Thai Journal of Mathematics
Volume 10 (2012) Number 2 : 321–336
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Certain Non-Linear Differential Polynomials Sharing 1-Points With Finite Weight

Abhijit Banerjee[1] and Sujoy Majumder

Department of Mathematics, West Bengal State University
Barasat, 24 Prgs. (N), Kolkata 700126, West Bengal, India
e-mail : abanerjee_kal@yahoo.co.in (A. Banerjee)
sujoy.katwa@gmail.com (S. Majumder)

Abstract : We employ the notion of weighted sharing of values to study the uniqueness of meromorphic functions when certain non-linear differential polynomials share the same 1-points. As a consequence of the main result we improve and supplement a result of Lahiri-Sarkar [I. Lahiri, A. Sarkar, Nonlinear differential polynomials sharing 1-points with weight two, Chinese J. Contemp. Math. 25 (3) (2004) 325–334] as well as a recent result of Zhang-Lin [X.Y. Zhang, W.C. Lin, Uniqueness and value sharing of entire functions, J. Math. Anal. Appl. 343 (2008) 938–950].

Keywords : Meromorphic function; Uniqueness; Non-linear differential polynomials; Weighted sharing.
2010 Mathematics Subject Classification : 30D35.

1 Introduction, Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring

[1]Corresponding author.

multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [1]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

During the last quarter of century or so a widely studied topic of the uniqueness theory has been to considering the shared value problems of different nonlinear differential polynomials and the uniqueness of their corresponding generating meromorphic functions and naturally a substantial number of authors have worked in this aspect (see [2–13]). In [14] Lahiri studied the problem of uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points. In the same paper regarding the nonlinear differential polynomials Lahiri [14] asked the following question.

What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 *CM?*

In 2001, Fang and Hong [6] proved the following result.

Theorem A. *Let f and g be two transcendental entire functions and $n(\geq 11)$ be an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share 1 CM, then $f \equiv g$.*

The above result created a lot of impulse among the researchers. In 2002, Fang and Fang [5] improved the above theorem by proving the following theorem.

Theorem B. *Let f and g be two non-constant entire functions and $n(\geq 8)$ be an integer . If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share 1 CM, then $f \equiv g$.*

In 2004, Lin and Yi [12] further improved *Theorem B* as follows.

Theorem C. *Let f and g be two transcendental entire functions and $n(\geq 7)$ be an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share 1 CM, then $f \equiv g$.*

In the same paper to investigate the uniqueness of meromorphic functions corresponding to the value sharing of their non linear differential polynomials Lin and Yi [12] proved the following result.

Theorem D. *Let f and g be two non-constant meromorphic functions and $n(\geq 13)$ be an integer. If $f^n(f-1)^2f^{'}$ and $g^n(g-1)^2g^{'}$ share 1 CM, then $f \equiv g$.*

In 2001, an idea of gradation of sharing of values was introduced in [15, 16] which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

Definition 1.1 ([15, 16]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In the mean time to investigate the uniqueness of meromorphic functions, Lahiri and Sarkar [9] considered two different types of nonlinear differential polynomials than those discussed earlier and proved the following.

Theorem E. *Let f and g be two non-constant meromorphic functions such that $f^n(f^2-1)f'$ and $g^n(g^2-1)g'$ share $(1,2)$, where $n(\geq 13)$ is an integer then either $f \equiv g$ or $f \equiv -g$. If n is an even integer then the possibility of $f \equiv -g$ does not arise.*

Recently, Zhang and Lin [17] considered the sharing value problem of more generalised differential polynomials namely the kth derivative of a linear expression but confined their investigation for entire functions only. Zhang and Lin [17] obtained the following result.

Theorem F. *Let f and g be two non-constant entire functions and n, m and k be three positive integers with $n > 2k + m + 4$. Suppose for two non zero constants a and b $[f^n (af^m + b)]^{(k)}$ and $[g^n (ag^m + b)]^{(k)}$ share $(1, \infty)$. Then $f \equiv g$.*

Remark 1.2. *The conclusion of the Theorem F is partially correct. Since in the proof of the theorem the possibility $f \equiv -g$ has not been considered.*

In the paper we will consider the value sharing of differential polynomials of the form given in *Theorem F* generated by a meromorphic functions and improve *Theorem F* and we will show that *Theorem E* can be obtained as a corollary of our result. Following theorem is the main result of the paper.

