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Weak and Strong Convergence Theorems for Four Nonexpansive Mappings in Uniformly Convex Banach Spaces

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Abstract : In this paper, we study three-step iterative algorithm with errors for four nonexpansive mappings in uniformly convex Banach spaces. Also we have proved strong convergence theorem for above said algorithm and mappings by using *condition* (GA) which is a generalization of *condition* (A) [1] and a weak convergence theorem by using Opial's condition [2]. The results presented in this paper improve and extend the corresponding results of Khan and Fukhar-ud-din [3], Takahashi and Tamura [4], Boonchari and Saejung [5] and many others.

Keywords : Nonexpansive mapping; Common fixed point; Condition (GA); Opial's condition; Three-step iterative algorithm with errors; Strong convergence; Uniformly convex Banach space; Weak convergence.
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1 Introduction

Let E be a normed space and K be a nonempty subset of E. A mapping $T: K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. Nonexpansive mappings have been widely and extensively studied by many authors in many aspects. One is to approximate a fixed point or a common fixed point of nonexpansive mappings by means of an iteratively constructed sequence.

Let $R, S, T, U\colon K \to K$ be four mappings. Xu [6] introduced the following iterative scheme,

(a) The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= a_n x_n + b_n T x_n + c_n u_n, \;\; n \geq 1 \end{aligned} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in K, is known as Mann iterative scheme with errors. This scheme reduces to Mann iterative scheme [7] if $c_n = 0$, i.e.,

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= a_n x_n + (1 - a_n) T x_n, \;\; n \geq 1 \end{aligned} \tag{1.2}$$

where $\{a_n\}$ is a sequence in $[0,1]$.

(b) The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &\in K, \\ y_n &= a'_n x_n + b'_n T x_n + c'_n v_n, \\ x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \;\; n \geq 1 \end{aligned} \tag{1.3}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are sequences in $[0,1]$ satisfying $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequence in K, is called the Ishikawa iterative scheme with errors. This scheme reduces to Ishikawa iterative scheme [8] if $c_n \equiv 0 \equiv c'_n$, i.e.,

$$\begin{aligned} x_1 &\in K, \\ y_n &= a'_n x_n + (1 - a'_n) T x_n, \\ x_{n+1} &= a_n x_n + (1 - a_n) T y_n, \;\; n \geq 1 \end{aligned} \tag{1.4}$$

where $\{a_n\}$, $\{a'_n\}$ are sequences in $[0,1]$.

A generalization of Mann and Ishikawa iterative schemes was given by Das and Debate [9] and Takahashi and Tamura [4]. This scheme dealt with two mappings:

$$\begin{aligned} x_1 &\in K, \\ y_n &= a'_n x_n + (1 - a'_n) T x_n, \\ x_{n+1} &= a_n x_n + (1 - a_n) S y_n, \;\; n \geq 1 \end{aligned} \tag{1.5}$$

where $\{a_n\}$, $\{a'_n\}$ are sequences in $[0,1]$.

(c) The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &\in K, \\ y_n &= a'_n x_n + b'_n T x_n + c'_n v_n, \\ x_{n+1} &= a_n x_n + b_n S y_n + c_n u_n, \;\; n \geq 1 \end{aligned} \tag{1.6}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are sequences in $[0,1]$ satisfying $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K, is studied by Khan and Fukhar-ud-din [3].

Recently, Boonchari and Saejung [5] generalized the scheme (1.6) to three nonexpansive mappings with errors as follows:

(d) The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &\in K, \\ y_n &= a'_n R x_n + b'_n T x_n + c'_n v_n, \\ x_{n+1} &= a_n R x_n + b_n S y_n + c_n u_n, \quad n \geq 1 \end{aligned} \tag{1.7}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are sequences in $[0,1]$ satisfying $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K. Also they have proved weak and strong convergence theorems for said scheme in uniformly convex Banach spaces.

