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Schatten Class Operators on Weighted Bergman Spaces

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Abstract : Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $dA(z)$ be the normalized area measure on $\mathbb{D}$. For $\alpha > -1$, let $d\lambda_\alpha(z) = \frac{dA_\alpha(z)}{(1-|z|^2)^{2+\alpha}}$ where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. In this paper we have shown that if the Toeplitz operator T_ϕ defined on the weighted Bergman space $L_a^2(dA_\alpha)$ belongs to the Schatten class $S_p, 1 \leq p < \infty$, then $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda_\alpha)$ where $\widetilde{\phi}$ is the Berezin transform of ϕ. Further, if $\phi \in L^p(\mathbb{D}, d\lambda_\alpha)$ then $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda_\alpha)$ and $T_\phi \in S_p$. Also, we find conditions on bounded linear operator C defined from $L_a^2(dA_\alpha)$ into itself such that $C \in S_p$ by comparing with or involving Toeplitz operators on weighted Bergman spaces. Applications of these results are also discussed.

Keywords : Schatten class operators; Little Hankel operators; Weighted Bergman spaces; Reproducing kernel; Berezin transform.
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1 Introduction

Let $dA(z)$ denote the Lebesgue area measure on the open unit disk $\mathbb{D}$, normalized so that the measure of the disk $\mathbb{D}$ equals 1. For $\alpha > -1$, the weighted Bergman space $L_a^2(dA_\alpha)$ is the Hilbert space consisting of analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA_\alpha)$ with respect to the measure $dA_\alpha = (\alpha+1)(1-|z|^2)^\alpha dA(z)$.

The reproducing kernel in $L^2_a(dA_\alpha)$ is given by

$$K^\alpha_w(z) = \frac{1}{(1-\overline{w}z)^{2+\alpha}},$$

for $z, w \in \mathbb{D}$. If $\langle , \rangle_\alpha$ denotes the inner product in $L^2(\mathbb{D}, dA_\alpha)$ then $\langle h, K^\alpha_w\rangle_\alpha = h(w)$, for every $h \in L^2_a(dA_\alpha)$ and $w \in \mathbb{D}$. Using the reproducing property of K^α_w we have

$$\|K^\alpha_w\|^2_\alpha = \langle K^\alpha_w, K^\alpha_w\rangle_\alpha = K^\alpha_w(w) = \frac{1}{(1-|w|^2)^{2+\alpha}},$$

thus the normalized reproducing kernel

$$k^\alpha_w(z) = \frac{(1-|w|^2)^{\frac{(2+\alpha)}{2}}}{(1-\overline{w}z)^{2+\alpha}},$$

for $z, w \in \mathbb{D}$. The sequence $\{e^\alpha_n(z)\}_{n=0}^\infty = \left\{\sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}}z^n\right\}_{n=0}^\infty$ forms an orthonormal basis for the weighted Bergman space $L^2_a(dA_\alpha)$. The orthogonal projection P_α of $L^2(\mathbb{D}, dA_\alpha)$ onto $L^2_a(dA_\alpha)$ is given by

$$(P_\alpha g)(w) = \langle g, K^\alpha_w\rangle_\alpha = \int_{\mathbb{D}} g(z)\frac{1}{(1-\overline{z}w)^{2+\alpha}}dA_\alpha(z),$$

for $g \in L^2(\mathbb{D}, dA_\alpha)$ and $w \in \mathbb{D}$. Given $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ is defined on $L^2_a(dA_\alpha)$ by

$$T_\phi h = P_\alpha(\phi h).$$

Thus we have

$$(T_\phi h)(w) = \int_{\mathbb{D}} \frac{\phi(z)h(z)}{(1-\overline{z}w)^{2+\alpha}}dA_\alpha(z), \text{ for } h \in L^2_a(dA_\alpha) \text{ and } w \in \mathbb{D}.$$

We define the Berezin transform of a bounded linear operator S on $L^2_a(dA_\alpha)$ to be the function $\widetilde{S}$ defined on $\mathbb{D}$ by

$$\widetilde{S}(w) = \langle Sk^\alpha_w, k^\alpha_w\rangle_\alpha, \text{ for } w \in \mathbb{D}.$$

