Thai Journal of Mathematics
Volume 10 (2012) Number 2 : 337–349
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

The Hybrid Method for Generalized Mixed Equilibrium Problems for an Infinite Family of Asymptotically Nonexpansive Mappings

Pornsak Yatakoat

Faculty of Liberal Arts and Science
Nakhon Phanom University, Nakhon Phanom 48000, Thailand
e-mail : p_yatakoat@npu.ac.th

Abstract : In this paper, we introduce a hybrid method for finding a common element of the set of common fixed points for an infinite family of asymptotically nonexpansive mappings and the set of solutions of a generalized mixed equilibrium problem in Hilbert spaces. The results obtained in this paper improve and extend the recently corresponding results.

Keywords : Generalized mixed equilibrium problem; Asymptotically nonexpansive mapping.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction

Let C be a closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of a real numbers, $A : C \to H$ a mapping and $\varphi : C \to \mathbb{R}$ a real-valued function. The *generalized mixed equilibrium problem* is for finding $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by GMEP(F, φ, A), that is,

$$GMEP(F, \varphi, A) = \{x \in C : F(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C\}. \tag{1.2}$$

If $F \equiv 0$, the problem (1.1) is reduced into the *mixed variational inequality of Browder type* [1], for finding $x \in C$ such that

$$\langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by MVI(C, φ, A).

If $A \equiv 0$ and $\varphi \equiv 0$, the problem (1.1) is reduced into the *equilibrium problem* [2] for finding $x \in C$ such that

$$F(x, y) \geq 0, \ \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by EP(F). This problem contains fixed point problems and includes as special cases numerous problems in physics, optimization, and economics. Some methods have been proposed to solve the equilibrium problem; see [3–5].

If $F \equiv 0$ and $\varphi \equiv 0$, the problem (1.1) is reduced into the *Harmann-Stampacchia variational inequility* [6] for finding $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \ \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by VI(C, A). The variational inequality has been extensively studied in the literature [7].

If $F \equiv 0$ and $A \equiv 0$, the problem (1.1) is reduced into the *minimize problem* for finding $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \ \forall y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted by Arg $min(\varphi)$.

Recall that a mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \ \forall x, y \in C. \tag{1.7}$$

A mapping A of C into H is called α-inverse strongly monotone, see [8–10], if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \ \forall x, y \in C. \tag{1.8}$$

It is obvious that any α-inverse strongly monotone mapping A is monotone and Lipschitz continuous.

Let C be a nonempty closed convex subset of a real Hilbert space H, $S : C \to C$ be a mapping. We denote $F(S)$ to be the set of fixed points of S, i.e. $F(S) = \{x \in C : x = Sx\}$. A mapping S is said to be

(i) *nonexpansive*, if $\|Sx - Sy\| \leq \|x - y\| \ \forall x, y \in C$;

(ii) *asymptotically nonexpansive*, if there exist a sequence $k_n \geq 1$ such that $\lim_{n\to\infty} k_n = 1$ and

$$\|S^n x - S^n y\| \leq k_n\|x - y\|, \ \forall x, y \in C, \ n \geq 1; \tag{1.9}$$

(iii) *uniformly L-Lipschitzian*, if there exist a constant $L > 0$ such that

$$\|S^n x - S^n y\| \leq L\|x - y\|, \ \forall x, y \in C, \ n \geq 1; \tag{1.10}$$

In 2003, Nakajo and Takahashi [11] proposed the following modification of the Mann iteration method for a nonexpansive mapping T in a Hilbert space H:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily,} \\ y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n &= \{v \in C : \|y_n - v\| \leq \|x_n - v\|\}, \\ Q_n &= \{v \in C : \langle x_n - v, x_0 - x_n \rangle \ \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \end{aligned} \tag{1.11}$$

where P_C is denoted the metric projection from H onto a closed and convex subset C of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ is defined by (1.11) converges strongly to $P_{F(T)} x_0$.