Theorem 1.3. *Let f and g be two transcendental meromorphic functions and n, $k(\geq 1)$, $m(\geq 2)$ be three positive integers with g.c.d. $(n+m, n) = 2$. Suppose for two non zero constants a and b, $[f^n (af^m + b)]^{(k)}$ and $[g^n (ag^m + b)]^{(k)}$ share $(1, l)$. Then $f \equiv g$ or $f \equiv -g$ or $[f^n(af^m + b)]^{(k)}[g^n(ag^m + b)]^{(k)} \equiv 1$ provided one of the following holds.*

(i) $l \geq 2$ *and* $n > 3k+m+8-2\{\Theta(\infty; f)+\Theta(\infty; g)\}-k\min\{\Theta(\infty; f), \Theta(\infty; g)\}$;

(ii) $l = 1$ *and* $n > 4k + \frac{3m}{2} + 9 - \left(\frac{k}{2} + \frac{5}{2}\right)\{\Theta(\infty; f) + \Theta(\infty; g)\}$;

(iii) $l = 0$ *and* $n > 9k+4m+14-(2k+3)\{\Theta(\infty; f)+\Theta(\infty; g)\}-\min\{\Theta(\infty; f), \Theta(\infty; g)\}$.

When $k = 1$ the possibility $[f^n(af^m + b)]^{(k)}[g^n(ag^m + b)]^{(k)} \equiv 1$ does not occur. Also the possibility $f \equiv -g$ does not arise if n and m are both odd or if n is odd and m is even or if n is even and m is odd.

Putting $n = s+1$, $m = 2$ $a = \frac{1}{s+3}$ $b = -\frac{1}{s+1}$ and $k = 1$ in the above theorem we can immediately deduce the following corollary.

Corollary 1.4. *Let f and g be two non-constant meromorphic functions and s be a positive integer. Suppose $f^s(f^2-1)f^{'}$ and $g^s(g^2-1)g^{'}$ share $(1,l)$. Then $f \equiv g$ or $f \equiv -g$ provided one of the following holds.*

(i) $l \geq 2$ and $s > 12 - 2\{\Theta(\infty; f) + \Theta(\infty; g)\} - \min\{\Theta(\infty; f), \Theta(\infty; g)\}$;

(ii) $l = 1$ and $s > 15 - 3\{\Theta(\infty; f) + \Theta(\infty; g)\}$;

(iii) $l = 0$ and $s > 30 - 5\{\Theta(\infty; f) + \Theta(\infty; g)\} - \min\{\Theta(\infty; f), \Theta(\infty; g)\}$.

If s is an even integer then the possibility of $f \equiv -g$ does not arise.

Remark 1.5. *Since* Theorem E *can be obtained as a special case of* Theorem 1.3*, clearly* Theorem 1.3 *improves and supplements* Theorem E*.*

Theorem 1.6. *Let f and g be two non-constant entire functions and n, $k(\geq 1)$, $m(\geq 2)$ be three positive integers with g.c.d. $(m+n, n) = 2$. Suppose for two non zero constants a and b, $[f^n(af^m+b)]^{(k)}$ and $[g^n(ag^m+b)]^{(k)}$ share $(1,l)$. Then $f \equiv g$ or $f \equiv -g$ provided one of the following holds.*

(i) $l \geq 2$ and $n > 2k + m + 4$;

(ii) $l = 1$ and $n > 3k + \frac{3m}{2} + 4$;

(iii) $l = 0$ and $n > 5k + 4m + 7$.

Also the possibility $f \equiv -g$ does not arise if n and m are both odd or if n is odd and m is even or if n is even and m is odd.

We now explain some definitions and notations which are used in the paper.

Definition 1.7 ([9])**.** Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid \geq p)$ $(\overline{N}(r, a; f \mid \geq p))$denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r, a; f \mid \leq p)$ $(\overline{N}(r, a; f \mid \leq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.8 ([15, 18, 19])**.** For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \cdots + \overline{N}(r, a; f \mid \geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.9. Let $a, b \in \mathbb{C} \cup \{\infty\}$. Let p be a positive integer. We denote by $\overline{N}(r, a; f \mid \geq p \mid g = b)$ $(\overline{N}(r, a; f \mid \geq p \mid g \neq b))$ the reduced counting function of those a-points of f with multiplicities $\geq p$, which are the b-points (not the b-points) of g.

Definition 1.10 ([2, 20]). Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p>q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p=q=1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p=q\geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g),\ N_E^{1)}(r,1;g),\ \overline{N}_E^{(2}(r,1;g)$.

Definition 1.11 ([2, 20]). Let k be a positive integer. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_{f>k}\ (r,1;g)$ the reduced counting function of those 1-points of f and g such that $p>q=k$. $\overline{N}_{g>k}\ (r,1;f)$ is defined analogously.