Inspired by [3, 5, 10], we extend the scheme (1.7) to the three-step iteration scheme for four nonexpansive mappings with errors. The scheme is as follows:

(e) The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &\in K, \\ z_n &= a''_n R x_n + b''_n U x_n + c''_n w_n, \\ y_n &= a'_n R x_n + b'_n T z_n + c'_n v_n, \\ x_{n+1} &= a_n R x_n + b_n S y_n + c_n u_n, \quad n \geq 1 \end{aligned} \tag{1.8}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$, $\{c''_n\}$ are sequences in $[0,1]$ satisfying $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in K.

2 Preliminaries

Let E be a Banach space and let K be a nonempty closed convex subset of E. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

A Banach space E is said to satisfy Opial's condition [2] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta_E(\varepsilon)$ of convexity of E by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon \right\}.$$

A mapping $T\colon K \to E$ is said to be demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in K and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ it follows that $x \in K$ and $Tx = y$.

Next, we state the following useful lemmas to prove our main results.

Lemma 2.1 ([11]). *Let E be a uniformly convex Banach space and $0 < \alpha \le t_n \le \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \le a, \quad \limsup_{n\to\infty} \|y_n\| \le a$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$$

hold for some $a \ge 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.2 ([12, Lemma 1]). *Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$\alpha_{n+1} \le \alpha_n + \beta_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists.

Lemma 2.3 ([13]). *Let E be a uniformly convex Banach space satisfying Opial's condition and let K be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of K into itself. Then $I - T$ is demiclosed with respect to zero.*

The purpose of this paper is to study the three-step iteration scheme (1.8) for four nonexpansive mappings with errors and prove weak and strong convergence theorems for said scheme. The results presented in this paper extend and improve the corresponding results of Khan and Fukhar-ud-din [3], Takahashi and Tamura [4], Boonchari and Saejung [5] and many others.

3 Main Results

In this section, we shall prove weak and strong convergence theorems of the iteration scheme defined by (1.8) to a common fixed point of the nonexpansive mappings R, S, T and U. Let $\mathcal{F}$ denote the set of all common fixed points of R, S, T and U.

Lemma 3.1. *Let E be a uniformly convex Banach space and K be its nonempty closed convex subset. Let R, S, T, $U\colon K \to K$ be nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (1.8) with the restrictions $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$ and $\sum_{n=1}^{\infty} c''_n < \infty$. If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U)$.*

Proof. Let $p \in \mathcal{F} = F(R)\cap F(S)\cap F(T)\cap F(U)$. Since R, S, T, U are nonexpansive mappings, from (1.8) we have

$$\begin{aligned}
\|z_n - p\| &= \|a''_n Rx_n + b''_n Ux_n + c''_n w_n - p\| \\
&\le a''_n \|Rx_n - p\| + b''_n \|Ux_n - p\| + c''_n \|w_n - p\| \\
&\le a''_n \|x_n - p\| + b''_n \|x_n - p\| + c''_n \|w_n - p\| \\
&\le (a''_n + b''_n) \|x_n - p\| + c''_n \|w_n - p\| \\
&= (1 - c''_n) \|x_n - p\| + c''_n \|w_n - p\| \\
&\le \|x_n - p\| + A_n \qquad (3.1)
\end{aligned}$$

where $A_n = c''_n \|w_n - p\|$. Since $\sum_{n=1}^{\infty} c''_n < \infty$, it follows that $\sum_{n=1}^{\infty} A_n < \infty$. Again from (1.8) and (3.1), we have