Inchan and Plubtieng [12] introduced the modified Ishikawa iteration process by shrinking hybrid method [13] for two asymototically nonexpansive mappings S and T, with a closed convex bounded subset C of a Hilbert space H. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, $\{x_n\}$ is defined as follows:

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T^n z_n, \\ z_n &= \beta_n x_n + (1 - \beta_n) S^n x_n, \\ C_{n+1} &= \{v \in C_n : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \ n \in \mathbb{N} \end{aligned} \tag{1.12}$$

where $\theta_n = (1 - \alpha_n)[(t_n^2 - 1) + (1 - \beta_n) t_n^2 (s_n^2 - 1)](diamC)^2 \to 0$, as $n \to \infty$ and $0 \leq \alpha_n \leq \alpha < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$. They proved that the sequence $\{x_n\}$ is generated by (1.12) converges strongly to a common fixed point of two asymptotically nonexpansive mappings S and T.

The purpose of this paper is to introduce the Mann iteration process for finding a common element of the set of common fixed points of an infinite family of asymptotically nonexpansive mappings and the set of solutions of a generalized mixed equilibrium problem under some control conditions. We prove that the strong convergence theorem which extends and improves the result of many others [11, 12].

2 Preliminaries

In this section, we present some useful lemmas which will be used in our main result and we will use the notation:

- $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
- $\omega_\omega(x_n) = \{x : \ x_{n_i} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

- $d(x, C) = \inf_{z \in C} \|x - z\|$.

Let H be a real Hilbert space. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \ \forall\ x, y \in H. \tag{2.1}$$

For each $x, y \in H$ and $\lambda \in \mathbb{R}$, we known that

$$\|\lambda x - (1 - \lambda)y\|^2 = \lambda\|x\|^2 - (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|. \tag{2.2}$$

Let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C, then

$$\|P_C x - P_C y\|^2 \leq \|x - y\|^2 - \|(I - P_C)x - (I - P_C)y\|^2,\ \forall x, y \in H, \tag{2.3}$$

where I is the identity mapping.

Lemma 2.1 (Opial's condition [14]). *For any sequence $\{x_n\}$ in a Hilbert space H with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \tag{2.4}$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.2 (The Kadec-Klee property [15]). *For any sequence $\{x_n\}$ in a Hilbert space H with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.*

Lemma 2.3 (Demiclosedness Principle [16]). *Suppose X is a Banach space satisfying the locally uniform Opial's condition, C is a nonempty weakly compact convex subset of X, and $T : C \to C$ is an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero, i.e. if $\{x_n\}$ is a sequence in C which converge weakly to x and if the sequence $\{x_n - Tx_n\}$ converge strongly to zero, then $x - Tx = 0$.*

Lemma 2.4 ([17]). *Let C be a nonempty closed convex subset of H and also give a real number $a \in \mathbb{R}$. The set $D = \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$ is convex and closed.*

Lemma 2.5 ([18]). *Assume that $\{a_n\}$ is sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,\ \forall n \geq 1, \tag{2.5}$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ is sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

For solving the generalized mixed equilibrium problem, let us assume that the bifunction $F : C \times C \to \mathbb{R}$, a continuous monotone $A : C \to H$, and $\varphi : C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for any $x, y \in C$;

(A3) For each fixed $y \in C$, $x \mapsto F(x,y)$ is weakly upper semicontinuous;

(A4) For each fixed $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous;

(B1) For each $x \in C$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that, for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{2.6}$$

(B2) C is a bounded set.

Lemma 2.6 ([19]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1) − (A4), and let $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be convex and proper lower semicontinuous function such that $C \cap dom\varphi \neq \emptyset$. For $r > 0$ and $x \in H$, define a mapping $K_r : H \to C$ as follows:*

$$K_r(x) = \left\{ u \in C : F(u,y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \;\geq 0,\; \forall y \in C \right\} \tag{2.7}$$

for all $x \in H$, Assume that either (B_1) or (B_2) holds. Then, the following hold:

(i) K_r *is single valued;*

(ii) K_r *is firmly nonexpansive, that is, $\|K_r x - K_r y\|^2 \leq \langle K_r x - K_r y, x - y\rangle$ for any $x, y \in H$;*

