unity.

The contour integral (2.4) converges, if

$$|\arg z_i| < \frac{1}{2}U_i\pi,\ U_i > 0,\ i = 1, \ldots, r, \tag{2.7}$$

where

$$\begin{aligned} U_i = &\sum_{j=1}^{n^{(i)}} \alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}} \alpha_j^{(i)} + \sum_{j=1}^{m^{(i)}} \beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}} \beta_j^{(i)} + \left(\sum_{j=1}^{n_2} \alpha_{2j}^{(i)} - \sum_{j=n_2+1}^{p_2} \alpha_{2j}^{(i)} \right) \\ &+ \cdots + \left(\sum_{j=1}^{n_r} \alpha_{rj}^{(i)} - \sum_{j=n_r+1}^{p_r} \alpha_{rj}^{(i)} \right) - \left(\sum_{j=1}^{q_2} \beta_{2j}^{(i)} + \cdots + \sum_{j=1}^{q_r} \beta_{rj}^{(i)} \right) \end{aligned} \tag{2.8}$$

and $I[z_1, \ldots, z_r] = O(|z_1|^{\alpha_1}, \ldots, |z_r|^{\alpha_r})$, $\max\{|z_1|, \ldots, |z_r|\} \to 0$, where $\alpha_i = \min_{1 \leq j \leq m^{(i)}} \Re(b_j^{(i)}/\beta_j^{(i)})$ and $\beta_i = \max_{1 \leq j \leq n^{(i)}} \Re((a_j^{(i)} - 1)/\alpha_j^{(i)})$; $i = 1, \ldots, r$.

For the conditions of convergence and analyticity of the multivariable I-function we refer to [2, 3].

Special case:

1. When $n_i = 0$; $i = 2, \ldots, r-1$ the I-function defined in (2.4) reduces to the multivariable H-function [1].

2. When $a_j^{(i)}, b_j^{(i)}, \alpha_j^{(i)}, \beta_j^{(i)}$'s are real numbers and $n_i = 0$; $i = 2, \ldots, r$, the multivariable I-function defined in (2.4) breaks up into a product of r H-functions.

Definition 2.4. The multidimensional analogue of a general class of polynomials $S_n^m(x)$ is defined by [1]

$$\begin{aligned} S_n^{m_1, \ldots, m_r}(x_1, \ldots, x_r) := &\sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \cdots + m_r k_r \leq n} (-n)_{m_1 k_1 + \cdots + m_r k_r} \\ &\times A(n; k_1, \ldots, k_r) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!}, \end{aligned} \tag{2.9}$$

where $m_1, \ldots, m_r$ are arbitrary positive integers, $n = 0, 1, 2, \ldots$ and the coefficients

$$A(n; k_1, \ldots, k_r),\ \ k_i \geq 0;\ i = 1, \ldots, r$$

are arbitrary constants, real or complex.

The order of the highest degree of the variables $x_1, \ldots, x_r$ of the multivariable polynomial (2.9) can be written as [9]

$$O\left(S_n^{m_1,\ldots,m_r}(x_1,\ldots,x_r)\right) = O\left(x_1^{[n/m_1]},\ldots,x_r^{[n/m_r]}\right), \tag{2.10}$$

where $[x]$ denotes the greatest integer $\leq x$.

Special case:

1. For $m_i = 1$, $i = 1, \ldots, r$ and $A(n; k_1, \ldots, k_r) = (1 + \alpha_1 + n)_{k_1}(1 + \alpha_2 + n_2)_{k_2} \cdots (1 + \alpha_r + n_r)_{k_r}$ the multivariable polynomial reduces to a multivariable Bessel polynomial [10].
2. For $m_i = 2, \sigma_i = 1\, i = 1, \ldots, r$, $A(n; k_1, \ldots, k_r) = 1$ and replacing $tx_1 \to \frac{1}{2(tx_1)^2}$, $tx_j \to \frac{tx_j}{2(tx_1)^2}$, $j = 2, \ldots, r$ the multivariable polynomial reduces to a multivariable Hermite polynomial [11].
3. For the case $r = 1$ of the multivariable polynomial (2.9) would give rise to the general class polynomials S_n^m introduce by Srivastava [12].

