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Normal Approximation of Number of Isolated Vertices in a Random Graph

Y. Punkla and N. Chaidee

Abstract : In this paper, we give bounds in normal approximation of number of isolated vertices in a random graph on n vertices. The technique we used here is the Stein's method.

Keywords : Random graph, isolated vertices, normal approximation, Stein's method.
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1 Introduction

A random graph is a graph generated by some random procedure. The study of random graphs has a long history. A systematic study began with the influential work of Erdös and Rényi in 1959-1961 ([2], [3], [4]) and has developed into one of the mainstays of modern discrete mathematics.

Let $G(n,p)$ be a random graph on n labeled vertices $\{1,2,\dots,n\}$ where each possible edge $\{i,j\}$ is present randomly and independently with the probabiity p, $0<p<1$. Let

$$X_i = \begin{cases} 1, & \text{if the } i-\text{th vertex is isolated;} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$W_n = X_1 + X_2 + \cdots + X_n.$$

Note that W_n is the number of isolated vertices in $G(n,p)$ and the expectation of W_n is

$$\lambda_n := EW_n = nq^{n-1},$$
$$\sigma_n^2 := \operatorname{Var} W_n = nq^{n-1}(1 + nq^{n-2}(p - \frac{1}{n}))$$

where $q = 1-p$ (see [8], p. 137).

In [7], Teerapabolarn, Neammanee and Chongcharoen showed that $X_1, X_2, \dots, X_n$ are not independent and the distribution of W_n can be approximated by Poisson with parameter λ_n. Here is their result.

Theorem 1.1

$$\left|P(W_n = 0) - e^{-\lambda_n}\right| \le (\lambda_n + e^{-\lambda_n} - 1)\left(\frac{(n-2)p+1}{n(1-p)}\right).$$

In 1987, Kordecki solved the problem by using normal approximation. His bound is as follows :

Theorem 1.2 *If $p = w_n/n, \log n - w_n \to \infty$ and $nw_n \to \infty$, then there exists a constant $C \equiv C(z)$ such that*

$$\sup_{z\in\mathbb{R}}\left|P\left(\frac{W_n - \lambda_n}{\sigma_n} \le z\right) - \Phi(z)\right| \le \frac{C}{\sigma_n} \tag{1.1}$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{t^2}{2}}\,dt$$

is the standard normal distribution function.

Notice that the constant C in the above theorem is not good enough because it depends on z which goes to infinity as $z \to \infty$ (see [8], Lemma 2.1(3)). In this work, we improve the upper bound in (1.1) to the case of non-uniform and uniform bounds. The followings are our main results.

Theorem 1.3 *(non-uniform) If $p = w_n/n, \log n - w_n \to \infty$ and $nw_n \to \infty$, then there exists a constant C, independent of z, such that*

$$\left|P\left(\frac{W_n - \lambda_n}{\sigma_n} \le z\right) - \Phi(z)\right| \le \frac{C}{(1+|z|)\sigma_n}. \tag{1.2}$$

Corollary 1.4 *(uniform) If $p = w_n/n, \log n - w_n \to \infty$ and $nw_n \to \infty$, then there exists a constant C, independent of z, such that*

$$\sup_{z\in\mathbb{R}}\left|P\left(\frac{W_n - \lambda_n}{\sigma_n} \le z\right) - \Phi(z)\right| \le \frac{C}{\sigma_n}.$$

Note that if $np = c$ where c is a constant, then the Poisson convergence is impossible since

$$\lim_{n\to\infty}(\lambda_n + e^{-\lambda_n} - 1)\left(\frac{(n-2)p+1}{n(1-p)}\right) = (1+c)e^{-c}$$

and in this case the rate of convergence to the normal distribution has the order $\dfrac{1}{\sqrt{n}}$.

2 Proof of Main Results

In this section, we give the proof of main results by using Stein's method for normal approximation and the idea from Kordecki [8].