(iii) $F(K_r) = MEP(F, \varphi)$*;*

(iv) $MEP(F, \varphi)$ *is closed and convex.*

Definition 2.7 ([20]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $\{S_n\}$ be a family of asymptotically nonexpansive mappings of C into itself, and let $\{\beta_{n,k} : n, k \in \mathbb{N},\ 1 \leq k \leq n\}$ be a real sequence of real numbers such that $0 \leq \beta_{i,j} \leq 1$ for every $i, j \in \mathbb{N}$ with $i \geq j$. For any $n \geq 1$, define a mapping $W_n : C \to C$ as follows:*

$$\begin{aligned}
U_{n,n} &= \beta_{n,n} S_n^n + (1 - \beta_{n,n}) I, \\
U_{n,n-1} &= \beta_{n,n-1} S_{n-1}^n U_{n,n} + (1 - \beta_{n,n-1}) I, \\
&\vdots \\
U_{n,k} &= \beta_{n,k} S_k^n U_{n,k+1} + (1 - \beta_{n,k}) I, \\
&\vdots \\
U_{n,2} &= \beta_{n,2} S_2^n U_{n,3} + (1 - \beta_{n,2}) I, \\
W_n = U_{n,1} &= \beta_{n,1} S_1^n U_{n,2} + (1 - \beta_{n,1}) I.
\end{aligned} \tag{2.8}$$

Such a mapping W_n is called the modified W-mapping generated by $S_n, S_{n-1}, ..., S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,2}, \beta_{n,1}$.

Lemma 2.8 ([21]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_m\}$ be a family of asymptotically nonexpansive mappings of C into itself with Lipschitz constants $\{t_{m,n}\}$, that is, $\|S_m^n x - S_m^n y\| \leq t_{m,n}\|x - y\|$, $(\forall m, n \in \mathbb{N},\ \forall x, y \in C)$ such that $F = \cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, and let $\{\beta_{n,k} : n, k \in \mathbb{N},\ 1 \leq k \leq n\}$ be a sequence of real numbers with $0 < a \leq \beta_{n,1} \leq 1$ for all $n \in \mathbb{N}$ and $0 < b \leq \beta_{n,i} \leq c < 1$ for every $n \in \mathbb{N}$ and $i = 2, \dots, n$ for some $a, b, c \in (0, 1)$. Let W_n be the modified W-mappings generated by $S_n, S_{n-1}, \dots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \dots, \beta_{n,2}, \beta_{n,1}$. Let $r_{n,k} = \{\beta_{n,k}(t_{k,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}t_{k,n}^2(t_{k+1,n}^2 - 1) + \cdots + \beta_{n,k}\beta_{n,k+1}\dots\beta_{n,n-1}t_{k,n}^1 t_{k+1,n}^2 \cdots t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-2,n}^2(t_{n-1,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1} \dots \beta_{n,n}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-1,n}^2(t_{n,n}^2)\}$ for every $n \in \mathbb{N}$ and $k = 1, 2, \dots, n$. Then, the followings hold:*

(i) $\|W_n x - z\|^2 \leq (1 + r_{n,1})\|x - z\|^2$ *for all* $n \in \mathbb{N},\ x \in C$ *and* $z \in \cap_{i=1}^{n} F(S_i)$;

(ii) if C is bounded and $\lim_{n\to\infty} r_{n,1} = 0$ for every sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0,\ \lim_{n\to\infty} \|z_n - W_n z_n\| = 0 \ \ \textit{imply}\ \omega_\omega(z_n) \subset F; \quad (2.9)$$

(iii) if $\lim_{n\to\infty} r_{n,1} = 0$, $F = \cap_{n=1}^{\infty} F(W_n)$ and F is closed convex.

3 Main Results

In this section, we prove a strong convergence for the set of common fixed points of an infinite family of asymptotically nonexpansive mappings and the set of solutions of a generalized mixed equilibrium problem in Hilbert space.

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H, let $F : C \times C \to \mathbb{R}$ be a bifunction, $A : C \to H$ be an $\alpha-$inverse strongly monotone, and $\varphi : C \to \mathbb{R}$ be a convex and lower semicontinuous function, satisfying the conditions $(A1) - (A4)$, $(B1)$ or $(B2)$, let $\{S_m\}$ be a family of asymptotically nonexpansive mappings of C into itself with Lipschitz constants $\{t_{m,n}\}$, that is, $\|S_m^n x - S_m^n y\| \leq t_{m,n}\|x - y\|$, $\forall m, n \in \mathbb{N}$, $\forall x, y \in C$ such that $F \cap GMEP \neq \emptyset$, where $F = \cap_{i=1}^{\infty} F(S_i)$, and let $\{\beta_{n,k} : n, k \in \mathbb{N}, 1 \leq k \leq n\}$ be a sequence of real numbers with $0 < a \leq \beta_{n,1} \leq 1$ for all $n \in \mathbb{N}$ and $0 < b \leq \beta_{n,i} \leq c < 1$ for every $n \in \mathbb{N}$ and $2 \leq i \leq n$ for some $a, b, c \in (0, 1)$. Let W_n be the modified W-mapping is generated by $S_n, S_{n-1}, \dots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \dots, \beta_{n,2}, \beta_{n,1}$. Assume that $r_{n,k} = \{\beta_{n,k}(t_{k,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}t_{k,n}^2(t_{k+1,n}^2 - 1) + \cdots + \beta_{n,k}\beta_{n,k+1} \cdots \beta_{n,n-1}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-2,n}^2(t_{n-1,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1} \cdots \beta_{n,n}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-1,n}^2(t_{n,n}^2)\}$, $\forall n \in \mathbb{N}$ and $k = 1, 2, \dots, n$, such that $\lim_{n\to\infty} r_{n,1} = 0$. Let $\{x_n\}$ and $\{u_n\}$ be sequences is* *generated by the following algorithm:*