Let $\widetilde{\phi}(w) = \langle T_\phi k^\alpha_w, k^\alpha_w\rangle_\alpha$ for $w \in \mathbb{D}$. That is, $\widetilde{\phi} = \widetilde{T_\phi}$. Let $d\lambda_\alpha(z) = K^\alpha_z(z)dA_\alpha(z) = \frac{dA_\alpha(z)}{(1-|z|^2)^{2+\alpha}}$, the Mobius invariant measure on $\mathbb{D}$. Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. Let $L^2_a(\mathbb{D})$ be the subspace of $L^2(\mathbb{D}, dA)$ consisting of analytic functions. The space $L^2_a(\mathbb{D})$ is called the Bergman space. The reproducing kernel of $L^2_a(\mathbb{D})$ is given by $K(z,\overline{w}) = \overline{K_z(w)} = \frac{1}{(1-z\overline{w})^2}$. Let $k_z(w) = \frac{(1-|z|^2)}{(1-\overline{z}w)^2}$. These functions k_z are called the normalized reproducing kernels of $L^2_a(\mathbb{D})$. Let $\phi : \mathbb{D} \to \mathbb{D}$ be analytic. Define the composition operator C_ϕ from $L^2_a(\mathbb{D})$ into itself by $C_\phi f = fo\phi$. The operator C_ϕ is a bounded linear operator on $L^2_a(\mathbb{D})$. The little Hankel operator $S_\phi : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$ is defined by $S_\phi f = PJ(\phi f)$ for $\phi \in L^\infty(\mathbb{D})$ where $J : L^2(\mathbb{D}, dA) \to L^2(\mathbb{D}, dA)$ is defined as $Jf(z) = f(\overline{z})$ and P is the orthogonal projection from $L^2(D, dA)$ onto $L^2_a(\mathbb{D})$. Similarly one can also define little Hankel operators on $L^2_a(\mathbb{D}, dA_\alpha)$. For $\phi \in L^\infty(\mathbb{D})$, the little Hankel operator S_ϕ on $L^2_a(dA_\alpha)$ with symbol ϕ is the operator defined by $S_\phi f = P_\alpha J_\alpha(\phi f)$ where $J_\alpha : L^2(\mathbb{D}, dA_\alpha) \longrightarrow L^2(\mathbb{D}, dA_\alpha)$ is defined as $J_\alpha f(z) = f(\overline{z})$. We can define for each $a \in \mathbb{D}$, an automorphism ϕ_a in $Aut(\mathbb{D})$ such that

(i) $(\phi_a \ o \ \phi_a)(z) \equiv z$;

(ii) $\phi_a(0) = a, \phi_a(a) = 0$;

(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.

In fact, $\phi_a(z) = \frac{a-z}{1-\overline{a}z}$ for all a and z in $\mathbb{D}$. Given $w \in \mathbb{D}$, and h any measurable function on $\mathbb{D}$, we define

$$U^\alpha_w h = (ho\phi_w)k^\alpha_w.$$

Using the identity

$$1 - \overline{\phi_w(z)}w = \frac{1-|w|^2}{1-\overline{z}w}$$

we have

$$k^\alpha_w(\phi_w(z)) = \frac{1}{k^\alpha_w(z)}.$$

Since $\phi_w o\phi_w(z) \equiv z$, we see that

$$(U^\alpha_w(U^\alpha_w h))(z) = h(z)$$

for all $z \in \mathbb{D}$ and $h \in L^2_a(dA_\alpha)$. Thus $(U^\alpha_w)^{-1} = U^\alpha_w$ and hence U^α_w is unitary on $L^2_a(dA_\alpha)$. Furthermore

$$T_{\phi o\phi_w} U^\alpha_w = U^\alpha_w T_\phi.$$

Recall the following : Suppose A is a positive operator on a Hilbert space H and x is a unit vector in H. Then

(i) $\langle A^p x, x\rangle \geq \langle Ax, x\rangle^p$ for all $p \geq 1$;

(ii) $\langle A^p x, x\rangle \leq \langle Ax, x\rangle^p$ for all $0 < p \leq 1$.

For proof see [1]. If T is a compact operator on a separable Hilbert space H, then there exist orthonormal sets $\{u_n\}_{n=0}^\infty$ and $\{\sigma_n\}_{n=0}^\infty$ in H such that $Tx = \sum_{n=0}^\infty \lambda_n \langle x, u_n\rangle \sigma_n$; $x \in H$ where λ_n is the nth singular value of T. Given $0 < p < \infty$, we define the Schatten p-class of H, denoted by $S_p(H)$ or simply S_p, to be the space of all compact operators T on H with its singular value sequence $\{\lambda_n\}$ belonging to l^p (the p-summable sequence space). We will be mainly concerned with the range $1 \leq p < \infty$. In this case, S_p is a Banach space with the norm $\|T\|_p = [\sum_n |\lambda_n|^p]^{\frac{1}{p}}$. The class S_1 is also called the trace class of H and S_2 is usually called the Hilbert-Schmidt class. It is not difficult to verify that if T is a compact operator on H and $p \geq 1$, then $T \in S_p$ if and only if $|T|^p = (T^*T)^{\frac{p}{2}} \in S_1$