Definition 1.12 ([15, 16]). Let f,g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly, $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H=\left(\frac{F^{(k+2)}}{F^{(k+1)}}-\frac{2F^{(k+1)}}{F^{(k)}-1}\right)-\left(\frac{G^{(k+2)}}{G^{(k+1)}}-\frac{2G^{(k+1)}}{G^{(k)}-1}\right). \tag{2.1}$$

Lemma 2.1 ([1]). *Let f be a non-constant meromorphic function, k a positive integer and let c be a non-zero finite complex number. Then*

$$\begin{aligned}T(r,f)&\leq\overline{N}(r,\infty;f)+N(r,0;f)+N\left(r,c;f^{(k)}\right)-N\left(r,0;f^{(k+1)}\right)+S(r,f)\\&\leq\overline{N}(r,\infty;f)+N_{k+1}(r,0;f)+\overline{N}\left(r,c;f^{(k)}\right)-N_0\left(r,0;f^{(k+1)}\right)+S(r,f),\end{aligned}$$

where $N_0\left(r,0;f^{(k+1)}\right)$ is the counting function of the zeros of $f^{(k+1)}$ which are not the zeros of $f(f^{(k)}-c)$

Lemma 2.2 ([21]). *Let f be a non-constant meromorphic function and p, k be positive integers, then*

$$N_p(r,0;f^{(k)})\leq N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.3 ([20]). *If f, g be two non-constant meromorphic functions such that they share $(1, 1)$. Then*

$$2\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>2}(r,1;g)\le N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 2.4 ([2]). *Let f, g share $(1, 1)$. Then*

$$\overline{N}_{f>2}(r,1;g)\le \frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}N_{\oslash}(r,0;f^{'})+S(r,f),$$

where $N_{\oslash}(r,0;f^{'})$ is the counting function of those zeros of $f^{'}$ which are not the zeros of $f(f-1)$.

Lemma 2.5 ([2]). *Let f and g be two non-constant meromorphic functions sharing $(1, 0)$. Then*

$$\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f)$$
$$\le N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 2.6 ([2]). *Let f, g share $(1, 0)$. Then*

$$\overline{N}_L(r,1;f)\le \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)$$

Lemma 2.7 ([2]). *Let f, g share $(1, 0)$. Then*

(i) $\overline{N}_{f>1}(r,1;g)\le \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_{\oslash}(r,0;f^{'})+S(r,f)$

(ii) $\overline{N}_{g>1}(r,1;f)\le \overline{N}(r,0;g)+\overline{N}(r,\infty;g)-N_{\oslash}(r,0;g^{'})+S(r,g)$.

Lemma 2.8 ([22]). *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + O(1)$.*

Lemma 2.9. *Let f and g be two non-constant meromorphic functions and a, b be two non zero constants. Then*

$$[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)}\not\equiv 1,$$

where n, $m \geq 2$, $k = 1$ be three positive integers and $n(\geq m + 3)$.

Proof. We note that when $k = 1$, according to the statement of the lemma we have to prove

$$[f^{n-1}\left(a(n+m)f^m+bn\right)f^{'}]\ \ [g^{n-1}\left(a(n+m)g^m+bn\right)g^{'}]\not\equiv 1.$$

If possible let us suppose that

$$[f^{n-1}\left(a(n+m)f^m+bn\right)f^{'}]\ \ [g^{n-1}\left(a(n+m)g^m+bn\right)g^{'}]\equiv 1. \qquad (2.2)$$

Let z_0 be a zero of f with multiplicity $p(\geq 1)$. So from (2.2) we get z_0 be a pole of g with multiplicity $q(\geq 1)$ such that

$$np - 1 = (n+m)q + 1, \tag{2.3}$$

i.e.,

$$mq = n(p-q) - 2 \geq n - 2.$$

Again from (2.3) we get

$$np = (n+m)q + 2 \geq (n+m)\frac{n-2}{m} + 2,$$

i.e.,

$$p \geq \frac{n+m-2}{m}.$$

Therefore

$$\Theta(0; f) \geq 1 - \frac{m}{n+m-2}.$$

Suppose $a(n+m)f^m + bn = a(n+m)\ (f-\alpha_1)(f-\alpha_2)\cdots(f-\alpha_m)$. Let z_1 be a zero of $(f-\alpha_i)$, $i = 1, 2, \ldots, m$ with multiplicity p. Then from (2.2) we have z_1 be a pole of g with multiplicity $q(\geq 1)$ such that

$$2p - 1 = (n+m)q + 1$$

i.e.,

$$p \geq \frac{n+m+2}{2}.$$

Hence

$$\Theta(\alpha_i; f) \geq 1 - \frac{2}{n+m+2}.$$

Since

$$\Theta(0; f) + \sum_{i=1}^{m} \Theta(\alpha_i; f) \leq 2,$$

it follows that

$$\frac{2m}{n+m+2} + \frac{m}{n+m-2} \geq m - 1,$$

which is a contradiction. □

Lemma 2.10. *Let f and g be two non-constant entire functions. Then*

$$[f^n(af^m + b)]^{(k)}[g^n(ag^m + b)]^{(k)} \not\equiv 1,$$

where a and b are nonzero complex numbers; n, m, k be three positive integers and $n(> 2k + m + 4)$.