$$\begin{aligned}
&x_1 \in C \quad \textit{chosen arbitrarily},\\
&u_n \in C,\\
&F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \tfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0,\ \forall y \in C,\\
&y_n = \alpha_n u_n + (1-\alpha_n) W_n u_n, \qquad (3.1)\\
&C_{n+1} = \{v \in C_n : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\},\\
&x_{n+1} = P_{C_{n+1}} x_1, \quad n \in \mathbb{N},
\end{aligned}$$

where $C_1 = C$ *and* $\theta_n = (1-\alpha_n) r_{n,1} (diamC)^2$ *and* $0 \leq \alpha_n \leq d < 1$ *and* $0 < e \leq r_n \leq f < 2\alpha$. *Then* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $P_{F\cap GMEP}(x_1)$.

Proof. We split the proof into 4 steps.
Step 1. Show that the sequences $\{x_n\}$ and $\{y_n\}$ are well defined.

By Lemma 2.4, we have that C_n is closed and convex. Let $x, y \in C$. Since A is $\alpha-$inverse strongly monotone and $r_n < 2\alpha,\ \forall \in \mathbb{N}$, we have

$$\begin{aligned}
\|(I - r_n A)x - (I - r_n A)y\|^2 &= \|x - y - r_n(Ax - Ay)\|^2\\
&= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2\\
&\leq \|x - y\|^2 - 2\alpha r_n\|Ax - Ay\|^2 + r_n^2\|Ax - Ay\|^2\\
&= \|x - y\|^2 + r_n(r_n - 2\alpha)\|Ax - Ay\|^2 \qquad (3.2)\\
&\leq \|x - y\|^2.
\end{aligned}$$

Thus $I - r_n A$ is nonexpansive. Since

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0,\ \forall y \in C, \quad (3.3)$$

we obtain

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (I - r_n A)x_n\rangle \geq 0,\ \forall y \in C. \quad (3.4)$$

It follows from Lemma 2.6 that $u_n = K_{r_n}(x_n - r_n A x_n)$, for all $n \in \mathbb{N}$.

Let $p \in F \cup GMEP$, by Lemma 2.6, we have $p = K_{r_n}(p - r_n Ap)$, for all $n \in \mathbb{N}$. Since $I - r_n A$ and K_{r_n} are nonexpansive, we have

$$\|u_n - p\| \leq \|K_{r_n}(x_n - r_n A x_n) - K_{r_n}(p - r_n Ap)\| \leq \|x_n - p\|,\ \forall n \in \mathbb{N}. \quad (3.5)$$

By Lemma 2.8 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(u_n - p) + (1-\alpha_n)(W_n u_n - p)\|^2\\
&\leq \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|W_n u_n - p\|^2\\
&\leq \alpha_n\|u_n - p\|^2 + (1-\alpha_n)(1 + r_{n,1})\|u_n - p\|^2 \qquad (3.6)\\
&= \|u_n - p\|^2 + (1-\alpha_n) r_{n,1}\|u_n - p\|^2\\
&\leq \|u_n - p\|^2 + \theta_n\\
&\leq \|x_n - p\|^2 + \theta_n.
\end{aligned}$$

So, $p \in C_n$ for all n and $F \cup GMEP \subset C_n$ for all n. This implies that $\{x_n\}$ is well defined and by Lemma 2.6, we have that $\{u_n\}$ is also well defined.