and $\|T\|_p^p = \||T|\|_p^p = \||T|^p\|_1$. Let $\mathcal{L}(L_a^2(dA_\alpha))$ be the set of all bounded linear operators from $L_a^2(dA_\alpha)$ into itself. Throughout we assume $p \geq 1$ and S_p is the Schatten p-ideal of $\mathcal{L}(L_a^2(dA_\alpha))$. In this paper we characterize bounded linear operators on $L_a^2(dA_\alpha)$ that belong to the class $S_p, 1 \leq p < \infty$. In section 2, we find conditions on ϕ such that the Toeplitz operators T_ϕ defined on the weighted Bergman spaces belong to the Schatten class $S_p, 1 \leq p < \infty$. In section 3, we find conditions on $C \in \mathcal{L}(L_a^2(dA_\alpha))$ such that $C \in S_p$, the Schatten p-class,$1 \leq p < \infty$ by comparing with positive Toeplitz operators defined on the weighted Bergman spaces $L_a^2(dA_\alpha)$ and applications of the result are also obtained. In section 4, we find necessary and sufficient conditions on $\phi \in L^2(\mathbb{D}, dA)$ such that the little Hankel operator $S_{\overline{\phi}}$ defined on $L_a^2(\mathbb{D})$ belong to the class $S_p, 2 \leq p < \infty$. In section 5, using the $p - C^*$ summing conditions, we obtain a characterization for bounded linear operators to belong to the class S_p. In fact, we have shown that if $A \in \mathcal{L}(L_a^2(dA_\alpha))$ then $T_{\overline{\phi}} A T_\phi \in S_p$ if $\phi \in H^\infty(\mathbb{D})$ and $|\phi|^2 \in L^p(\mathbb{D}, d\lambda_\alpha)$. Also using the concept of m-Berezin transform, we find conditions on ϕ such that the composition operators defined on $L_a^2(\mathbb{D})$ belong to the Schatten class $S_p, 1 \leq p < \infty$.

2 Schatten Class Toeplitz Operators

In this section, we find conditions on ϕ such that the Toeplitz operators T_ϕ defined on the weighted Bergman spaces belong to the Schatten class $S_p, 1 \leq p < \infty$. Let

$$BT = \left\{ f \in L^1(\mathbb{D}, dA) : \|f\|_{BT} = \sup_{z \in \mathbb{D}} \widetilde{|f|}(z) < \infty \right\}.$$

The space L^∞ is properly contained in BT (see [2]) and if $\phi \in BT$ then T_ϕ is bounded on $L_a^2(dA_\alpha)$ and there is a constant C such that $\|T_\phi\| \leq C\|\phi\|_{BT}$.

Theorem 2.1. *Suppose* $1 \leq p < \infty$ *and* $d\lambda_\alpha(z) = \frac{dA_\alpha(z)}{(1-|z|^2)^{2+\alpha}}, \alpha > -1$ *Then the following hold: (1) If* $T_\phi \in S_p$, *then* $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda_\alpha)$. *(2) If* $\phi \in L^p(\mathbb{D}, d\lambda_\alpha)$ *then* $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda_\alpha)$ *and* $T_\phi \in S_p$.

Proof. Suppose $T_\phi \in S_p$. Then

$$\int_{\mathbb{D}} \langle |T_\phi|^p k_w^\alpha, k_w^\alpha \rangle_\alpha \, d\lambda_\alpha(w) < \infty.$$

Hence, $\int_{\mathbb{D}} \left\langle (T_\phi^* T_\phi)^{\frac{p}{2}} k_w^\alpha, k_w^\alpha \right\rangle_\alpha d\lambda_\alpha(w) < \infty$. If $2 \leq p < \infty$, then

$$\int_{\mathbb{D}} \langle T_\phi^* T_\phi k_w^\alpha, k_w^\alpha \rangle_\alpha^{\frac{p}{2}} \, d\lambda_\alpha(w) \leq \int_{\mathbb{D}} \left\langle (T_\phi^* T_\phi)^{\frac{p}{2}} k_w^\alpha, k_w^\alpha \right\rangle_\alpha d\lambda_\alpha(w) < \infty.$$

It follows therefore that

$$\begin{aligned}\int_{\mathbb{D}} \|P_\alpha(\phi o \phi_w)\|_\alpha^p d\lambda_\alpha(w) &= \int_{\mathbb{D}} \|P_\alpha(U_w^\alpha(\phi k_w^\alpha))\|_\alpha^p d\lambda_\alpha(w)\\ &= \int_{\mathbb{D}} \|U_w^\alpha T_\phi k_w^\alpha\|_\alpha^p d\lambda_\alpha(w)\\ &= \int_{\mathbb{D}} \|T_\phi k_w^\alpha\|_\alpha^p d\lambda_\alpha(w)\\ &= \int_{\mathbb{D}} \langle T_\phi^* T_\phi k_w^\alpha, k_w^\alpha\rangle_\alpha^{\frac{p}{2}} d\lambda_\alpha(w) < \infty.\end{aligned}$$

Now

$$\begin{aligned}|P_\alpha(\phi o \phi_w)(0)| &= |\langle P_\alpha(\phi o \phi_w), 1\rangle_\alpha|\\ &= |\langle U_w^\alpha(T_\phi k_w^\alpha), 1\rangle_\alpha|\\ &= |\langle T_\phi k_w^\alpha, U_w^\alpha 1\rangle_\alpha|\\ &= |\langle T_\phi k_w^\alpha, k_w^\alpha\rangle_\alpha|\\ &\leq \|T_\phi k_w^\alpha\|_\alpha\\ &= \|P_\alpha(\phi o \phi_w)\|_\alpha.\end{aligned}$$

Thus

$$\int_{\mathbb{D}} |P_\alpha(\phi o \phi_w)(0)|^p d\lambda_\alpha(w) < \infty.$$