Stein's method was given by Stein [5] in 1972. His technique was relied on the elementary differential equation

$$f'(w) - wf(w) = I_{\{w \le z\}} - \Phi(z) \tag{2.1}$$

where I_A is defined by

$$I_A(w) = \begin{cases} 1, & \text{if } w \in A; \\ 0, & \text{if } w \notin A. \end{cases}$$

It is well-known that the unique solution φ_z of Stein's equation (2.1) is of the form

$$\varphi_z(x) = \begin{cases} \sqrt{2\pi} e^{\frac{x^2}{2}} \Phi(x)[1-\Phi(z)], & \text{if } x \le z; \\ \\ \sqrt{2\pi} e^{\frac{x^2}{2}} \Phi(z)[1-\Phi(x)], & \text{if } x \ge z. \end{cases} \tag{2.2}$$

Lemma 2.1 *For $z > 0$ and $x \neq z$, we have*

(i) $|\varphi_z''(x)| \le 4.32$ *for* $0 < z < 1$,

(ii) $|\varphi_z''(x)| \le \dfrac{C}{1+z}$ *for* $0 < x < \dfrac{z}{2}$ *and some constant* C *and*

(iii) $|\varphi_z''(x)| \le \dfrac{e^{-z^2/2}}{\sqrt{2\pi}} + \Phi(z)(\dfrac{1}{z} + z)$ *for* $0 < x < z$ *and* $z \ge 1$.

Proof. From (2.2), we note that

$$\varphi_z'(x) = \begin{cases} \sqrt{2\pi} e^{\frac{x^2}{2}} (1-\Phi(x))\,\Phi(z), & \text{if } x \ge z; \\ \\ \sqrt{2\pi} e^{\frac{x^2}{2}} (1-\Phi(z))\,\Phi(x), & \text{if } x < z, \end{cases}$$

and

$$\varphi_z''(x) = \begin{cases} -\Phi(z)[x - \sqrt{2\pi}(1-\Phi(x))(1+x^2)e^{x^2/2}], & \text{if } x > z; \\ \\ (1-\Phi(z))[x + \sqrt{2\pi}\,\Phi(x)(1+x^2)e^{x^2/2}], & \text{if } x < z. \end{cases}$$

(i) Let $0 < z < 1$.

Case 1 $0 < x < z$.

Since

$$\varphi_z'''(x) = \begin{cases} -\Phi(z)[2 + x^2 - \sqrt{2\pi} x e^{x^2/2}(1-\Phi(x))(3+x^2)], & \text{if } x > z; \\ \\ (1-\Phi(z))[2 + x^2 + \sqrt{2\pi} x e^{x^2/2}\,\Phi(x)(3+x^2)], & \text{if } x < z, \end{cases}$$

$\varphi_z'''(x) > 0$ for $0 < x < z$, so that φ_z'' is an increasing function on $(0, z)$. Hence

$$0 \le \sqrt{\frac{\pi}{2}}\,(1 - \Phi(z)) = \varphi_z''(0) \le \varphi_z''(x) \le \frac{1}{2}[1 + 2\sqrt{2\pi}\,\Phi(1)e^{1/2}] \le 3.98. \qquad (2.3)$$

Case 2 $x < -1$.
Since

$$\frac{1}{x^3} < x + \sqrt{2\pi}\,\Phi(x)(1 + x^2)e^{x^2/2} < -\frac{1}{x} \quad \text{if} \quad x \le 0, \qquad (2.4)$$

([8], p. 133), we have

$$\left|x + \sqrt{2\pi}\,\Phi(x)(1 + x^2)e^{x^2/2}\right| < \left|\frac{1}{x}\right| < 1.$$

Therefore,

$$|\varphi_z''(x)| \le 1. \qquad (2.5)$$

Case 3 $-1 < x < 0$.

$$\begin{aligned} |\varphi_z''(x)| &\le \frac{1}{2}[|x| + \sqrt{2\pi}\,\Phi(x)(1 + x^2)e^{1/2}] \\ &\le \frac{1}{2}(1 + 2\sqrt{2\pi}\,\Phi(0)e^{1/2}) \\ &\le 2.57. \qquad (2.6) \end{aligned}$$