Step 2. We show that $\|x_{n+1} - x_n\| \to 0,\ \|x_n - u_n\| \to 0,\ \|u_{n+1} - u_n\| \to 0,\ \|u_n - W_n u_n\| \to 0$. as $n \to \infty$. From $x_n = P_{C_n} x_1$, we have that

$$\langle x_1 - x_n, x_n - v\rangle \geq 0, \quad \text{for each} \ \ v \in F \cap GMEP \subset C_n,\ n \in \mathbb{N}. \tag{3.7}$$

So, for $p \in F \cap GMEP$, we have

$$\begin{aligned} 0 \ &\leq \langle x_1 - x_n, x_n - p\rangle = -\langle x_n - x_1, x_n - x_1\rangle + \langle x_1 - x_n, x_1 - p\rangle \\ &\leq \|x_n - x_1\|^2 + \|x_n - x_1\|\|x_1 - p\|. \end{aligned} \tag{3.8}$$

This implies that

$$\|x_n - x_1\|^2 \leq \|x_n - x_1\|\|x_1 - p\|, \tag{3.9}$$

and hence

$$\|x_n - x_1\| \leq \|x_1 - p\|. \tag{3.10}$$

Since C is bounded, then $\{x_n\}$ and $\{u_n\}$ are bounded. From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have

$$\langle x_1 - x_n, x_n - x_{n+1}\rangle \ \geq 0, \quad \forall n \in \mathbb{N}. \tag{3.11}$$

So,

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1}\rangle = -\langle x_n - x_1, x_n - x_1\rangle + \langle x_1 - x_n, x_1 - x_{n+1}\rangle \\ &\leq -\|x_n - x_1\|^2 + \|x_n - x_1\|\|x_1 - x_{n+1}\|. \end{aligned} \tag{3.12}$$

This implies that

$$\|x_n - x_1\| \leq \|x_1 - x_{n+1}\|, \quad \forall n \in \mathbb{N}. \tag{3.13}$$

Hence, $\{\|x_n - x_1\|\}$ is nondecreasing, it follows that $\lim_{n\to\infty} \|x_n - x_1\|$ exists. From (2.1) and (3.11), we have

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_1) - (x_n - x_1)\|^2 \\ &= \|(x_{n+1}-x_1)\|^2 - \|x_n-x_1\|^2 - 2\langle x_{n+1}-x_n, x_n-x_1\rangle \\ &\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2. \end{aligned} \tag{3.14}$$

Since $\lim_{n\to\infty} \|x_n - x_1\|$ exists, we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.15}$$

On the other hand, it follows from $x_{n+1} \in C_{n+1}$ that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n \ \to 0, \quad \text{as } n \to \infty. \tag{3.16}$$

It follows that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad \text{as } n \to \infty. \tag{3.17}$$

Next, we claim that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Let $p \in F \cap GMEP$, it follows from (3.6) that

$$\begin{aligned} \|y_n - p\|^2 &\leq \|u_n - p\|^2 + \theta_n \\ &= \|K_{r_n}(I - r_n A)x_n - K_{r_n}(I - r_n A)p\|^2 + \theta_n \\ &\leq \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 + \theta_n. \end{aligned} \tag{3.18}$$

This implies that

$$\begin{aligned} e(2\alpha - f)\|Ax_n - Ap\|^2 &\leq \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n \\ &\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \theta_n. \end{aligned} \tag{3.19}$$

It follows from (3.17) that

$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0. \tag{3.20}$$

From Lemma 2.6, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|K_{r_n}(I - r_n A)x_n - K_{r_n}(I - r_n A)p\|^2 \\ &\leq \langle (x_n - r_n Ax_n) - (p - r_n Ap), u_n - p\rangle \\ &= \frac{1}{2}\{\|x_n - r_n Ax_n - (p - r_n Ap)\|^2 + \|u_n - p\|^2 \\ &\quad - \|x_n - r_n Ax_n - (p - r_n Ap) - (u_n - p)\|^2\} \\ &\leq \frac{1}{2}\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r_n(Ax_n - Ap)\|^2\} \\ &= \frac{1}{2}\{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n\langle x_n - u_n, Ax_n - Ap\rangle - r_n^2\|Ax_n - Ap\|^2\}. \end{aligned} \tag{3.21}$$

This implies that

$$\begin{aligned} \|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Ap\rangle - r_n^2\|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Ap\rangle \\ &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\|. \end{aligned} \tag{3.22}$$