That is, $\int_{\mathbb{D}} |\widetilde{\phi}(w)|^p d\lambda_\alpha(w) < \infty$ and $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda_\alpha)$. Suppose $1 \leq p < 2$. Then by Heinz inequality [3], it follows that

$$\begin{aligned}\infty > \int_{\mathbb{D}} \langle |T_\phi|^p k_w^\alpha, k_w^\alpha\rangle_\alpha d\lambda_\alpha(w) &= \int_{\mathbb{D}} \langle |T_\phi|^{2\cdot\frac{p}{2}} k_w^\alpha, k_w^\alpha\rangle_\alpha d\lambda_\alpha(w)\\ &\geq \int_{\mathbb{D}} \frac{|\langle T_\phi k_w^\alpha, k_w^\alpha\rangle_\alpha|^2}{\left\langle |T_\phi^*|^{2(1-\frac{p}{2})} k_w^\alpha, k_w^\alpha\right\rangle_\alpha} d\lambda_\alpha(w)\\ &= \int_{\mathbb{D}} \frac{|\widetilde{\phi}(w)|^2}{\|P_\alpha(\overline{\phi} o \phi_w)\|_\alpha^{2-p}} d\lambda_\alpha(w)\\ &= \int_{\mathbb{D}} |\widetilde{\phi}(w)|^2 \|P_\alpha(\overline{\phi} o \phi_w)\|_\alpha^{p-2} d\lambda_\alpha(w)\\ &\geq \int_{\mathbb{D}} \frac{|\widetilde{\phi}(w)|^2}{\|P_\alpha(\overline{\phi} o \phi_w)\|_\alpha^2} \|P_\alpha(\overline{\phi} o \phi_w)\|_\alpha^p d\lambda_\alpha(w)\\ &\geq \int_{\mathbb{D}} \frac{|\widetilde{\phi}(w)|^2}{C^2\|\phi\|_{BT}^2} |P_\alpha(\phi o \phi_w)(0)|^p d\lambda_\alpha(w)\\ &= \int_{\mathbb{D}} \frac{|\widetilde{\phi}(w)|^2}{C^2\|\phi\|_{BT}^2} |\widetilde{\phi}(w)|^p d\lambda_\alpha(w)\end{aligned}$$

since

$$\begin{aligned}\left\langle |T_\phi^*|^{2-p} k_w^\alpha, k_w^\alpha \right\rangle_\alpha &= \left\langle |T_\phi^*|^{2\cdot\frac{(2-p)}{2}} k_w^\alpha, k_w^\alpha \right\rangle_\alpha \\ &\leq \left\langle |T_\phi^*|^2 k_w^\alpha, k_w^\alpha \right\rangle_\alpha^{\frac{(2-p)}{2}} \\ &= \left\langle T_\phi T_\phi^* k_w^\alpha, k_w^\alpha \right\rangle_\alpha^{\frac{(2-p)}{2}} \\ &= \|T_\phi^* k_w^\alpha\|_\alpha^{2-p} \\ &= \|P_\alpha(\overline{\phi} o \phi_w)\|_\alpha^{2-p}.\end{aligned}$$

Hence

$$\int_{\mathbb{D}} |\widetilde{\phi}(w)|^{p+2} d\lambda_\alpha(w) < \infty,$$

and therefore $\int_{\mathbb{D}} |\widetilde{\phi}(w)|^p d\lambda_\alpha(w) < \infty$. Thus $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda_\alpha)$.

Now suppose $\phi \in L^1(\mathbb{D}, d\lambda_\alpha)$. Then

$$\begin{aligned}\int_{\mathbb{D}} |\widetilde{\phi}(w)| d\lambda_\alpha(w) &= \int_{\mathbb{D}} |\widetilde{\phi}(w)| \frac{dA_\alpha(w)}{(1-|w|^2)^{2+\alpha}} \\ &\leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |\phi(z)| \frac{(1-|w|^2)^{(2+\alpha)}}{|1-\overline{w}z|^{4+2\alpha}} dA_\alpha(z) \right) \frac{dA_\alpha(w)}{(1-|w|^2)^{2+\alpha}} \\ &= \int_{\mathbb{D}} |\phi(z)| \int_{\mathbb{D}} \frac{dA_\alpha(w)}{|1-\overline{w}z|^{4+2\alpha}} dA_\alpha(z) \\ &= \int_{\mathbb{D}} |\phi(z)| \langle K_z^\alpha, K_z^\alpha \rangle_\alpha dA_\alpha(z) \\ &= \int_{\mathbb{D}} |\phi(z)| \frac{dA_\alpha(z)}{(1-|z|^2)^{2+\alpha}},\end{aligned}$$