3 Main Results

In this section, we have evaluated certain multiple integrals involving the product of the multidimensional analogue of a general class of polynomials with the multivariable I-function.

Theorem 3.1. *Suppose the conditions (2.7) to be satisfied. The Multiple Mellin transform of the multidimensional analogue of a general class of polynomials (2.9) with the I-function of r variables (2.4) is defined as follows*

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^r t_i^{s_i-1} S_n^{m_1,\ldots,m_r}(x_1 t_1^{\sigma_1},\ldots,x_r t_r^{\sigma_r}) I[z_1 t_1^{\mu_1},\ldots,z_r t_r^{\mu_r}] dt_1 \cdots dt_r \\
&\qquad = \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r\leq n} (-n)_{m_1k_1+\cdots+m_rk_r} A(n;k_1,\ldots,k_r) \\
&\qquad\quad \times \prod_{i=1}^r \left\{ \frac{x_i^{k_i} z_i^{-\omega_i}}{k_i!\mu_i} \phi_i(-\omega_i) \right\} \psi(-\omega_1,\ldots,-\omega_r)
\end{aligned} \tag{3.1}$$

where $\mu_i > 0$, $\sigma_i > 0$, $n \geq 0$, $\omega_i = \frac{s_i+\sigma_i k_i}{\mu_i}$,

$$\Re\left\{s_i + \mu_i \min_{1\leq j\leq m^{(i)}} \left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right\} > 0,\ \Re\left\{s_i + \sigma_i[n/m_i] + \mu_i \min_{1\leq j\leq n^{(i)}} \left(\frac{1-a_j^{(i)}}{\alpha_j^{(i)}}\right)\right\} < 0,$$

$\psi(t_1,\ldots,t_r)$ *and* $\phi(t_i)$, $i = 1,\ldots,r$ *are given in (2.5) and (2.6) respectively.*

Proof. For the sake of convenience, take the left side of (3.1) as Ω, using the Definition 2.4 and changing the orders of integrations and summations, we get

$$\Omega = \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r\le n} (-n)_{m_1k_1+\cdots+m_rk_r} A(n;k_1,\ldots,k_r)$$
$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^r \left\{ \frac{x_i^{k_i}}{k_i!} t_i^{s_i+\sigma_i k_i-1} \right\} I[z_1 t_1^{\mu_1},\ldots,z_r t_r^{\mu_r}] dt_1\cdots dt_r. \tag{3.2}$$

Now express the formula (3.2) in terms of the multiple Mellin-Barnes contour integrals defined by Definition 2.3. We finally arrive at the right side of (3.1) after changing the orders of integrations and evaluating the integrals by the Mellin inversion theorem [13]. □

Corollary 3.2. *For $x_i = 0$, $i = 1,\ldots,r$, the formula* (2.4) *coincides with the result of [14].*

Corollary 3.3. *For $x_i = 0$, $i = 1,\ldots,r$ and $n_i = p_i = q_i = 0$, $i = 2,\ldots,r-1$, the formula* (2.4) *coincides with the result of [1].*

Theorem 3.4. *Suppose the conditions (2.7) to be satisfied. The Multiple Laplace transform of the multidimensional analogue of a general class of polynomials (2.9) with the I-function of r variables (2.4) is defined as follows*

$$\int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^r s_i t_i\right) \prod_{i=1}^r t_i^{\rho_i-1} S_n^{m_1,\ldots,m_r}(x_1 t_1^{\sigma_1},\ldots,x_r t_r^{\sigma_r})$$
$$\times I[z_1 t_1^{\mu_1},\ldots,z_r t_r^{\mu_r}] dt_1\cdots dt_r$$
$$= \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r\le n} (-n)_{m_1k_1+\cdots+m_rk_r} A(n;k_1,\ldots,k_r)$$
$$\times \prod_{i=1}^r \left\{ \frac{x_i^{k_i} s_i^{-\sigma_i k_i-\rho_i}}{k_i!} \right\} I^{\{0,n_i\}_{2,r}:\{(m^{(i)},n^{(i)}+1)\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\{(p^{(i)}+1,q^{(i)})\}^{1,r}} \left[\begin{matrix} z_1 s_1^{-\mu_1} \\ \vdots \\ z_r s_r^{-\mu_1} \end{matrix} \middle| \begin{matrix} \mathcal{A} : \mathcal{B} \\ \mathcal{C} : \mathcal{D} \end{matrix} \right] \tag{3.3}$$