$$\begin{aligned}
\|y_n - p\| &= \|a'_n Rx_n + b'_n Tz_n + c'_n v_n - p\| \\
&\le a'_n \|Rx_n - p\| + b'_n \|Tz_n - p\| + c'_n \|v_n - p\| \\
&\le a'_n \|x_n - p\| + b'_n \|z_n - p\| + c'_n \|v_n - p\| \\
&\le a'_n \|x_n - p\| + b'_n [\|x_n - p\| + A_n] + c'_n \|v_n - p\| \\
&\le (a'_n + b'_n) \|x_n - p\| + b'_n A_n + c'_n \|v_n - p\| \\
&= (1 - c'_n) \|x_n - p\| + b'_n A_n + c'_n \|v_n - p\| \\
&\le \|x_n - p\| + B_n \qquad (3.2)
\end{aligned}$$

where $B_n = b'_n A_n + c'_n \|v_n - p\|$. Since $\sum_{n=1}^{\infty} c'_n < \infty$ and $\sum_{n=1}^{\infty} A_n < \infty$ it follows that $\sum_{n=1}^{\infty} B_n < \infty$. From (1.8) and (3.2), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|a_n Rx_n + b_n Sy_n + c_n u_n - p\| \\
&\le a_n \|Rx_n - p\| + b_n \|Sy_n - p\| + c_n \|u_n - p\| \\
&\le a_n \|x_n - p\| + b_n \|y_n - p\| + c_n \|u_n - p\| \\
&\le a_n \|x_n - p\| + b_n [\|x_n - p\| + B_n] + c_n \|u_n - p\| \\
&\le (a_n + b_n) \|x_n - p\| + b_n B_n + c_n \|u_n - p\| \\
&= (1 - c_n) \|x_n - p\| + b_n B_n + c_n \|u_n - p\| \\
&\le \|x_n - p\| + D_n \qquad (3.3)
\end{aligned}$$

where $D_n = b_n B_n + c_n \|u_n - p\|$. Since $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$ it follows that $\sum_{n=1}^{\infty} D_n < \infty$. Hence by Lemma 2.2, $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof. □

Lemma 3.2. *Let E be a uniformly convex Banach space and K be its nonempty closed convex subset. Let R, S, T, $U\colon K \to K$ be nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (1.8) with the restrictions $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$, $\sum_{n=1}^{\infty} c''_n < \infty$ and $0 < \alpha \le b_n, b'_n, b''_n \le \beta < 1$ for some $\alpha, \beta \in (0,1)$. If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$ and*

$$\|x - Sy\| \le \|Rx - Sy\|, \ \forall x, y \in K, \qquad (3.4)$$

and

$$\|x - Rx\| \leq \|Ux - Rx\|, \ \forall x \in K, \tag{3.5}$$

then

$$\lim_{n\to\infty} \|Rx_n - x_n\| = \lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Ux_n - x_n\| = 0,$$

for all $p \in \mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U)$.

Proof. From Lemma 3.1 we get $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = r$. Then if $r = 0$, we are done. Assume that $r > 0$. Next, we want to show that $\lim_{n\to\infty} \|Rx_n - Sy_n\| = 0$. We note that $\{u_n - Rx_n - p\}$ is a bounded sequence, so $\lim_{n\to\infty} c_n \|u_n - Rx_n - p\| = 0$. From (3.2) we have

$$\|y_n - p\| \leq \|x_n - p\| + B_n, \ n \geq 1,$$

where $B_n = b'_n A_n + c'_n \|v_n - p\|$ such that $\sum_{n=1}^{\infty} B_n < \infty$.

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = r. \tag{3.6}$$

Note that

$$\limsup_{n\to\infty} \|Ty_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| = r. \tag{3.7}$$

Also,

$$\limsup_{n\to\infty} \|Rx_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = r. \tag{3.8}$$

Next, consider

$$\begin{aligned} r &= \lim_{n\to\infty} \|x_{n+1} - p\| \\ &= \lim_{n\to\infty} \|a_n Rx_n + b_n Sy_n + c_n u_n - p\| \\ &= \lim_{n\to\infty} \|(1 - b_n)Rx_n + b_n Sy_n + c_n u_n - c_n Rx_n - p\| \\ &= \lim_{n\to\infty} \|(1 - b_n)(Rx_n - p) + b_n(Sy_n - p) + c_n(u_n - Rx_n - p)\| \\ &= \lim_{n\to\infty} \|(1 - b_n)(Rx_n - p) + b_n(Sy_n - p)\|. \end{aligned} \tag{3.9}$$