Proof. We omit the proof since the proof can be found in the proof of Theorem 1 in [17]. □

Lemma 2.11. *Let f and g be two non-constant meromorphic functions such that $F = f^n(af^m + b)$ and $G = g^n(ag^m + b)$, where $m \geq 2$ and $n + m \geq 9$ is an integer with g.c.d. $(n + m, n) = 2$ and a, b are non-zero constants. Then*

$$F \equiv G$$

implies either $f \equiv g$ or $f \equiv -g$. Also if n and m are both odd or if n is odd and m is even or if n is even and m is odd then the possibility $f \equiv -g$ does not arise.

Proof. Clearly if n and m are both odd or if n is odd and m is even or if n is even and m is odd then $f \equiv -g$ contradicts $F \equiv G$. Let neither $f \equiv g$ nor $f \equiv -g$. We put $h = \frac{g}{f}$. Then $h \not\equiv 1$ and $h \not\equiv -1$. Also $F \equiv G$ implies

$$f^m = -\frac{b}{a}\,\frac{h^n - 1}{h^{n+m} - 1}.$$

If n and m are both even then the numerator and the denominator has two common factors namely $h + 1$ and $h - 1$. Also we observe that since a non-constant meromorphic function can not have more than two Picard exceptional values h can take at least $n+m-4$ values among $u_j = exp(\frac{2j\pi i}{n+m})$, where $j = 1, 2, \ldots, n+m-1$. Since f is non-constant it follows that h is non constant. Again since f^m has no simple pole $h - u_j$ has no simple zero for at least $n + m - 4$ values of u_j, for $j = 1, 2, \ldots, n+m-1$ and for these values of j we have $\Theta(u_j; h) \geq \frac{1}{2}$, which leads to a contradiction. Therefore either $f \equiv g$ or $f \equiv -\, g$. This proves the lemma. □

3 Proofs of the Theorems

Proof of Theorem 1.3. Let $F = f^n(af^m + b)$ and $G = g^n(ag^m + b)$. It follows that $F^{(k)}$ and $G^{(k)}$ share $(1, l)$.
Case 1 Let $H \not\equiv 0$.
Subcase 1.1 $l \geq 1$
From (2.1) we get

$$\begin{aligned} N(r,\infty;H) \leq\ & \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*\left(r,1;F^{(k)},G^{(k)}\right) \\ & + \overline{N}\left(r,0;F^{(k)} \mid\geq 2\right) + \overline{N}\left(r,0;G^{(k)} \mid\geq 2\right) \\ & + \overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right) + \overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right), \end{aligned} \tag{3.1}$$

where $\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)$ is the reduced counting function of those zeros of $F^{(k+1)}$ which are not the zeros of $F^{(k)}\left(F^{(k)} - 1\right)$ and $\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)$ is similarly defined.

Let z_0 be a simple zero of $F^{(k)} - 1$. Then z_0 is a simple zero of $G^{(k)} - 1$ and a zero of H. So

$$N\left(r,1;F^{(k)} \mid= 1\right) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,F) + S(r,G). \tag{3.2}$$

While $l \geq 2$, using (3.1) and (3.2) we get

$$\begin{aligned}\overline{N}\left(r,1;F^{(k)}\right) &\leq N\left(r,1;F^{(k)}\mid=1\right)+\overline{N}\left(r,1;F^{(k)}\mid\geq 2\right) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F^{(k)}\mid\geq 2)+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right) \\ &\quad+\overline{N}_*\left(r,1;F^{(k)},G^{(k)}\right)+\overline{N}\left(r,1;F^{(k)}\mid\geq 2\right)+\overline{N}_{\otimes}(r,0;F^{(k+1)}) \\ &\quad+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G).\end{aligned} \tag{3.3}$$

So from Lemmas 2.1 and 2.8 we have

$$\begin{aligned}T(r,F)+T(r,G) &\leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_{k+1}(r,0;F)+N_{k+1}(r,0;G) \\ &\quad+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right) \\ &\quad+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+\overline{N}\left(r,1;G^{(k)}\right)+\overline{N}\left(r,1;F^{(k)}\mid\geq 2\right) \\ &\quad+\overline{N}_*\left(r,1;F^{(k)},G^{(k)}\right)-N_0\left(r,0;F^{(k+1)}\right)-N_0\left(r,0;G^{(k+1)}\right) \\ &\quad+S(r,F)+S(r,G).\end{aligned} \tag{3.4}$$

We note that

$$\begin{aligned}&N_{k+1}(r,0;F)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right) \\ &\leq N_{k+1}(r,0;F)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\mid F=0\right) \\ &\quad+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\mid F\neq 0\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right) \\ &\leq N_{k+1}(r,0;F)+\overline{N}\left(r,0;F\mid\geq k+2\right)+\overline{N}_0\left(r,0;F^{(k+1)}\right) \\ &\leq N_{k+2}(r,0;F)+\overline{N}_0\left(r,0;F^{(k+1)}\right).\end{aligned} \tag{3.5}$$