where $\mu_i > 0, \Re(s_i) > 0$ *and* $\Re\left\{\rho_i + \sigma_i[n/m_i] + \mu_i \min_{1\le j\le m^{(i)}} \left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)\right\} > 0$, $i = 1,\ldots,r$,

$$\mathcal{A} \equiv \left\{ (a_{rj};\alpha_{rj}^{(1)},\ldots,\alpha_{rj}^{(r)})_{1,p_i}^{2,r} \right\}, \quad \mathcal{B} \equiv \left\{ (1-\rho_i-\sigma_i k_i,\mu_i)^{1,r}, \left(a_j^{(i)},\alpha_j^{(i)}\right)_{1,p^{(i)}}^{1,r} \right\},$$
$$\mathcal{C} \equiv \left\{ (b_{rj};\beta_{rj}^{(1)},\ldots,\beta_{rj}^{(r)})_{1,q_i}^{2,r} \right\}, \quad \mathcal{D} \equiv \left\{ \left(b_j^{(i)},\beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r} \right\}.$$

Proof. For the sake of convenience, take the left hand side of (3.3) as Δ, then use Definition 2.4 and change the orders of integrations and summations to get

$$\Delta = \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r\le n} (-n)_{m_1k_1+\cdots+m_rk_r}\, A(n;k_1,\cdots,k_r) \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^r s_i t_i\right)$$
$$\prod_{i=1}^r \left\{ \frac{x_i^{k_i}}{k_i!} t_i^{s_i+\sigma_i k_i -1} \right\} I[z_1 t_1^{\mu_1},\ldots,z_r t_r^{\mu_r}] dt_1\cdots dt_r. \tag{3.4}$$

Now express Formula (3.4) in terms of multiple Mellin-Barnes contour integrals in Definition 2.3 by changing the orders of integration, which is permissible under the stated conditions and an appeal to Euler's integral of the first kind.

$$\Delta = \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r\le n} (-n)_{m_1k_1+\cdots+m_rk_r}\, A(n;k_1,\ldots,k_r) \prod_{i=1}^r \left\{ \frac{x_i^{k_i}}{k_i!} \right\}$$
$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} \psi(\zeta_1,\ldots,\zeta_r) \prod_{i=1}^r \left\{ \phi_i(\zeta_i) z_i^{\zeta_i} \frac{\Gamma(\sigma_i k_i + \mu_i\zeta_i + \rho_i)}{s^{\sigma_i k_i + \mu_i \zeta_i + \rho_i}} \right\} d\zeta_1\cdots d\zeta_r. \tag{3.5}$$

Using Definition 2.3 and a little rearrangement we finally at arrive at the right hand side of (3.3). □

Corollary 3.5. *For $x_i = 0$, $i = 1,\ldots,r$, the Formula* (3.3) *coincides with the result of [14].*

Theorem 3.6. *Suppose the conditions (2.7) to be satisfied. The Multiple Laplace transform of the multidimensional analogue of a general class of polynomials (2.9) with the I-function of r variables (2.4) is defined as follows*

$$\int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^r s_i t_i\right) \prod_{i=1}^r t_i^{\rho_i -1} S_n^{m_1,\ldots,m_r}(x_1 t_1^{\sigma_1},\ldots,x_r t_r^{\sigma_r})$$
$$\times I[z_1 t_1^{-\mu_1},\ldots,z_r t_r^{-\mu_r}] dt_1\cdots dt_r$$
$$= \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r\le n} (-n)_{m_1k_1+\cdots+m_rk_r}\, A(n;k_1,\ldots,k_r)$$
$$\times \prod_{i=1}^r \left\{ \frac{x_i^{k_i} s_i^{-\sigma_i k_i - \rho_i}}{k_i!} \right\} I^{\{0,n_i\}_{2,r}:\left\{(m^{(i)}+1,n^{(i)})\right\}^{1,r}}_{\{p_i,q_i\}_{2,r}:\left\{(p^{(i)},q^{(i)}+1)\right\}^{1,r}} \left[\begin{array}{c|c} z_1 s_1^{\mu_1} & \\ \vdots & \mathcal{A}:\mathcal{B} \\ & \mathcal{C}:\mathcal{D} \\ z_r s_r^{\mu_1} & \end{array} \right] \tag{3.6}$$