By (3.21) and (3.22), we obtain

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\| + \theta_n, \tag{3.23}$$

which implies that

$$\begin{aligned} \|x_n - u_n\|^2 &\leq \|x_n - p\|^2 - \|y_n - p\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\| + \theta_n \\ &\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + 2r_n\|x_n - u_n\|\|Ax_n - Ap\| + \theta_n. \end{aligned} \tag{3.24}$$

This implies by (3.17) and (3.24) that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.25}$$

From (3.15) and (3.25), we have

$$\|u_n - u_{n+1}\| \le \|u_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - u_{n+1}\| \to 0, \text{ as } n \to \infty. \tag{3.26}$$

Similarly, from (3.17) and (3.25), we have

$$\|y_n - u_n\| \le \|y_n - x_n\| + \|x_n - u_n\| \to 0, \text{ as } n \to \infty. \tag{3.27}$$

Since

$$(1-\alpha_n)\|w_n u_n - u_n\| = \|y_n - u_n\|, \tag{3.28}$$

it implies by $0 \le \alpha_n \le d < 1$ that

$$\|W_n u_n - u_n\| = \frac{\|y_n - u_n\|}{1-\alpha_n} < \frac{\|y_n - u_n\|}{1-d} \to 0, \text{ as } n \to \infty. \tag{3.29}$$

Step 3. We show that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converge weakly to z, where $z \in F \cap GMEP$.

Since $\{x_n\}$ is bounded and C is closed, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $z \in C$. From (3.25), It follows by (3.26), (3.29) and Lemma 2.8 that $z \in F$. Next, we prove that $z \in GMEP$. Indeed, we observe that $u_n = K_{r_n}(x_n - r_n A x_n)$ and

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \ \forall y \in C. \tag{3.30}$$

By (A_2), we get

$$\varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n). \tag{3.31}$$

Replacing n by n_i, we obtain

$$\varphi(y) - \varphi(u_{n_i}) + \langle Ax_n, y - u_n\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \ge F(y, u_{n_i}). \tag{3.32}$$

Put $z_t = ty + (1-t)z$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So, we have

$$\begin{aligned}\langle z_t - u_{n_i}, Az_t\rangle &\ge \langle z_t - u_{n_i}, Az_t\rangle - \langle Ax_n, z_t - u_{n_i}\rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \\ &\quad + F(z_t, u_{n_i}) - \varphi(z_t) + \varphi(u_{n_i}) \\ &= \langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle + \langle z_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\ &\quad - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{i_n}}\right\rangle + F(z_t, u_{n_i}) - \varphi(z_t) + \varphi(u_{n_i}).\end{aligned} \tag{3.33}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Au_{n_i} - Ax_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle \geq 0$. So, by (A_4) and the weakly lower semicontinuity of φ, we have

$$\langle z_t - z, Az_t\rangle \geq F(z_t, z) - \varphi(z_t) + \varphi(z). \tag{3.34}$$

It follows by (A_1) and (A_4) that

$$\begin{aligned} 0 &= F(z_t, z_t) - \varphi(z_t) + \varphi(z_t) \\ &\leq tF(z_t, y) + (1-t)F(z_t, z) + t\varphi(y) + (1-t)\varphi(z) - \varphi(z_t) \\ &= t\big(F(z_t, y) + \varphi(y) - \varphi(z_t)\big) + (1-t)\big(F(z_t, z) + \varphi(z) - \varphi(z_t)\big) \\ &\leq t\big(F(z_t, y) + \varphi(y) - \varphi(z_t)\big) + (1-t)\langle z_t - z, Az_t\rangle \\ &= t\big(F(z_t, y) + \varphi(y) - \varphi(z_t)\big) + (1-t)t\langle y - z, Az_t\rangle \end{aligned} \tag{3.35}$$

and hence

$$0 \leq F(z_t, y) + \varphi(y) - \varphi(z_t) + (1-t)\langle y - z, Az_t\rangle. \tag{3.36}$$

Letting $t \to 0$, we have, for each $y \in C$, that

$$0 \leq F(z, y) + \varphi(y) - \varphi(z) + \langle y - z, Az\rangle. \tag{3.37}$$

This implies that $z \in GMEP$.

Step 4. We prove that $x_n \to z,\ u_n \to z$, where $z = P_{F\cap GMEP}x_1$.