Case 4 $x > z$.
We note that

$$-\frac{1}{x} < x - \sqrt{2\pi}(1 - \Phi(x))(1 + x^2)e^{x^2/2} < \frac{1}{x^3} \quad \text{if} \quad x > 0 \qquad (2.7)$$

([8], p. 133).
If $x \ge 1$, then, by (2.7),

$$\left|x - \sqrt{2\pi}(1 - \Phi(x))(1 + x^2)e^{x^2/2}\right| < \left|\frac{1}{x}\right| \le 1$$

which implies $|\varphi_z''(x)| \le 1$.
If $x < 1$, then

$$|\varphi_z''(x)| \le \Phi(1)[|x| + 2\sqrt{2\pi}(1 - \Phi(0))e^{1/2}] \le \Phi(1)[1 + \sqrt{2\pi}e^{1/2}] \le 4.32. \qquad (2.8)$$

By (2.3), (2.5), (2.6) and (2.8), we completed the proof.

(ii) For $0 < x \le \frac{z}{2}$, we have

$$\begin{aligned}
|\varphi_z''(x)| &\le |1-\Phi(z)||x+\Phi(x)\sqrt{2\pi}(1+x^2)e^{x^2/2}| \\
&\le \frac{e^{-z^2/2}}{z\sqrt{2\pi}}(\frac{z}{2}+\sqrt{2\pi}(1+\frac{z^2}{4})e^{z^2/8}) \\
&\le 0.2e^{-z^2/2}+\frac{1+\frac{z^2}{4}}{z}e^{-3z^2/8} \\
&\le 0.2e^{-z^2/2}+Ce^{-z/2} \\
&\le Ce^{-z/2} \\
&\le \frac{C}{1+z}.
\end{aligned}$$

(iii) If $0 < x < z$ and $z \ge 1$, then

$$\begin{aligned}
0 \le \varphi_z''(x) &= (1-\Phi(z))[x+\sqrt{2\pi}\Phi(x)(1+x^2)e^{\frac{x^2}{2}}] \\
&\le \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}[z+\sqrt{2\pi}\Phi(z)(1+z^2)e^{\frac{z^2}{2}}] \\
&= \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}+\Phi(z)(\frac{1}{z}+z)
\end{aligned}$$

where we have used the fact that $1-\Phi(z) \le \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}$ for $z > 0$ in the first inequality. □

Lemma 2.2 *If* $p = \frac{w_n}{n}$, $\log n - w_n \to \infty$ *and* $nw_n \to \infty$, *then*

(i) $E(W_{n-1}-W_n+1)^2 = \mathrm{O}(1)$ *and*

(ii) $E(W_{n-1}-W_n+1)^4 = \mathrm{O}(1)$.

Proof. (i) See [8].
(ii) For $i = 1, 2, \ldots, n-1$, let

$$Y_i = \begin{cases} 1, & \text{if the } i\text{-th vertex has degree 1 and it is jointed with the } n\text{-th vertex;} \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned}
E(W_{n-1}-W_n+1)^4 &= E(Y_1+Y_2+\cdots+Y_{n-1}-X_n+1)^4 \\
&\le C\Big\{E(Y_1+Y_2+\cdots+Y_{n-1})^4+EX_n^4+1\Big\}. \qquad (2.9)
\end{aligned}$$

Note that $EX_n^4 = q^{n-1} \leq 1$ and

$$EY_i^4 = P(Y_i = 1) = pq^{n-2}.$$

If $i \neq j$, then

$$EY_i^3 Y_j = P(Y_i = 1, Y_j = 1) = p^2 q^{n-2} q^{n-3} = p^2 q^{2n-5} = EY_i^2 Y_j^2.$$

If $i \neq j \neq k$, then

$$EY_i^2 Y_j Y_k = P(Y_i = 1, Y_j = 1, Y_k = 1) = p^3 q^{n-2} q^{n-3} q^{n-4} = p^3 q^{3n-9}.$$