the change of the order of integration being justified by the positivity of the integrand. Hence $\widetilde{\phi} \in L^1(\mathbb{D}, d\lambda_\alpha)$. Similarly if $\phi \in L^\infty(\mathbb{D})$ then $\widetilde{\phi} \in L^\infty(\mathbb{D})$ as $|\widetilde{\phi}(w)| = |\langle \phi k_w^\alpha, k_w^\alpha \rangle_\alpha| \leq \|\phi k_w^\alpha\|_2 \|k_w^\alpha\|_2 \leq \|\phi\|_\infty \|k_w^\alpha\|_2^2 = \|\phi\|_\infty$. By Marcinkiewicz interpolation theorem it follows that if $\phi \in L^p(\mathbb{D}, d\lambda_\alpha)$ then $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda_\alpha)$ for $1 \leq p \leq \infty$. Now suppose $\phi \in L^p(\mathbb{D}, d\lambda_\alpha), 1 \leq p \leq \infty$. We shall prove $T_\phi \in S_p$. The case $p = +\infty$ is trivial. By interpolation we need only to prove the result for $p = 1$. Suppose $\phi \in L^1(\mathbb{D}, d\lambda_\alpha)$ and $\{e_n^\alpha\} = \left\{ \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} z^n \right\}_{n=0}^\infty$ is the standard orthonormal basis for $L_a^2(dA_\alpha)$. Now $\langle T_\phi e_n^\alpha, e_n^\alpha \rangle_\alpha = \int_{\mathbb{D}} |e_n^\alpha(z)|^2 \phi(z) dA_\alpha(z)$ and

$$\begin{aligned}\sum_{n=0}^\infty |\langle T_\phi e_n^\alpha, e_n^\alpha \rangle_\alpha| &\leq \int_{\mathbb{D}} \sum_{n=0}^\infty |e_n^\alpha(z)|^2 |\phi(z)| dA_\alpha(z) \\ &\leq \int_{\mathbb{D}} K_z^\alpha(z) |\phi(z)| dA_\alpha(z) \\ &= \int_{\mathbb{D}} |\phi(z)| d\lambda_\alpha(z).\end{aligned}$$

Thus $T_\phi \in S_1$ and $\|T_\phi\|_{S_1} \leq \int_{\mathbb{D}} |\phi(z)| d\lambda_\alpha(z)$. This proves the claim. □

3 Bounded Linear Operators on Weighted Bergman Spaces

In this section, we find conditions on $C \in \mathcal{L}(L_a^2(dA_\alpha))$ such that $C \in S_p$, the Schatten p-class, $1 \leq p < \infty$ by comparing with positive Toeplitz operators defined on the weighted Bergman spaces $L_a^2(dA_\alpha)$ and applications of the result are also obtained.

Theorem 3.1. *Let $\phi \in L^p(\mathbb{D}, d\lambda_\alpha), \psi \in L^q(\mathbb{D}, d\lambda_\alpha)$ where $1 \leq p, q < \infty$. Let $C \in \mathcal{L}(L_a^2(dA_\alpha))$ is such that*

$$|\langle CK_y^\alpha, K_x^\alpha \rangle_\alpha|^2 \leq \langle T_{|\phi|} K_y^\alpha, K_y^\alpha \rangle_\alpha \langle T_{|\psi|} K_x^\alpha, K_x^\alpha \rangle_\alpha \tag{3.1}$$

for all $x, y \in \mathbb{D}$. Then $C \in S_{2r}$ and $\|C\|_{2r}^2 \leq \|T_{|\phi|}\|_p \|T_{|\psi|}\|_q$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Proof. First we show that (3.1) implies

$$|\langle Cf, g \rangle_\alpha|^2 \leq \langle T_{|\phi|} f, f \rangle_\alpha \langle T_{|\psi|} g, g \rangle_\alpha$$

for all $f, g \in L_a^2(dA_\alpha)$. Let $f = \sum_{j=1}^n c_j K_{y_j}^\alpha$ where c_j are constants, $y_j \in \mathbb{D}$ for $j = 1, 2, ..., n$ and $g = \sum_{i=1}^m d_i K_{x_i}^\alpha$ where d_i are constants, $x_i \in \mathbb{D}$ for $i = 1, 2, ..., m$. Then

$$\begin{aligned}
|\langle Cf, g \rangle_\alpha| &= \left| \left\langle C\left(\sum_{j=1}^n c_j K_{y_j}^\alpha\right), \sum_{i=1}^m d_i K_{x_i}^\alpha \right\rangle_\alpha \right| \\
&= \left| \sum_{i=1,j=1}^{m,n} c_j \overline{d_i} \left\langle CK_{y_j}^\alpha, K_{x_i}^\alpha \right\rangle_\alpha \right| \\
&\leq \sum_{i=1,j=1}^{m,n} |c_j||d_i| \left| \left\langle CK_{y_j}^\alpha, K_{x_i}^\alpha \right\rangle_\alpha \right| \\
&\leq \sum_{i=1,j=1}^{m,n} |c_j||d_i| \left\langle T_{|\phi|} K_{y_j}^\alpha, K_{y_j}^\alpha \right\rangle_\alpha^{\frac{1}{2}} \left\langle T_{|\psi|} K_{x_i}^\alpha, K_{x_i}^\alpha \right\rangle_\alpha^{\frac{1}{2}} \\
&= \left\langle T_{|\phi|}\left(\sum_{j=1}^n c_j K_{y_j}^\alpha\right), \sum_{j=1}^n c_j K_{y_j}^\alpha \right\rangle_\alpha^{\frac{1}{2}} \left\langle T_{|\psi|}\left(\sum_{i=1}^m d_i K_{x_i}^\alpha\right), \sum_{i=1}^m d_i K_{x_i}^\alpha \right\rangle_\alpha^{\frac{1}{2}} \\
&= \left\langle T_{|\phi|} f, f \right\rangle_\alpha^{\frac{1}{2}} \left\langle T_{|\psi|} g, g \right\rangle_\alpha^{\frac{1}{2}}.
\end{aligned}$$