Putting $z' = P_{F\cap GMEP}x_1$ and consider the sequence $\{x_1 - x_{n_i}\}$. Then we have $x_1 - x_{n_i} \rightharpoonup x_1 - z$ and by the weak lower semicontinuity of the norm and $\|x_1 - x_{n+1}\| \leq \|x_1 - z'\|$ for all $n \in \mathbb{N}$ which is implied by the fact that $x_{n+1} = P_{C_{n+1}}x_1$, we have

$$\begin{aligned} \|x_1 - z'\| &\leq \|x_1 - z\| \\ &\leq \liminf_{n\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{n\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \|x_1 - z'\|. \end{aligned} \tag{3.38}$$

This implies that $\|x_1 - z'\| = \|x_1 - z\|$. By the uniqueness of the nearest point projection of x_1 onto $F \cap GMEP$ that

$$\|x_1 - x_{n_i}\| \to \|x_1 - z'\|. \tag{3.39}$$

This implies that $x_{n_i} \to z'$. Since $\{x_n\}$ is an arbitrary sequence of C, we can conclude that $x_n \to z'$. By (3.25), we have that $u_n \to z'$ also. This proof is completed. □

Acknowledgements : I would like to thank the referees for his comments and suggestions on the manuscript. This work was supported by the Faculty of Liberal Arts and Science, Nakhon Phanom University, Nakhon Phanom 48000, Thailand.

References

[1] F.E. Browder, Existence and approximation of solutions of nonlinear variational inequalities, Proceedings of the Nation Academy of Science of the United States of America 56 (1966) 1080–1086.

[2] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, The Mathematics Student 63 (1-4) (1994) 123–145.

[3] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, Journal of Nonlinear and Convex Analysis 6 (1) (2005) 117–136.

[4] S.D. Flam, A.S. Antipin, Equilibrium programming using proximal-like algorithms, Mathematical Programming 78 (1) (1997) 29–41.

[5] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problem in Hilbert spaces, J. Math. Anal. Appl. 331 (1) (2007) 506–515.

[6] P. Hartman, G. Stampacchia, On some non-linear elliptic differential-functional equations, Acta Mathematica 115 (1966) 271–310.

[7] J.-C. Yoa, O. Chadli, Pseudomonotone complementarity problems and variational inequalities, Handbook of Generalized Convexity and Generalized Monotonicity 76 (2005) 501–558.

[8] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (1967) 197–228.

[9] F. Liu, M.Z. Nashed, Regularization of nonlinear ill-posed variational inequalities and convergence rates, Set-Valued Analysis 6 (1998) 313–344.

[10] K. Nakajo, W. Takahashi, Strong and weak convergence theorems by an improved splitting method, Communications on Applied Nonlinear Analysis 9 (2002) 99–107.

[11] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003) 372–379.

[12] I. Inchan, S. Plubtieng, Strong convergence theorems of hybrid methods for two asymptotically nonexpansive mappings in Hilbert spaces, Nonlinear Anal. : Hybrid System 2 (2008) 1125–1135.

[13] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008) 276–286.

[14] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.

[15] K. Goebel, W.A. Kirk, Topics in Metrics Fixed Point Theory, Cambridge Studies in Advanced Mathematics, 28 (1990).

[16] P.K. Lin, K.K. Tan, H.K. Xu, Demiclosedness Principle and Asymptotic behavior for Asymptotically nonexpansive mappings, Nonlinear Anal. 24 (1995) 929–946.

[17] C. Martinez-Yanes, H.K. Xu, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal. 64 (2006) 2400–2411.

[18] H.K. Xu, Iterative algorithm for nonlinear operators, Journal of the London Mathematical Society: Second Series 66 (2002) 240–256.

[19] J.W. Peng, Y.C. Liou, J.C. Yao, An iterative algorithm combining viscosity method with parallel method for a generalized equilibrium problem and strict pseudocontractions, Fixed Point Theory and Applications 2009, 2009:794178.

[20] W. Takahashi, Weak and strong convergence theorems for families of nonexpansive mappings and thier applications, Annales Universitatis Mariae Curie-Sklodowska: Section A 51 (2) (1997) 277–292.

[21] K. Nakajo, K. Shimoji, W. Takahashi, On strong convergence by the hybrid method for families of mappings in Hilbert spaces, Nonlinear Anal. 71 (2009) 112–119.

(Received 2 March 2012)
(Accepted 14 May 2012)