If $i \neq j \neq k \neq l$, then

$$EY_i Y_j Y_k Y_l = P(Y_i = 1, Y_j = 1, Y_k = 1, Y_l = 1) = p^4 q^{n-2} q^{n-3} q^{n-4} q^{n-5} = p^4 q^{4n-14}.$$

Therefore,

$$\begin{aligned} E(Y_1 + Y_2 + \cdots + Y_{n-1})^4 &= \sum_{i=1}^{n-1} EY_i^4 + \sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n-1} \Big\{ EY_i^3 Y_j + EY_i^2 Y_j^2 \Big\} \\ &\quad + \sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n-1} \sum_{\substack{k=1 \\ k \neq i,j}}^{n-1} EY_i^2 Y_j Y_k + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{\substack{k=1 \\ i \neq j \neq k \neq l}}^{n-1} \sum_{l=1}^{n-1} EY_i Y_j Y_k Y_l \\ &= (n-1)pq^{n-2} + (n-1)(n-2)(p^2 q^{2n-5} + p^2 q^{2n-5}) \\ &\quad + (n-1)(n-2)(n-3)p^3 q^{3n-9} \\ &\quad + (n-1)(n-2)(n-3)(n-4)p^4 q^{4n-14} \\ &= (n-1)pq^{n-2} \Big\{ 1 + 2(n-2)pq^{n-3} + (n-2)(n-3)p^2 q^{2n-7} \\ &\qquad + (n-2)(n-3)(n-4)p^3 q^{3n-10} \Big\}. \end{aligned}$$

Let $p = \frac{w_n}{n}$. Then $q^n = (1 - \frac{w_n}{n})^n \approx e^{-w_n} = e^{-np}$. Therefore,

$$(n-1)pq^{n-2} = \frac{(n-1)p}{q^2} e^{-np} \leq \frac{1}{q^2} = \frac{1}{(1-p)^2}$$

and

$$(n-2)pq^{n-3} = \frac{(n-2)p}{q^3} e^{-np} \leq \frac{1}{q^3} = \frac{1}{(1-p)^3}.$$

Similarly,

$$(n-2)(n-3)p^2 q^{2n-7} \leq \frac{1}{(1-p)^7} \quad \text{and} \quad (n-2)pq^{n-3} \leq \frac{1}{(1-p)^{10}}.$$

Hence

$$E(Y_1 + Y_2 + \cdots + Y_{n-1})^4 \le \frac{1}{(1-p)^{12}}. \tag{2.10}$$

Therefore, by (2.9) and (2.10),

$$E(W_{n-1} - W_n + 1)^4 = \mathrm{O}(1).$$

□

Proof of Theorem 1.3.

To prove Theorem 1.3, we let

$$U_n = \frac{W_n - \lambda_n}{\sigma_n}.$$

It suffices to consider $z \ge 0$ as we can apply the result to $-U_n$ when $z < 0$. Let $z > 0$. By Stein equation (2.1),

$$E(\varphi'(U_n) - U_n\varphi(U_n)) = P(U_n \le z) - \Phi(z).$$

In [8], Kordecki showed that

$$\begin{aligned}|E(\varphi_z'(U_n) - U_n\varphi_z(U_n))| \le & \left|E\left(\varphi_z'(U_n)\left(1 - \frac{\lambda_n}{\sigma_n^2}(W_{n-1} - W_n + 1)\right)\right)\right| \\ & + \frac{\lambda_n}{2\sigma_n^3}E\left(\left|\varphi_z''\left(U_n + \nu\frac{W_{n-1} - W_n + 1}{\sigma_n}\right)\right|(W_{n-1} - W_n + 1)^2\right) \\ & := A_1 + A_2\end{aligned}$$

where $0 < \nu < 1$ and

$$E\left(\varphi_z'(U_n)(W_{n-1} - W_n + 1)\right) = E\left(\varphi_z'(U_n)E(W_{n-1} - W_n + 1|W_n)\right).$$