Since the set of vectors $\{\sum c_j K_{x_j}^\alpha, x_j \in \mathbb{D}, j = 1, 2, ..., n\}$ is dense in $L_a^2(dA_\alpha)$, hence

$$|\langle Cf, g \rangle_\alpha|^2 \leq \langle T_{|\phi|} f, f \rangle_\alpha \langle T_{|\psi|} g, g \rangle_\alpha$$

for all $f, g \in L_a^2(dA_\alpha)$. If $\phi \in L^p(\mathbb{D}, d\lambda_\alpha)$, then $T_{|\phi|} \in S_p$ and

$$\|T_{|\phi|}\|_p = (trace T_{|\phi|}^p)^{\frac{1}{p}} < \infty.$$

Similarly since $\psi \in L^q(\mathbb{D}, d\lambda_\alpha)$ then

$$\|T_{|\psi|}\|_q = (trace T^q_{|\psi|})^{\frac{1}{q}} < \infty.$$

Let $\{u_n\}_{n=0}^{\infty}$ and $\{\sigma_n\}_{n=0}^{\infty}$ be two orthonormal sequences in $L^2_a(dA_\alpha)$. Then using Holder's inequality, we obtain that

$$\begin{aligned}
\sum_{n=0}^{\infty} |\langle Cu_n, \sigma_n\rangle_\alpha|^{2r} &\leq \sum_{n=0}^{\infty} \langle T_{|\phi|}u_n, u_n\rangle_\alpha^r \langle T_{|\psi|}\sigma_n, \sigma_n\rangle_\alpha^r \\
&\leq \left(\sum_{n=0}^{\infty} \langle T_{|\phi|}u_n, u_n\rangle_\alpha^p\right)^{\frac{r}{p}} \left(\sum_{n=0}^{\infty} \langle T_{|\psi|}\sigma_n, \sigma_n\rangle_\alpha^q\right)^{\frac{r}{q}} \\
&\leq \left(\sum_{n=0}^{\infty} \langle T^p_{|\phi|}u_n, u_n\rangle_\alpha\right)^{\frac{r}{p}} \left(\sum_{n=0}^{\infty} \langle T^q_{|\psi|}\sigma_n, \sigma_n\rangle_\alpha\right)^{\frac{r}{q}} \\
&\leq \left(trace T^p_{|\phi|}\right)^{\frac{r}{p}} \left(trace T^q_{|\psi|}\right)^{\frac{r}{q}} \\
&= \|T_{|\phi|}\|_p^r \|T_{|\psi|}\|_q^r \quad \text{ if } \; \frac{1}{r} = \frac{1}{p} + \frac{1}{q}\ .
\end{aligned}$$

Thus

$$\|C\|_{2r} \leq \|T_{|\phi|}\|_p^{\frac{1}{2}} \|T_{|\psi|}\|_q^{\frac{1}{2}}.$$

□

Corollary 3.2. *If $\phi, \psi \in L^p(\mathbb{D}, d\lambda_\alpha)$ and $C \in \mathcal{L}(L^2_a(dA_\alpha))$ is such that*

$$|\langle CK^\alpha_y, K^\alpha_x\rangle_\alpha|^2 \leq \langle T_{|\phi|}K^\alpha_y, K^\alpha_y\rangle_\alpha \langle T_{|\psi|}K^\alpha_x, K^\alpha_x\rangle_\alpha$$

for all $x, y \in \mathbb{D}$ then $\|C\|_p^2 \leq \|T_{|\phi|}\|_p \|T_{|\psi|}\|_p$.

Proof. The proof follows from the Theorem 3.1 if we assume $p = q$. □

Corollary 3.3. *If A, B are two positive operators in $\mathcal{L}(L^2_a(dA_\alpha))$ and $A \in S_p, B \in S_q, 1 \leq p, q < \infty$ and $C \in \mathcal{L}(L^2_a(dA_\alpha))$ is such that*

$$|\langle CK^\alpha_y, K^\alpha_x\rangle_\alpha|^2 \leq \langle AK^\alpha_y, K^\alpha_y\rangle_\alpha \langle BK^\alpha_x, K^\alpha_x\rangle_\alpha$$

for all $x, y \in \mathbb{D}$ then $\|C\|_{2r}^2 \leq \|A\|_p \|B\|_q$ if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $p = q$, then $\|C\|_p^2 \leq \|A\|_p \|B\|_p$.