where $\mu_i > 0, \Re(s_i) > 0$ *and* $\Re\left\{\rho_i + \sigma_i[n/m_i] + \mu_i \min_{1\le j\le m^{(i)}} \left(\frac{a_j^{(i)}-1}{\alpha_j^{(i)}}\right)\right\} >$

0, $i = 1, \ldots, r$,

$$\mathcal{A} \equiv \left\{ (a_{rj}; \alpha_{rj}^{(1)}, \ldots, \alpha_{rj}^{(r)})_{1,p_i}^{2,r} \right\}, \qquad \mathcal{B} \equiv \left\{ \left(a_j^{(i)}, \alpha_j^{(i)} \right)_{1,p^{(i)}}^{1,r} \right\},$$
$$\mathcal{C} \equiv \left\{ (b_{rj}; \beta_{rj}^{(1)}, \ldots, \beta_{rj}^{(r)})_{1,q_i}^{2,r} \right\}, \qquad \mathcal{D} \equiv \left\{ \left(b_j^{(i)}, \beta_j^{(i)} \right)_{1,q^{(i)}}^{1,r}, (\rho_i + \sigma_i k_i, \mu_i)^{1,r} \right\}.$$

Proof. The proof of this theorem as the same as Theorem 3.4. □

Corollary 3.7. *For $x_i = 0$, $i = 1, \ldots, r$, the Formula* (3.6) *coincides with the result of [14].*

Theorem 3.8. *Suppose the conditions (2.7) to be satisfied. The Multiple Mellin transform of the multidimensional analogue of a general class of polynomials (2.9) with the I-function of r variables (2.4) is defined as follows*

$$\begin{aligned} &\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^r t_i^{s_i - 1} S_n^{m_1, \ldots, m_r}(x_1 t_1^{\sigma_1}, \ldots, x_r t_r^{\sigma_r}) \\ &\qquad\qquad \times I[z_1 t_1^{\mu_1}, \ldots, z_r t_r^{\mu_r}] I'[\eta_1 t_1^{\nu_1}, \ldots, \eta_r t_r^{\nu_r}] dt_1 \cdots dt_r \\ &= \sum_{k_1, \ldots, k_r = 0}^{m_1 k_1 + \cdots + m_r k_r \leq n} (-n)_{m_1 k_1 + \cdots + m_r k_r} A(n; k_1, \ldots, k_r) \prod_{i=1}^r \left\{ \frac{x_i^{k_i} \eta_i^{-\omega_i}}{k_i! \nu_i} \right\} \\ &\quad \times I^{\{0,0\}_{2,r} : \left\{ (m^{(i)} + n'^{(i)}, m'^{(i)} + n^{(i)}) \right\}^{1,r}}_{\left\{ p_i + q_i', p_i' + q_i \right\}_{2,r} : \left\{ (p^{(i)} + q'^{(i)}, p'^{(i)} + q^{(i)}) \right\}^{1,r}} \left[\begin{matrix} z_1 \eta_1^{-\omega_1} \\ \vdots \\ z_r \eta_r^{-\omega_r} \end{matrix} \middle| \begin{matrix} \mathcal{A} : \mathcal{B} \\ \mathcal{C} : \mathcal{D} \end{matrix} \right] \end{aligned} \tag{3.7}$$

where $\omega_i = \frac{s_i + \sigma_i k_i}{\nu_i}$, $n_i = 0 = n_i'$, $\sigma_i \geq 0, \nu_i > 0, \mu_i > 0$.