By Chen and Shao [1], we have

$$E|\varphi_z'(U_n)| \le \frac{C}{(1+z)^2}$$

for some constant C. Therefore

$$\begin{aligned}&\left|E\left(\varphi_z'(U_n)E(1 - \frac{\lambda_n}{\sigma_n^2}(W_{n-1} - W_n + 1|W_n)) - \varphi_z'(U_n)E(1 - \frac{\lambda_n}{\sigma_n^2}(W_{n-1} - W_n + 1))\right)\right| \\ &\le E|\varphi_z'(U_n)|E|E(1 - \frac{\lambda_n}{\sigma_n^2}(W_{n-1} - W_n + 1|W_n)) - E(1 - \frac{\lambda_n}{\sigma_n^2}(W_{n-1} - W_n + 1))| \\ &\le \frac{C}{(1+z)^2}\left\{\mathrm{Var}(E(1 - \frac{\lambda_n}{\sigma_n^2}(W_{n-1} - W_n + 1|W_n)))\right\}^{1/2} \\ &= \frac{C}{(1+z)^2}\frac{\lambda_n}{\sigma_n^2}\left\{\mathrm{Var}(E((W_{n-1} - W_n + 1|W_n)))\right\}^{1/2} \\ &= \frac{C}{(1+z)^2}\frac{\lambda_n}{\sigma_n^2}d_n'\end{aligned}$$

where

$$d'_n = \{\mathrm{Var}(E((W_{n-1} - W_n + 1|W_n)))\}^{1/2}.$$

Hence we obtain that

$$\begin{aligned} A_1 &\le \left| E\varphi'_z(U_n)E(1 - \frac{\lambda_n}{\sigma_n^2}(W_{n-1} - W_n + 1)) \right| + \frac{C}{(1+z)^2}\frac{\lambda_n}{\sigma_n^2}d'_n \\ &\le \frac{C}{(1+z)^2}\left| 1 - \frac{\lambda_n(\lambda_{n-1} - \lambda_n + 1)}{\sigma_n^2} \right| + \frac{C}{(1+z)^2}\frac{\lambda_n}{\sigma_n^2}d'_n \\ &= \frac{C}{(1+z)^2\sigma_n} \end{aligned}$$

where we have used the fact that $|E(1 - \frac{\lambda_n}{\sigma_n^2}(W_{n-1} - W_n + 1))| = 0$ and $\frac{\lambda_n}{\sigma_n^2}d'_n = \mathrm{O}(\sigma_n^{-1})$ (see [8], p. 139) in the last inequality.
Consider A_2: if $0 < z < 1$, then by Lemma 2.1(i) and Lemma 2.2(i),

$$A_2 \le \frac{\lambda_n}{2\sigma_n^3}E(W_{n-1} - W_n + 1)^2 = \frac{\lambda_n}{2\sigma_n^3}\frac{C}{1+z}.$$

If $z \ge 1$, then

$$\begin{aligned} A_2 = &\frac{\lambda_n}{2\sigma_n^3}E\big(|\varphi''_z(U_n + \nu\frac{W_{n-1} - W_n + 1}{\sigma_n})| \\ &\qquad \times (W_{n-1} - W_n + 1)^2 I(U_n + \nu\frac{W_{n-1} - W_n + 1}{\sigma_n} < 0)\big) \\ &+ \frac{\lambda_n}{2\sigma_n^3}E\big(|\varphi''_z(U_n + \nu\frac{W_{n-1} - W_n + 1}{\sigma_n})| \\ &\qquad \times (W_{n-1} - W_n + 1)^2 I(0 \le U_n + \nu\frac{W_{n-1} - W_n + 1}{\sigma_n} \le \frac{z}{2})\big) \\ &+ \frac{\lambda_n}{2\sigma_n^3}E\big(|\varphi''_z(U_n + \nu\frac{W_{n-1} - W_n + 1}{\sigma_n})| \\ &\qquad \times (W_{n-1} - W_n + 1)^2 I(U_n + \nu\frac{W_{n-1} - W_n + 1}{\sigma_n} > \frac{z}{2})\big) \\ := &A_{21} + A_{22} + A_{23}. \end{aligned}$$