Clearly similar expression holds for G. Also

$$\begin{aligned}&\overline{N}\left(r,1;F^{(k)}\mid\geq 2\right)+\overline{N}_*\left(r,1;F^{(k)},G^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right) \\ &\leq \overline{N}\left(r,1;G^{(k)}\mid=2\right)+2\overline{N}_L\left(r,1;F^{(k)}\right)+2\overline{N}_L\left(r,1;G^{(k)}\right) \\ &\quad+\overline{N}_E^{(3}\left(r,1;G^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right) \\ &\leq N\left(r,1;G^{(k)}\right) \\ &\leq T\left(r,G^{(k)}\right)+O(1) \\ &\leq T(r,G)+k\overline{N}(r,\infty;G)+S(r,G).\end{aligned} \tag{3.6}$$

Using Lemma 2.8, (3.5) and (3.6) in (3.4) we obtain for $\varepsilon > 0$

$$\begin{aligned}(n+m)T(r,f) &= T(r,F) + O(1) \qquad (3.7)\\ &\le N_{k+2}(r,0;F) + N_{k+2}(r,0;G) + 2\overline{N}(r,\infty;F)\\ &\quad + (k+2)\overline{N}(r,\infty;G) + S(r,F) + S(r,G)\\ &\le N_{k+2}\left(r,0;f^n\right) + N_{k+2}(r,0;af^m+b) + N_{k+2}\left(r,0;g^n\right)\\ &\quad + N_{k+2}(r,0;ag^m+b) + 2\overline{N}(r,\infty;f) + (k+2)\overline{N}(r,\infty;g)\\ &\quad + S(r,f) + S(r,g)\\ &\le \left(4+m+k-2\Theta(\infty;f)+\varepsilon\right)T(r,f)\\ &\quad + \left(4+m+2k-(2+k)\Theta(\infty;g)+\varepsilon\right)T(r,g) + S(r,f) + S(r,g)\\ &\le (8+2m+3k-2\Theta(\infty;f)-2\Theta(\infty;g)\\ &\qquad - k\min\{\Theta(\infty;f),\Theta(\infty;g)\} + 2\varepsilon)T(r) + S(r,f) + S(r,g).\end{aligned}$$

In a similar way we can obtain

$$\begin{aligned}&(n+m)T(r,g) \qquad (3.8)\\ &\le (8+2m+3k-2\Theta(\infty;f)-2\Theta(\infty;g)-k\min\{\Theta(\infty;f),\Theta(\infty;g)\}\\ &\qquad +2\varepsilon)\,T(r) + S(r,f) + S(r,g).\end{aligned}$$

So from (3.7) and (3.8) we get

$$\begin{aligned}&(n-m-3k-8+2\Theta(\infty;f)+2\Theta(\infty;g) \qquad (3.9)\\ &\qquad\qquad +k\min\{\Theta(\infty;f),\Theta(\infty;g)\}-2\varepsilon)\,T(r) \le S(r).\end{aligned}$$

Since $\varepsilon > 0$ be arbitrary, (3.9) gives a contradiction.

While $l = 1$, using Lemmas 2.2, 2.3 and 2.4, (3.1) and (3.2) we get

$$\begin{aligned}&\overline{N}\left(r,1;F^{(k)}\right) + \overline{N}\left(r,1;G^{(k)}\right) \qquad (3.10)\\ &\le N\left(r,1;F^{(k)} \mid= 1\right) + \overline{N}_L\left(r,1;F^{(k)}\right) + \overline{N}_L\left(r,1;G^{(k)}\right)\\ &\quad + \overline{N}_E^{(2}\left(r,1;G^{(k)}\right) + \overline{N}\left(r,1;G^{(k)}\right)\\ &\le N\left(r,1;F^{(k)} \mid= 1\right) + N\left(r,1;G^{(k)}\right) - \overline{N}_L\left(r,1;F^{(k)}\right) - \overline{N}_L\left(r,1;G^{(k)}\right)\\ &\quad + \overline{N}_{F^{(k)}>2}\left(r,1;G^{(k)}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)\\
&\quad+\overline{N}_{*}\left(r,1;F^{(k)},G^{(k)}\right)-\overline{N}_{L}\left(r,1;F^{(k)}\right)-\overline{N}_{L}\left(r,1;G^{(k)}\right)\\
&\quad+\frac{1}{2}\overline{N}\left(r,0;F^{(k)}\right)+\frac{1}{2}\overline{N}\left(r,\infty;F^{(k)}\right)+T\left(r,G^{(k)}\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)\\
&\quad+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G)\\
&\leq\left(\frac{k}{2}+\frac{3}{2}\right)\overline{N}(r,\infty;F)+(k+1)\overline{N}(r,\infty;G)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)\\
&\quad+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)+\frac{1}{2}N_{k+1}(r,0;F)+T(r,G)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)\\
&\quad+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G).
\end{aligned}
$$