$$\Re \left\{ s_i + \sigma_i [n/m_i] + \mu_i \min_{1 \leq j \leq m^{(i)}} \left(\frac{b_j^{(i)}}{\beta_j^{(i)}} \right) + \nu_i \min_{1 \leq j \leq m'^{(i)}} \left(\frac{b_j'^{(i)}}{\beta_j'^{(i)}} \right) \right\} > 0,$$

$$\Re \left\{ s_i + \sigma_i [n/m_i] - \mu_i \min_{1 \leq j \leq n^{(i)}} \left(\frac{1 - a_j^{(i)}}{\alpha_j^{(i)}} \right) - \nu_i \min_{1 \leq j \leq n'^{(i)}} \left(\frac{1 - a_j'^{(i)}}{\alpha_j'^{(i)}} \right) \right\} < 0,$$

$i = 1, \ldots, r$ *and*

$$\mathcal{A} \equiv \left\{ \left(a_{rj}; \alpha_{rj}^{(1)}, \ldots, \alpha_{rj}^{(r)} \right)_{1,p_i}^{2,r}, \left(1 - b_{rj}' - \sum_{k=1}^r \omega_k \beta_{rj}'^{(k)}; \frac{\mu_1}{\nu_1} \beta_{rj}'^{(1)}, \ldots, \frac{\mu_r}{\nu_r} \beta_{rj}'^{(r)} \right)_{1,q_i'}^{2,r} \right\},$$
$$\mathcal{B} \equiv \left\{ \left(a_j^{(i)}, \alpha_j^{(i)} \right)_{1,p^{(i)}}^{1,r}, \left(1 - b_j'^{(i)} - \omega_i \beta_j'^{(i)}, \frac{\mu_i}{\nu_i} \beta_j'^{(i)} \right)_{1,q'^{(i)}}^{1,r} \right\},$$

$$\mathcal{C}\equiv\left\{\left(b_{rj};\beta_{rj}^{(1)},\ldots,\beta_{rj}^{(r)}\right)_{1,q_i}^{2,r},\left(1-a'_{rj}-\sum_{k=1}^{r}\omega_k\alpha_{rj}^{'(k)};\frac{\mu_1}{\nu_1}\alpha_{rj}^{'(1)},\ldots,\frac{\mu_r}{\nu_r}\alpha_{rj}^{'(r)}\right)_{1,p'_i}^{2,r}\right\},$$

$$\mathcal{D}\equiv\left\{\left(b_j^{(i)},\beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r},\left(1-a_j^{'(i)}-\omega_i\alpha_j^{'(i)},\frac{\mu_i}{\nu_i}\alpha_j^{'(i)}\right)_{1,p'^{(i)}}^{1,r}\right\}.$$

Proof. For the sake of convenience, take the left hand side of (3.7) as Λ and express the first I-function in terms of multiple Mellin-Barnes contour integrals by using Denition 2.3 and changing the orders of integrations, which is permissible under the stated conditions. We obtain

$$\Lambda=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\zeta_1,\ldots,\zeta_r)\prod_{i=1}^{r}\left\{\phi_i(\zeta_i)z_i^{\zeta_i}\right\}\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{r}\left\{t_i^{s_i+\nu_i\zeta_i-1}\right\}$$
$$\times S_n^{m_1,\ldots,m_r}(x_1t_1^{\sigma_1},\ldots,x_rt_r^{\sigma_r})I'[\eta_1t_1^{\nu_1},\ldots,\eta_rt_r^{\nu_r}]dt_1\cdots dt_rd\zeta_1\cdots d\zeta_r \tag{3.8}$$

Using Theorem 3.1, after a straightforward calculation we finally arrive at formula (3.7). □

Corollary 3.9. *For $x_i=0$, $i=1,\ldots,r$, the Formula* (3.7) *coincides with the result of [14].*