From (3.7), (3.8) and (3.9), using Lemma 2.1 we have

$$\lim_{n\to\infty} \|Rx_n - Sy_n\| = 0. \tag{3.10}$$

Using (3.4), it follows then that

$$\begin{aligned} \|Rx_n - x_n\| &\leq \|Rx_n - Sy_n\| + \|Sy_n - x_n\| \\ &\leq 2\|Rx_n - Sy_n\| \to 0 \ \textit{as} \ n \to \infty, \end{aligned} \tag{3.11}$$

and hence

$$\|Sy_n - x_n\| \leq \|Sy_n - Rx_n\| + \|Rx_n - x_n\| \to 0 \ as \ n \to \infty. \tag{3.12}$$

Again, we observe that for each $n \geq 1$,

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - Sy_n\| + \|Sy_n - p\| \\ &\leq \|x_n - Sy_n\| + \|y_n - p\|, \end{aligned} \tag{3.13}$$

using (3.12), we obtain

$$r = \lim_{n\to\infty} \|x_n - p\| \leq \liminf_{n\to\infty} \|y_n - p\|.$$

This together with (3.6) gives

$$\lim_{n\to\infty} \|y_n - p\| = r. \tag{3.14}$$

Now from (3.1) we have

$$\|z_n - p\| \leq \|x_n - p\| + A_n, \ \ n \geq 1,$$

where $A_n = c''_n \|w_n - p\|$ such that $\sum_{n=1}^{\infty} A_n < \infty$.

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = r. \tag{3.15}$$

Also,

$$\limsup_{n\to\infty} \|Tz_n - p\| \leq \limsup_{n\to\infty} \|z_n - p\| = r, \tag{3.16}$$

and

$$\limsup_{n\to\infty} \|Rx_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = r. \tag{3.17}$$

Now from (3.14) and the boundedness of the sequence $\{v_n - Rx_n - p\}$, we have

$$\begin{aligned} r &= \lim_{n\to\infty} \|y_n - p\| \\ &= \lim_{n\to\infty} \|a'_n Rx_n + b'_n Tz_n + c'_n v_n - p\| \\ &= \lim_{n\to\infty} \|(1 - b'_n)Rx_n + b'_n Tz_n + c'_n v_n - c'_n Rx_n - p\| \\ &= \lim_{n\to\infty} \|(1 - b'_n)(Rx_n - p) + b'_n(Tz_n - p) + c'_n(v_n - Rx_n - p)\| \\ &= \lim_{n\to\infty} \|(1 - b'_n)(Rx_n - p) + b'_n(Tz_n - p)\|. \end{aligned} \tag{3.18}$$

From (3.16), (3.17) and (3.18), using Lemma 2.1 we have

$$\lim_{n\to\infty} \|Rx_n - Tz_n\| = 0. \tag{3.19}$$

and hence

$$\|Tz_n - x_n\| \le \|Tz_n - Rx_n\| + \|Rx_n - x_n\| \to 0 \ \ as \ \ n \to \infty. \tag{3.20}$$