Proof. Proceeding similarly as in Theorem 3.1 and Corollary 3.2 by replacing $T_{|\phi|}$ by A and $T_{|\psi|}$ by B, the corollary follows. □

Corollary 3.4. *If $A, B \in \mathcal{L}(L^2_a(dA_\alpha)), 0 \leq A \in S_p, 1 \leq p < \infty$ and (3.1) holds for $x, y \in \mathbb{D}$, then*

$$\|C\|_{2p}^2 \leq \|A\|_p \|B\|.$$

Proof. Let $\{u_n\}_{n=0}^{\infty}$ and $\{\sigma_n\}_{n=0}^{\infty}$ be two orthonormal bases for $L^2_a(dA_\alpha)$, then

$$\begin{aligned}|\langle Cu_n, \sigma_n\rangle_\alpha|^2 &\le \langle Au_n, u_n\rangle_\alpha \langle B\sigma_n, \sigma_n\rangle_\alpha\\ &\le \langle Au_n, u_n\rangle_\alpha \|B\|.\end{aligned}$$

Then $|\langle Cu_n, \sigma_n\rangle_\alpha|^{2p} \le \|B\|^p \langle Au_n, u_n\rangle_\alpha^p$. Hence

$$\sum_{n=0}^{\infty} |\langle Cu_n, \sigma_n\rangle_\alpha|^{2p} \le \|B\|^p \sum_{n=0}^{\infty} \langle Au_n, u_n\rangle_\alpha^p$$

and $\|C\|_{2p}^2 \le \|B\| \|A\|_p$. □

If $\phi \in L^p(\mathbb{D}, d\lambda_\alpha)$ then $T_\phi \in S_p$. Hence $|T_\phi| \in S_p$. Thus if $B \in \mathcal{L}(L^2_a(dA_\alpha)), C \in \mathcal{L}(L^2_a(dA_\alpha))$ are such that $|\langle CK_y^\alpha, K_x^\alpha\rangle_\alpha|^2 \le \langle |T_\phi| K_y^\alpha, K_y^\alpha\rangle_\alpha \langle BK_x^\alpha, K_x^\alpha\rangle_\alpha$ for all $x, y \in \mathbb{D}$ then $C \in S_{2p}$ and $\|C\|_{2p}^2 \le \|B\| \| |T_\phi| \|_p$.

Corollary 3.5. *Let $\phi \in L^p(\mathbb{D}, d\lambda_\alpha), 1 < p < \infty$ and $\phi = \phi^+$ where $\phi^+(z) = \overline{\phi(\overline{z})}$. Then there exists an operator $S \in \mathcal{L}(L^2_a(dA_\alpha))$ such that $T_{|\phi|}S = ST_{|\phi|}$ and $\|T_{|\phi|}S\|_p \le r(S)\|T_{|\phi|}\|_p$ where $r(S)$ is the spectral radius of S.*

Proof. Since $\phi \in L^p(\mathbb{D}, d\lambda_\alpha)$ and $\phi^+ = \phi$, hence $T_{|\phi|}$ and S_ϕ are self- adjoint operators, $T_{|\phi|} \in S_p$ and $S_\phi \in S_p$. For details see [1]. Let $\mathcal{U}$ be the group of unitary operators on $L^2_a(\mathbb{D})$. Let $\mathcal{U}_A = \{UAU^* : U \in \mathcal{U}\}$, the unitary orbit of an operator $A \in \mathcal{L}(L^2_a(\mathbb{D}))$.

Define $f(X) = \|T_{|\phi|} - X\|_p$ for all $X \in S_p$. Then f attains its minimum at some $S \in S_p$ on $\mathcal{U}_{S_\phi} = \{US_\phi U^* : U \in \mathcal{U}\}$ and $T_{|\phi|}S = ST_{|\phi|}$. This follows from [4]. The operator S is self-adjoint. To prove the corollary we have to show that for any two orthonormal sequences $\{u_n\}_{n=0}^{\infty}$ and $\{\sigma_n\}_{n=0}^{\infty}$ in $L^2_a(dA_\alpha)$,

$$\sum_{n=0}^{\infty} |\langle T_{|\phi|}Su_n, \sigma_n\rangle_\alpha|^p \le r(S)^p \|T_{|\phi|}\|_p^p.$$

Notice that since $T_{|\phi|}S = ST_{|\phi|}$ and $S = S^*$ we obtain

$$\begin{aligned}|\langle T_{|\phi|}Su_n, \sigma_n\rangle_\alpha|^2 &= |\langle T_{|\phi|}(Su_n), \sigma_n\rangle_\alpha|^2\\ &\le \langle T_{|\phi|}(Su_n), Su_n\rangle_\alpha \langle T_{|\phi|}\sigma_n, \sigma_n\rangle_\alpha\\ &= \langle S^*T_{|\phi|}Su_n, u_n\rangle_\alpha \langle T_{|\phi|}\sigma_n, \sigma_n\rangle_\alpha\\ &= \langle T_{|\phi|}S^2u_n, u_n\rangle_\alpha \langle T_{|\phi|}\sigma_n, \sigma_n\rangle_\alpha\ .\end{aligned}$$