So in view of Lemmas 2.1, 2.8, (3.5) and (3.10) we get for $\varepsilon>0$

$$
\begin{aligned}
(n+m)T(r,f)&=T(r,F)+O(1) \qquad (3.11)\\
&\leq\left(\frac{k}{2}+\frac{5}{2}\right)\overline{N}(r,\infty;F)+(k+2)\overline{N}(r,\infty;G)+\frac{1}{2}N_{k+1}(r,0;F)\\
&\quad+N_{k+2}(r,0;F)+N_{k+2}(r,0;G)+S(r,F)+S(r,G)\\
&\leq\left(2k+5+\frac{3m}{2}-\left(\frac{k}{2}+2\right)\Theta(\infty;f)-\frac{1}{2}\Theta(\infty;f)+\varepsilon\right)T(r,f)\\
&\quad+\left(2k+4+m-\left(\frac{k}{2}+2\right)\Theta(\infty;g)-\frac{k}{2}\Theta(\infty;g)+\varepsilon\right)T(r,g)\\
&\quad+S(r,f)+S(r,g)\\
&\leq\left(4k+9+\frac{5m}{2}-\left(\frac{k}{2}+\frac{5}{2}\right)(\Theta(\infty;f)+\Theta(\infty;g))+2\varepsilon\right)T(r)\\
&\quad+S(r).
\end{aligned}
$$

In a similar manner we can get

$$
(n+m)T(r,g)\leq\left(4k+9+\frac{5m}{2}-\left(\frac{k}{2}+\frac{5}{2}\right)(\Theta(\infty;f)+\Theta(\infty;g))+2\varepsilon\right)T(r)+S(r). \tag{3.12}
$$

Combining (3.11) and (3.12) we get

$$
\left(n-4k-9-\frac{3m}{2}+\left(\frac{k}{2}+\frac{5}{2}\right)(\Theta(\infty;f)+\Theta(\infty;g))-2\varepsilon\right)T(r)\leq S(r). \tag{3.13}
$$

Since $\varepsilon>0$ be arbitrary, (3.13) implies a contradiction.

Subcase 1.2 $l=0$. Here (3.2) changes to

$$
N_{E}^{1)}\left(r,1;F^{(k)}\mid=1\right)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,F)+S(r,G) \tag{3.14}
$$

Using Lemmas 2.2, 2.5, 2.6, 2.7 and (3.1) and (3.14) we get

$$\begin{aligned}
&\overline{N}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right) \qquad (3.15)\\
&\le N_E^{1)}\left(r,1;F^{(k)}\right)+\overline{N}_L\left(r,1;F^{(k)}\right)+\overline{N}_L\left(r,1;G^{(k)}\right)+\overline{N}_E^{(2}\left(r,1;F^{(k)}\right)\\
&\quad+\overline{N}\left(r,1;G^{(k)}\right)\\
&\le N_E^{1)}\left(r,1;F^{(k)}\right)+N\left(r,1;G^{(k)}\right)-\overline{N}_L\left(r,1;G^{(k)}\right)+\overline{N}_{F^{(k)}>1}\left(r,1;G^{(k)}\right)\\
&\quad+\overline{N}_{G^{(k)}>1}\left(r,1;F^{(k)}\right)\\
&\le \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}\left(r,0;F^{(k)}\mid\ge 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\ge 2\right)\\
&\quad+\overline{N}_*\left(r,1;F^{(k)},G(k)\right)+T\left(r,G^{(k)}\right)-\overline{N}_L\left(r,1;G^{(k)}\right)\\
&\quad+\overline{N}_{F^{(k)}>1}\left(r,1;G^{(k)}\right)+\overline{N}_{G^{(k)}>1}\left(r,1;F^{(k)}\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)\\
&\quad+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G)\\
&\le (2k+3)\overline{N}(r,\infty;F)+(2k+2)\overline{N}(r,\infty;G)+\overline{N}\left(r,0;F^{(k)}\mid\ge 2\right)\\
&\quad+\overline{N}\left(r,0;G^{(k)}\mid\ge 2\right)+2N_{k+1}(r,0;F)+N_{k+1}(r,0;G)+T(r,G)\\
&\quad+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G).
\end{aligned}$$

So in view of Lemmas 2.1, 2.8, (3.5) and (3.15) we get for $\varepsilon>0$

$$\begin{aligned}
(n+m)T(r,f) &= T(r,F)+O(1) \qquad (3.16)\\
&\le (2k+4)\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)+2N_{k+1}(r,0;F)\\
&\quad+N_{k+1}(r,0;G)+N_{k+2}(r,0;F)+N_{k+2}(r,0;G)+S(r,f)+S(r,g)\\
&\le (9k+14+5m-(2k+3)\Theta(\infty;f)-(2k+3)\Theta(\infty;g)\\
&\qquad-\min\{\Theta(\infty;f),\Theta(\infty;g)\}+2\varepsilon)\,T(r)+S(r).
\end{aligned}$$