Again note that

$$\limsup_{n\to\infty} \|Ux_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = r, \tag{3.21}$$

and

$$\limsup_{n\to\infty} \|Rx_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = r, \tag{3.22}$$

also,

$$\begin{aligned} \|x_n - p\| &\le \|x_n - Tz_n\| + \|Tz_n - p\| \\ &\le \|x_n - Tz_n\| + \|z_n - p\|, \end{aligned}$$

using (3.20), we obtain

$$r = \lim_{n\to\infty} \|x_n - p\| \le \liminf_{n\to\infty} \|z_n - p\|.$$

This together with (3.15) gives

$$\lim_{n\to\infty} \|z_n - p\| = r. \tag{3.23}$$

Now from (3.23) and the boundedness of the sequence $\{w_n - Rx_n - p\}$, we have

$$\begin{aligned} r &= \lim_{n\to\infty} \|z_n - p\| \\ &= \lim_{n\to\infty} \|a''_n Rx_n + b''_n Ux_n + c''_n w_n - p\| \\ &= \lim_{n\to\infty} \|(1 - b''_n)Rx_n + b''_n Ux_n + c''_n w_n - c''_n Rx_n - p\| \\ &= \lim_{n\to\infty} \|(1 - b''_n)(Rx_n - p) + b''_n(Ux_n - p) + c''_n(w_n - Rx_n - p)\| \\ &= \lim_{n\to\infty} \|(1 - b''_n)(Rx_n - p) + b''_n(Ux_n - p)\|. \end{aligned} \tag{3.24}$$

From (3.21), (3.22) and (3.24), using Lemma 2.1 we have

$$\lim_{n\to\infty} \|Rx_n - Ux_n\| = 0. \tag{3.25}$$

Using (3.5), it follows then that

$$\begin{aligned} \|Ux_n - x_n\| &\le \|Ux_n - Rx_n\| + \|Rx_n - x_n\| \\ &\le 2\|Ux_n - Rx_n\| \to 0 \ \ as \ \ n \to \infty. \end{aligned} \tag{3.26}$$

Consequently, we have

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - Tz_n\| + \|Tz_n - Tx_n\| \\
&\leq \|x_n - Tz_n\| + \|z_n - x_n\| \\
&\leq \|x_n - Tz_n\| + \|a''_n Rx_n + b''_n Ux_n + c''_n w_n - x_n\| \\
&\leq \|x_n - Tz_n\| + a''_n \|Rx_n - x_n\| + b''_n \|Ux_n - x_n\| \\
&\quad + c''_n \|w_n - x_n\|, \qquad (3.27)
\end{aligned}$$

using (3.11), (3.20) and (3.26) in (3.27), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \qquad (3.28)$$

And

$$\begin{aligned}
\|x_n - Sx_n\| &\leq \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \\
&\leq \|x_n - Sy_n\| + \|y_n - x_n\| \\
&\leq \|x_n - Sy_n\| + \|a'_n Rx_n + b'_n Tz_n + c'_n v_n - x_n\| \\
&\leq \|x_n - Sy_n\| + a'_n \|Rx_n - x_n\| + b'_n \|Tz_n - x_n\| \\
&\quad + c'_n \|v_n - x_n\|, \qquad (3.29)
\end{aligned}$$

using (3.11), (3.12) and (3.20) in (3.29), we have

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \qquad (3.30)$$

Thus from (3.11), (3.30), (3.28) and (3.26), we have

$$\lim_{n\to\infty} \|Rx_n - x_n\| = \lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Ux_n - x_n\| = 0.$$

This completes the proof. □

We first establish the weak convergence theorem for the iteration scheme (1.8).

Theorem 3.3. *Let E be a uniformly convex Banach space satisfies the Opial's condition and K, R, S, T, U and $\{x_n\}$ be as in Lemma 3.2. If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$, $0 < \alpha \leq b_n, b'_n, b''_n \leq \beta < 1$ for some $\alpha, \beta \in (0,1)$ and R, S, U satisfy the conditions (3.4) and (3.5), then $\{x_n\}$ converges weakly to a common fixed point of the mappings R, S, T and U.*

Proof. Let $p \in \mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U)$, then as proved in Lemma 3.1, we get $\lim_{n\to\infty} \|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U)$. To prove this, let q_1 and q_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Lemma 3.2, $\lim_{n\to\infty} \|x_n - Rx_n\| = 0$ and $I - R$ is demiclosed with respect to zero by Lemma 2.3, therefore we obtain $Rq_1 = q_1$. Similarly, $Sq_1 = q_1$, $Tq_1 = q_1$ and $Uq_1 = q_1$. Again in the same way as above, we can prove that $q_2 \in F(R) \cap F(S) \cap$