First we estimate A_{21}. If $-1 < x < 0$, then

$$|\varphi''_z(x)| \le C(1 - \Phi(z)) \le \frac{Ce^{-z^2/2}}{\sqrt{2\pi}z}.$$

Suppose that $x < -1$. By (2.4), we have

$$|\varphi''_z(x)| \le (1 - \Phi(z)) \le \frac{e^{-z^2/2}}{\sqrt{2\pi}z}$$

which implies that,

$$A_{21} \le \frac{\lambda_n}{2\sigma^3} \frac{Ce^{-z^2/2}}{\sqrt{2\pi}z}.$$

By Lemma 2.1(ii) and Lemma 2.2(i),

$$A_{22} \le \frac{\lambda_n}{2\sigma_n^3} \cdot \frac{C}{1+z} E(W_{n-1} - W_n + 1)^2 = \frac{\lambda_n}{2\sigma_n^3} \cdot \frac{C}{1+z}.$$

By (2.7), we have $|\varphi_z''(x)| \le 2$ for $x > \frac{z}{2}$ and $z \ge 1$, so

$$\begin{aligned}
A_{23} &\le \frac{\lambda_n}{\sigma_n^3} |E(W_{n-1} - W_n + 1)^2 I(U_n + \nu \frac{W_{n-1} - W_n + 1}{\sigma_n} > \frac{z}{2})| \\
&\le \frac{\lambda_n}{\sigma_n^3} \{E(W_{n-1} - W_n + 1)^4\}^{1/2} \{P(U_n + \nu \frac{W_{n-1} - W_n + 1}{\sigma_n} > \frac{z}{2}))\}^{1/2} \\
&\le \frac{\lambda_n}{\sigma_n^3} \{E(W_{n-1} - W_n + 1)^4\}^{1/2} \{\frac{E(U_n + \nu \frac{W_{n-1} - W_n + 1}{\sigma_n})^2}{\frac{z^2}{4}}\}^{1/2} \\
&\le \frac{2\sqrt{2}\lambda_n}{\sigma_n^3 z} \{E(W_{n-1} - W_n + 1)^4\}^{1/2} \{E(U_n^2) + E(\nu \frac{W_{n-1} - W_n + 1}{\sigma_n})^2\}^{1/2} \\
&\le \frac{C\lambda_n}{\sigma_n^3} \frac{1}{1+z} \{E(W_{n-1} - W_n + 1)^4\}^{1/2} \{1 + \frac{\mathrm{O}(1)}{\sigma_n^2}\}^{1/2}.
\end{aligned}$$

Since $\sigma_n^2 \to \infty$(see [8], p.137) and by Lemma 2.2(2),

$$A_{23} \le \frac{C\lambda_n}{2\sigma_n^3} \frac{1}{1+z} \{1 + \frac{\mathrm{O}(1)}{\sigma_n^2}\}^{1/2} = \frac{C\lambda_n}{2\sigma_n^3} \frac{1}{1+z}.$$

Since $\lambda_n/\sigma_n^2 = \mathrm{O}(1)$(see [8], p. 137),

$$A_2 \le \frac{C\lambda_n}{2\sigma_n^3} \frac{1}{1+z} = \frac{C}{(1+z)\sigma_n}.$$

The comparison of A_1 and A_2 gives that

$$|P(\frac{W_n - \lambda_n}{\sigma_n} \le z) - \Phi(z)| \le \frac{C}{(1+|z|)\sigma_n}.$$

□

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Y. Punkla and N. Chaidee
Department of Mathematics
Chulalongkorn University
Bangkok 10330, Thailand.
e-mail: Yuwaree.P@Chula.ac.th, Nattakarn.C@Chula.ac.th