Similarly we can obtain

$$\begin{aligned}
(n+m)T(r,g) &= T(r,G)+O(1) \qquad (3.17)\\
&\le (9k+14+5m-(2k+3)\Theta(\infty;f)-(2k+3)\Theta(\infty;g)\\
&\qquad-\min\{\Theta(\infty;f),\Theta(\infty;g)\}+2\varepsilon)\,T(r)+S(r).
\end{aligned}$$

Combining (3.16) and (3.17) we get

$$\begin{aligned}
&(n-9k-14+4m+(2k+3)\Theta(\infty;f)+(2k+3)\Theta(\infty;g) \qquad (3.18)\\
&\qquad+\min\{\Theta(\infty;f),\Theta(\infty;g)\}-2\varepsilon)\,T(r)\le S(r).
\end{aligned}$$

(3.18) implies a contradiction for $\varepsilon>0$.

Case 2 Next we suppose that $H \equiv 0$. Then by integration we get from (2.1)

$$\frac{1}{F^{(k)}-1} \equiv \frac{bG^{(k)}+a-b}{G^{(k)}-1}, \tag{3.19}$$

where a, b are constants and $a \neq 0$. From (3.19) it is clear that $F^{(k)}$ and $G^{(k)}$ share $(1,\infty)$ and hence they share $(1,2)$. So in this case always $n > 3k+m+8-2\{\Theta(\infty;f)+\Theta(\infty;g)\} - k\min\{\Theta(\infty;f),\Theta(\infty;g)\}$. We now consider the following subcases.

Subcase 2.1 Let $b \neq 0$ and $a \neq b$. If $b=-1$, then from (3.19) we have

$$F^{(k)} = \frac{-a}{G^{(k)}-a-1}.$$

Therefore

$$\overline{N}\left(r,a+1;G^{(k)}\right) = \overline{N}\left(r,\infty;F^{(k)}\right) = \overline{N}(r,\infty;f).$$

Since $a \neq b = -1$, from Lemma 2.1 we have

$$\begin{aligned}(n+m)T(r,g) &= T(r,G)+O(1)\\ &\leq \overline{N}(r,\infty;G)+N_{k+1}(r,0;G)+\overline{N}(r,a+1;G^{(k)})+S(r,G)\\ &\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+N_{k+1}(r,0;G)+S(r,G)\\ &\leq (1-\Theta(\infty;f)+\varepsilon)T(r,f)+(k+2+m-\Theta(\infty;g)+\varepsilon)T(r,g)\\ &\quad + S(r,g).\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$. So for $r \in I$ we have

$$(n-k-3+\Theta(\infty;f)+\Theta(\infty;g)-2\varepsilon)T(r,g) \leq S(r,g),$$

which is a contradiction for arbitrary $\varepsilon > 0$.

If $b \neq -1$, from (3.19) we obtain that

$$F^{(k)} - \left(1+\frac{1}{b}\right) = \frac{-a}{b^2[G^{(k)}+(a-b)/b]}.$$

Therefore

$$\overline{N}\left(r,(b-a)/b;G^{(k)}\right) = \overline{N}\left(r,\infty;F^{(k)}-(1+1/b)\right) = \overline{N}(r,\infty;f).$$

Using Lemma 2.1 and the same argument as used in the case when $b=-1$ we can get a contradiction.

Subcase 2.2 Let $b \neq 0$ and $a = b$. If $b = -1$, then from (3.19) we have

$$F^{(k)}G^{(k)} \equiv 1,$$

that is

$$[f^n(af^m+b)]^{(k)}[g^n(ag^m+b)]^{(k)} \equiv 1,$$

which is impossible by Lemma 2.9 when $k=1$.

If $b \neq -1$, from (3.19) we have

$$\frac{1}{F^{(k)}} = \frac{bG^{(k)}}{(1+b)G^{(k)}-1}.$$

Hence from Lemma 2.2 we have

$$\begin{aligned}\overline{N}\left(r, 1/(1+b); G^{(k)}\right) &= \overline{N}\left(r, 0; F^{(k)}\right)\\ &\leq N_{k+1}(r,0;F) + k\overline{N}(r,\infty;f).\end{aligned}$$

From Lemma 2.1 we have

$$\begin{aligned}(n+m)T(r,g)+O(1) &= T(r,G)\\ &\leq \overline{N}(r,\infty;G) + N_{k+1}(r,0;G) + \overline{N}\left(r, \frac{1}{b+1}; G^{(k)}\right) + S(r,G)\\ &\leq k\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N_{k+1}(r,0;F) + N_{k+1}(r,0;G)\\ &\quad + S(r,G)\\ &\leq (2k+1+m-k\Theta(\infty;f)+\epsilon)T(r,f)\\ &\quad + (k+2+m-\Theta(\infty;g)+\epsilon)T(r,g) + S(r,g).\end{aligned}$$

For $r \in I$ we have

$$(n-3k-3-m+k\Theta(\infty;f)+\Theta(\infty;g)-2\epsilon)T(r,g) \leq S(r,g),$$

which is a contradiction for $n \geq 3k+9$.