$F(T) \cap F(U)$. Next, we prove the uniqueness. For this we suppose that $q_1 \neq q_2$, then by the Opial's condition

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q_1\| &= \lim_{i\to\infty} \|x_{n_i} - q_1\| \\
&< \lim_{i\to\infty} \|x_{n_i} - q_2\| \\
&= \lim_{n\to\infty} \|x_n - q_2\| \\
&= \lim_{j\to\infty} \left\|x_{n_j} - q_2\right\| \\
&< \lim_{j\to\infty} \left\|x_{n_j} - q_1\right\| \\
&= \lim_{n\to\infty} \|x_n - q_1\|.
\end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a common fixed point in $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U)$. This completes the proof. □

Our next aim to prove strong convergence theorems. Recall that the following: A mapping $T: K \to K$ where K is a subset of E, is said to satisfy *condition* (A) [1] if there exists a nondecreasing function $f: [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$ and $F(T)$ denote the set of all fixed points of T.

Senter and Dotson [1] approximated fixed points of nonexpansive mapping T by Mann iterates. Later on, Maiti and Ghosh [14] and Tan and Xu [12] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same condition (A) which is weaker than the requirement that T is demicompact.

We modify the condition (A) for four mappings $R, S, T, U: K \to K$ as follows:

Four mappings $R, S, T, U: K \to K$ where K is a subset of E, are said to satisfy condition (GA) if there exists a nondecreasing function $f: [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\frac{1}{4}\Big(\|x - Rx\| + \|x - Sx\| + \|x - Tx\| + \|x - Ux\| \Big) \geq f(d(x, \mathcal{F}))$$

for all $x \in K$ where $d(x, \mathcal{F}) = \inf\{\|x - x^*\| : x^* \in \mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U)\}$.

Note that condition (GA) reduces to condition (A) when $R = S = T = U$. We shall use condition (GA) instead of the compactness of K to study the strong convergence of $\{x_n\}$ defined as in (1.8).

Theorem 3.4. *Let E be a uniformly convex Banach space and K, $\{x_n\}$ be as in Lemma 3.2. Let $R, S, T, U: K \to K$ be four nonexpansive mappings satisfying condition (GA). If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$, $0 < \alpha \leq b_n, b'_n, b''_n \leq \beta < 1$ for some $\alpha, \beta \in (0,1)$ and R, S, U satisfy the conditions (3.4) and (3.5), then $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U.*

Proof. By Lemma 3.1, we know that $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in \mathcal{F}$. Let $\lim_{n\to\infty}\|x_n - p\| = r$ for some $r \geq 0$. If $r = 0$, we are done. Suppose that $r > 0$. By Lemma 3.2 we know that

$$\lim_{n\to\infty}\|Rx_n - x_n\| = \lim_{n\to\infty}\|Sx_n - x_n\| = \lim_{n\to\infty}\|Tx_n - x_n\| = \lim_{n\to\infty}\|Ux_n - x_n\| = 0.$$

Let $M = \sup_{n\geq 1}\{\|u_n - x_n\|\}$. Moreover, from (3.3), we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq \|x_n - p\| + D_n\\ &= \|x_n - p\| + b_n B_n + c_n\|u_n - p\|\\ &\leq \|x_n - p\| + B_n + c_n(\|u_n - x_n\| + \|x_n - p\|)\\ &\leq (1 + c_n)\|x_n - p\| + B_n + c_n M\\ &\leq (1 + c_n)\|x_n - p\| + (B_n + c_n)M \end{aligned} \tag{3.31}$$

where $D_n = b_n B_n + c_n\|u_n - p\|$ with $\sum_{n=1}^{\infty} B_n < \infty$ and $\sum_{n=1}^{\infty} D_n < \infty$.