Theorem 3.10. *Suppose the conditions (2.7) to be satisfied. The Multiple Mellin transform of the product of a general class of polynomials (2.9) with the I-function of r variables (2.4) is defined as follows*

$$\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{r}t_i^{s_i-1}S_n^{m_1,\ldots,m_r}(x_1t_1^{\sigma_1},\ldots,x_rt_r^{\sigma_r})$$
$$\times I[z_1t_1^{-\mu_1},\ldots,z_rt_r^{-\mu_r}]I'[\eta_1t_1^{\nu_1},\cdots,\eta_rt_r^{\nu_r}]dt_1\cdots dt_r$$
$$=\sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r\le n}(-n)_{m_1k_1+\cdots+m_rk_r}A(n;k_1,\ldots,k_r)\prod_{i=1}^{r}\left\{\frac{x_i^{k_i}\eta_i^{-\omega_i}}{k_i!\nu_i}\right\}$$
$$\times I^{\{0,0\}_{2,r}:\left\{(m^{(i)}+m'^{(i)},n'^{(i)}+n^{(i)})\right\}^{1,r}}_{\left\{p_i+p'_i,q'_i+q_i\right\}_{2,r}:\left\{(p^{(i)}+p'^{(i)},q'^{(i)}+q^{(i)})\right\}^{1,r}}\left[\begin{array}{c|c}z_1\eta_1^{\omega_1}&\mathcal{A}:\mathcal{B}\\ \vdots&\\ z_r\eta_r^{\omega_r}&\mathcal{C}:\mathcal{D}\end{array}\right] \tag{3.9}$$

where $\omega_i=\frac{s_i+\sigma_ik_i}{\nu_i}$, $n_i=0=n'_i,\sigma_i\ge0,\nu_i>0,\mu_i>0.$

$$\Re\left\{s_i+\sigma_i[n/m_i]+\mu_i\min_{1\le j\le n^{(i)}}\left(\frac{1-a_j^{(i)}}{\alpha_j^{(i)}}\right)+\nu_i\min_{1\le j\le m'^{(i)}}\left(\frac{b_j^{'(i)}}{\beta_j^{'(i)}}\right)\right\}>0,$$

$$\Re\left\{s_i+\sigma_i[n/m_i]-\mu_i\min_{1\le j\le m^{(i)}}\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)-\nu_i\min_{1\le j\le n'^{(i)}}\left(\frac{1-a_j^{'(i)}}{\alpha_j^{'(i)}}\right)\right\}<0,$$

$i = 1, \dots, r$ *and*

$$\mathcal{A} \equiv \left\{ \left(a_{rj}; \alpha_{rj}^{(1)}, \dots, \alpha_{rj}^{(r)}\right)_{1,p_i}^{2,r}, \left(a'_{rj} + \sum_{k=1}^{r} \omega_k \alpha_{rj}^{'(k)}; \frac{\mu_1}{\nu_1} \alpha_{rj}^{'(1)}, \dots, \frac{\mu_r}{\nu_r} \alpha_{rj}^{'(r)}\right)_{1,p'_i}^{2,r} \right\},$$

$$\mathcal{B} \equiv \left\{ \left(a_j^{(i)}, \alpha_j^{(i)}\right)_{1,p^{(i)}}^{1,r}, \left(a_j^{'(i)} + \omega_i \alpha_j^{'(i)}, \frac{\mu_i}{\nu_i} \alpha_j^{'(i)}\right)_{1,p'^{(i)}}^{1,r} \right\},$$

$$\mathcal{C} \equiv \left\{ \left(b_{rj}; \beta_{rj}^{(1)}, \dots, \beta_{rj}^{(r)}\right)_{1,q_i}^{2,r}, \left(b'_{rj} + \sum_{k=1}^{r} \omega_k \beta_{rj}^{'(k)}; \frac{\mu_1}{\nu_1} \beta_{rj}^{'(1)}, \dots, \frac{\mu_r}{\nu_r} \beta_{rj}^{'(r)}\right)_{1,q'_i}^{2,r} \right\},$$

$$\mathcal{D} \equiv \left\{ \left(b_j^{(i)}, \beta_j^{(i)}\right)_{1,q^{(i)}}^{1,r}, \left(b_j^{'(i)} + \omega_i \beta_j^{'(i)}, \frac{\mu_i}{\nu_i} \beta_j^{'(i)}\right)_{1,q'^{(i)}}^{1,r} \right\}.$$