Subcase 2.3 Let $b=0$. From (3.19) we obtain

$$F^{(k)} = \frac{G^{(k)}+a-1}{a}. \tag{3.20}$$

If $a-1 \neq 0$ then From (3.20) we obtain

$$\overline{N}\left(r, 1-a; G^{(k)}\right) = \overline{N}\left(r, 0; F^{(k)}\right).$$

We can similarly deduce a contradiction as in Subcase 2.2. Therefore $a=1$ and from (3.20) we obtain

$$F = G + p(z), \tag{3.21}$$

where $p(z)$ is a polynomial of degree at most $k-1$. We claim that $p(z) \equiv 0$. Otherwise noting that f is transcendental when $k \geq 2$, in view of Lemma 2.8 we have

$$\begin{aligned}(n+m)T(r,f) &= T(r,F)+O(1) \\ &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;f)+\overline{N}(r,p;F)+S(r,F) \\ &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;f)+\overline{N}(r,0;G)+S(r,F) \\ &\leq 3T(r,f)+2T(r,g)+S(r,f).\end{aligned} \tag{3.22}$$

Also from (3.21) we get

$$T(r,f)=T(r,g)+S(r,f),$$

which together with (3.22) implies a contradiction. So

$$F\equiv G.$$

So from Lemma 2.11 we get the conclusion of the theorem. □

Proof of Theorem 1.6. We omit the proof since instead of Lemma 2.9 using Lemma 2.10 and proceeding in the same way the proof of the theorem can be carried out in the line of proof of Theorem 1.3 and Theorem 1 of [17]. □

References

[1] W.K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[2] A. Banerjee, Meromorphic functions sharing one value, Int. J. Math. Math. Sci. 22 (2005) 3587–3598.

[3] A. Banerjee, On uniqueness for non-linear differential polynomials sharing the same 1-points, Ann. Polon. Math. 89 (3) (2006) 259–272.

[4] A. Banerjee, A uniqueness result on some differential polynomials sharing 1 points, Hiroshima Math. J. 37 (3) 2007 397–408.

[5] C.Y. Fang, M.L. Fang, Uniqueness of meromorphic functions and differential polynomials, Comput. Math. Appl. 44 (2002) 607–617.

[6] M.L. Fang, W. Hong, A unicity theorem for entire functions concerning differential polynomials, Indian J. Pure Appl. Math. 32 (9) (2001) 1343–1348.

[7] I. Lahiri, N. Mandal, Uniqueness of nonlinear differential polynomials sharing simple and double 1-points, Int. J. Math. Math. Sci. 12 (2005) 1933–1942.

[8] I. Lahiri, R. Pal, Nonlinear differential polynomials sharing 1-points, Bull. Korean Math. Soc. 43 (1) (2006) 161–168.

[9] I. Lahiri, A.Sarkar, Nonlinear differential polynomials sharing 1-points with weight two, Chinese J. Contemp. Math. 25 (3) (2004) 325–334.

[10] I. Lahiri, P.Sahoo, Uniqueness of non-linear differential polynomials sharing 1-points, Georgian Math. J. 12 (1) (2005) 131–138.

[11] W.C. Lin, Uniqueness of differential polynomials and a problem of Lahiri, Pure Appl. Math. (in Chinese) 17 (2) (2001) 104–110.

[12] W.C. Lin, H.X. Yi, Uniqueness theorems for meromorphic function, Indian J. Pure Appl. Math. 35 (2) (2004) 121–132.

[13] H. Qiu, M. Fang, On the uniqueness of entire functions, Bull. Korean Math. Soc. 41 (1) (2004) 109–116.

[14] I. Lahiri, Uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points, Ann. Polon. Math. 71 (2) (1999) 113–128.

[15] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, Complex Var. Theory Appl. 46 (2001) 241–253.

[16] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, Nagoya Math. J. 161 (2001) 193–206.

[17] X.Y. Zhang, W.C. Lin, Uniqueness and value sharing of entire functions, J. Math. Anal. Appl. 343 (2008) 938–950.

[18] H.X. Yi, On characteristic function of a meromorphic function and its derivative, Indian J. Math. 33 (2) (1991) 119–133.

[19] I. Lahiri, On a question of Hong Xun Yi, Arch. Math. (Brno) 38 (2002) 119–128.

[20] T.C. Alzahary, H.X. Yi, Weighted value sharing and a question of I.Lahiri, Complex Var. Theory Appl. 49 (15) (2004) 1063–1078.

[21] Q.C. Zhang, Meromorphic function that shares one small function with its derivative, J. Inequal. Pure Appl. Math. 6 (4) (2005), Art.116.

[22] C.C. Yang, On deficiencies of differential polynomials II, Math. Z. Vol. 125 (1972) 107–112.

(Received 11 October 2010)
(Accepted 30 November 2011)