This implies that $d(x_{n+1}, \mathcal{F}) \leq (1 + c_n)d(x_n, \mathcal{F}) + (B_n + c_n)M$ and hence $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists by virtue of Lemma 2.2. By condition (GA), we have

$$\lim_{n\to\infty} f(d(x_n, \mathcal{F})) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in E.

Let $\varepsilon > 0$. We choose a positive integer N_1 such that

$$d(x_{N_1}, \mathcal{F}) < \frac{\varepsilon}{4}. \tag{3.32}$$

We next choose $p^* \in \mathcal{F}$ such that

$$\|x_{N_1} - p^*\| < \frac{\varepsilon}{4}. \tag{3.33}$$

By $\lim_{n\to\infty}\|x_n - p\|$ exists, the sequence $\{\|x_n - p\|\}$ is bounded. Let $M^* = \sup_{n\geq 1}\{\|x_n - p\|, M\}$. Then from (3.31), we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| + (B_n + c_n)M^*. \tag{3.34}$$

Since $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$, there exists a positive integer N_2 such that

$$\sum_{k=N_2}^{\infty} \theta_k < \frac{\varepsilon}{4}, \tag{3.35}$$

where $\theta_k = (B_k + c_k)M^*$. We take $N = \max\{N_1, N_2\}$. Let $n \geq N$ and $m \geq 1$. It follows from (3.33), (3.34) and (3.35) that

$$\begin{aligned}
\|x_{n+m}-x_n\| &\le \|x_{n+m}-p^*\| + \|p^*-x_n\| \\
&\le \|x_n-p^*\| + \sum_{k=n}^{n+m-1}\theta_k + \|p^*-x_n\| \\
&= 2\,\|x_n-p^*\| + \sum_{k=n}^{n+m-1}\theta_k \\
&\le 2\,\|x_N-p^*\| + 2\left\{\sum_{k=N}^{n-1}\theta_k + \sum_{k=n}^{n+m-1}\theta_k\right\} \\
&\le 2\,\|x_N-p^*\| + 2\sum_{k=N}^{n+m-1}\theta_k \\
&\le 2\frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in E. Since K is closed, $x_n \to x \in K$. By the continuities of R, S, T, U and (3.11), (3.30), (3.28), (3.26), we get $Rx = Sx = Tx = Ux = x$. So $x \in \mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U)$. This shows that $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U. This completes the proof. □

For our next result, we shall need the following definition.

Definition 3.5. Let K be a nonempty closed subset of a Banach space E. A mapping $T\colon K \to K$ is said to be *semi-compact*, if for any bounded sequence $\{x_n\}$ in K such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\lim_{n\to\infty} x_{n_j} = x \in K$.

Theorem 3.6. *Let E be a uniformly convex Banach space and K, $\{x_n\}$ be as in Lemma 3.2. Let $R, S, T, U\colon K \to K$ be four nonexpansive mappings. If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset$, $0 < \alpha \le b_n, b'_n, b''_n \le \beta < 1$ for some $\alpha, \beta \in (0,1)$ and R, S, U satisfy the conditions (3.4) and (3.5). Suppose one of the mappings in $\{R, S, T, U\}$ is semi-compact. Then $\{x_n\}$ converges strongly to a common fixed of the mappings R, S, T and U.*