Proof. The proof of this theorem as the same as Theorem 3.8. □

Corollary 3.11. *For $x_i = 0$, $i = 1, \dots, r$ then Formula* (3.9) *coincides with the result of [14].*

Corollary 3.12. *By putting $x_i = 0$, $i = 1, \dots, r$ and $n_i = p_i = q_i = 0$, $i = 2, \dots, r-1$ in the Formula* (3.9)*, we get the integral established by Garg [15].*

4 Application

In this section we establish the multiple Mellin convolution of the I-function transform. For the Mellin convolution, we shall first required the following special case-

On putting $x_i = s_i = 0$, $\mu_i = \nu_i = \eta_i = 1$, $i = 1, \dots, r$ in the Formula (3.9) we get

$$\begin{aligned} &\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} t_i^{-1} I[z_1 t_1^{-1}, \dots, z_r t_r^{-1}] I'[t_1, \dots, t_r] dt_1 \cdots dt_r \\ &\qquad = I^{\{0,0\}_{2,r} : \left\{(m^{(i)}+m'^{(i)}, n'^{(i)}+n^{(i)})\right\}^{1,r}}_{\left\{p_i+p'_i, q'_i+q_i\right\}_{2,r} : \left\{(p^{(i)}+p'^{(i)}, q'^{(i)}+q^{(i)})\right\}^{1,r}} \left[\begin{array}{c|c} z_1 & \mathcal{A} : \mathcal{B} \\ \vdots & \\ z_r & \mathcal{C} : \mathcal{D} \end{array} \right] \\ &\qquad = \mathbf{I}[z_1, \dots, z_r], \text{ (say)}, \end{aligned} \tag{4.1}$$

the $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and the validity conditions easily establish from formula (3.9).

The multidimensional I-function transform will be defined and represented in the following manner

$$\begin{aligned}
&\mathcal{I}\left\{f(t_1,\ldots,t_r);s_1,\ldots,s_r\right\}\\
&\qquad\equiv I^{\{0,0\}_{2,r};\left\{(m^{(i)},n^{(i)})\right\}^{1,r};\mathcal{A}:\mathcal{B}}_{\{p_i,q_i\}_{2,r}:\left\{(p^{(i)},q^{(i)})\right\}^{1,r};\mathcal{C}:\mathcal{D}}\left\{f(t_1,\ldots,t_r);s_1,\ldots,s_r\right\}\\
&\qquad=\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{r}t_i^{-1}I[s_1t_1,\ldots,s_rt_r]f(t_1,\ldots,t_r)dt_1\cdots dt_r, \qquad (4.2)
\end{aligned}$$

provided that the integral (4.2) is absolutely convergent.

Theorem 4.1. *Suppose that the multidimensional Mellin transform of* $\mathcal{I}$, $\mathcal{I}^{'}$, $\mathcal{I}^{''}$ *and* $(g*\cdots*h)$ *exists*

$$\begin{aligned}
&\mathcal{I}\left\{g(t_1,\ldots,t_r);s_1,\ldots,s_r\right\}*\cdots*\mathcal{I}^{'}\left\{h(t_1,\cdots,t_r);s_1,\ldots,s_r\right\}\\
&\qquad\qquad=\mathcal{I}^{''}\left\{(g*\cdots*h)(t_1,\ldots,t_r);s_1,\ldots,s_r\right\}, \qquad (4.3)
\end{aligned}$$

where $\mathcal{I}$, $\mathcal{I}^{'}$, $\mathcal{I}^{''}$ *are the multidimensional* I*-function transforms defined in* (4.2).

To prove the result (4.3), we have to take the multiple Mellin transform (2.1) on the left side and required the special case (4.1). A similar proof is already described in Srivastava and Buschman [16] and Garg [15]. We therefore omit the details.

5 Conclusion

The I-function of r variables defined by (2.4) in terms of the Mellin-Barnes type of basic integrals is most general in character and involves a number of special functions including the G-function and the H-function. These extended results may provide better multiple Mellin and Laplace transforms of a general class of polynomials together with certain special functions.

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