Proof. Suppose R is semi-compact. By Lemma 3.2, we have

$$\lim_{n\to\infty}\|x_n - Rx_n\| = 0.$$

So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = x^* \in K$. Now Lemma 3.2 guarantees that $\lim_{n_j\to\infty}\left\|x_{n_j} - Rx_{n_j}\right\| = 0$, $\lim_{n_j\to\infty}\left\|x_{n_j} - Sx_{n_j}\right\| = 0$, $\lim_{n_j\to\infty}\left\|x_{n_j} - Tx_{n_j}\right\| = 0$, $\lim_{n_j\to\infty}\left\|x_{n_j} - Ux_{n_j}\right\| = 0$ and so $\|x^* - Rx^*\| = 0$, $\|x^* - Sx^*\| = 0$, $\|x^* - Tx^*\| = 0$, $\|x^* - Ux^*\| = 0$. This implies that $x^* \in \mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U)$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, it follows, as in the proof of Theorem 3.4, that $\{x_n\}$ converges strongly to a common fixed point of the mappings R, S, T and U. This completes the proof. □

If $b''_n = c''_n = 0$ and $R = U = I$ (the identity map), then (3.4) and (3.5) are automatically satisfied and we have the following.

Corollary 3.7 ([3, Theorem 1, Theorem 2])**.** *Let E be a uniformly convex Banach space and K, S, T and $\{x_n\}$ be as in Theorem 3.4. Suppose $F(S) \cap F(T) \neq \emptyset$. Then*

1. *If E has the Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of the mappings S and T.*
2. *If the mappings S and T satisfy condition (A'), then $\{x_n\}$ converges strongly to a common fixed point of the mappings S and T.*

Remark 3.8.

(i) Theorem 3.3 and 3.4 extend Theorem 6 and 7 of Boonchari and Saejung [5] to the case of three-step iteration scheme with errors for four nonexpansive mappings considered in this paper.

(ii) Theorem 3.3 and 3.4 also extend and improve the corresponding results of Khan and Fukhar-ud-din [3] in the following ways:

- *(a) We remove the boundedness of K.*
- *(b) The identity mapping in [3] is replaced by the more general nonexpansive mapping.*
- *(c) The two-step iteration scheme with errors in [3] for two nonexpansive mappings are extended to the three-step iteration scheme with errors for four nonexpansive mappings.*

(iii) Our results also extend and improve the corresponding results of Takahashi and Tamura [4] to the case of three-step iteration scheme with errors for four nonexpansive mappings considered in this paper.

The following example shows that our results extend substantially the results in [3].

Example 3.9 ([10])**.** *Let E be the real line with the usual norm $|\cdot|$ and let $K = [-1, 1]$. Define $R, S, T, U \colon K \to K$ by*

$$R(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ -x, & \text{if } x \in [-1,0). \end{cases}, \quad S(x) = \begin{cases} -\sin x, & \text{if } x \in [0,1], \\ \sin x, & \text{if } x \in [-1,0). \end{cases}$$

$$T(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,1], \\ -\frac{x}{2}, & \text{if } x \in [-1,0). \end{cases} \quad \text{and } U(x) = \begin{cases} \frac{x}{3}, & \text{if } x \in [0,1], \\ -\frac{x}{3}, & \text{if } x \in [-1,0). \end{cases}$$

for $x \in K$. Obviously, $F(R) \cap F(S) \cap F(T) \cap F(U) = \{0\}$. Now we check that S is nonexpansive. In fact, if x and $y \in [0,1]$ or if x and $y \in [-1,0)$, then

$|Sx - Sy| = |sin\,x - sin\,y| = 2|cos\,\frac{x+y}{2} sin\,\frac{x-y}{2}| = 2|sin\,\frac{x-y}{2}| \leq 2|\frac{x-y}{2}| = |x - y|$; *if* $x \in [0, 1]$ *and* $y \in [-1, 0)$ *or* $x \in [-1, 0)$ *and* $y \in [0, 1]$, *then*

$$|Sx - Sy| = |sin\,x + sin\,y| = 2\left|sin\,\frac{x+y}{2}\ cos\,\frac{x-y}{2}\right| \leq |x + y| \leq |x - y|.$$

That is, S is nonexpansive. Similarly, we can verify that R, T and U are nonexpansive. Moreover, it is not difficult to see that nonexpansive mappings R, S, T and U satisfy condition (GA).

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