Repeating this process we obtain

$$\begin{aligned}|\langle T_{|\phi|}Su_n, \sigma_n\rangle_\alpha|^{2^{m+1}} &= \left(|\langle T_{|\phi|}Su_n, \sigma_n\rangle_\alpha|^{2^m}\right)^2\\ &\le \left[\langle T_{|\phi|}S^{2^m}u_n, u_n\rangle_\alpha \langle T_{|\phi|}u_n, u_n\rangle_\alpha^{2^{m-1}-1} \langle T_{|\phi|}\sigma_n, \sigma_n\rangle_\alpha^{2^{m-1}}\right]^2\\ &\le \langle T_{|\phi|}S^{2^m}u_n, S^{2^m}u_n\rangle_\alpha \langle T_{|\phi|}u_n, u_n\rangle_\alpha \langle T_{|\phi|}u_n, u_n\rangle_\alpha^{2^m-2} \langle T_{|\phi|}\sigma_n, \sigma_n\rangle_\alpha^{2^m}\\ &= \langle {S^*}^{2^m} T_{|\phi|}S^{2^m}u_n, u_n\rangle_\alpha \langle T_{|\phi|}u_n, u_n\rangle_\alpha^{2^m-1} \langle T_{|\phi|}\sigma_n, \sigma_n\rangle_\alpha^{2^m}\\ &= \langle T_{|\phi|}S^{2^{m+1}}u_n, u_n\rangle_\alpha \langle T_{|\phi|}u_n, u_n\rangle_\alpha^{2^m-1} \langle T_{|\phi|}\sigma_n, \sigma_n\rangle_\alpha^{2^m}.\end{aligned}$$

Thus

$$|\langle T_{|\phi|}Su_n,\sigma_n\rangle_\alpha|^{2^m} \leq \|T_{|\phi|}\|\|S^{2^m}\|\|u_n\|^2\langle T_{|\phi|}u_n,u_n\rangle_\alpha^{2^{m-1}-1}\langle T_{|\phi|}\sigma_n,\sigma_n\rangle_\alpha^{2^{m-1}}$$

and

$$|\langle T_{|\phi|}Su_n,\sigma_n\rangle_\alpha| \leq \|T_{|\phi|}\|^{\frac{1}{2^m}}\|S^{2^m}\|^{\frac{1}{2^m}}\|u_n\|^{\frac{2}{2^m}}\langle T_{|\phi|}u_n,u_n\rangle_\alpha^{\frac{1}{2}-\frac{1}{2^m}}\langle T_{|\phi|}\sigma_n,\sigma_n\rangle_\alpha^{\frac{1}{2}}.$$

Letting $m\to\infty$, we obtain

$$|\langle T_{|\phi|}Su_n,\sigma_n\rangle_\alpha|^2 \leq [r(S)]^2\langle T_{|\phi|}u_n,u_n\rangle_\alpha\langle T_{|\phi|}\sigma_n,\sigma_n\rangle_\alpha\ .$$

Hence proceeding as in Theorem 3.1 and Corollary 3.2, one can show that $\|T_{|\phi|}S\|_p \leq r(S)\|T_{|\phi|}\|_p$. □

4 Schatten Class Little Hankel Operators

In this section, we find necessary and sufficient conditions on $\phi \in L^2(\mathbb{D}, dA)$ such that the little Hankel operator $S_{\overline{\phi}}$ defined on $L_a^2(\mathbb{D})$ belong to the class $S_p, 2 \leq p < \infty$. For $\phi \in L^2(\mathbb{D}, dA)$, define

$$(V\phi)(z) = 3(1-|z|^2)^2\int_{\mathbb{D}}\frac{\phi(w)}{(1-z\overline{w})^4}dA(w).$$

Under the complex integral paring with respect to dA, we have $V = P_2^*$, where $P_2f(z) = 3\int_{\mathbb{D}}\frac{(1-|w|^2)^2}{(1-z\overline{w})^4}f(w)dA(w)$ is a projection from $L^1(\mathbb{D}, dA)$ onto $L_a^1(\mathbb{D})$. These operators V play crucial role in obtaining the Schatten class characterization for $S_{\overline{\phi}}$.

The little Hankel operator S_ϕ can also be defined for $\phi \in L^2(\mathbb{D}, dA)$ as $S_\phi f = PJ(\phi f)$ for $f \in L_a^2(\mathbb{D})$. Notice that if $\phi \in L^2(\mathbb{D}, dA)$, then $S_{\overline{\phi}} = S_{\overline{P\phi}}$ in the sense that $S_{\overline{\phi}}g = S_{\overline{P\phi}}g$ for all $g \in H^\infty(\mathbb{D})$ (which is dense in $L_a^2(\mathbb{D})$), where P is the Bergman projection. The operator V has the following property: VP=V, PV=P and $V^2 = V$ on $L^2(\mathbb{D}, dA)$. We verify now that if $\phi \in L^2(\mathbb{D}, dA)$, then $S_{\overline{\phi}}$ is bounded if and only if $V\phi(z)$ is bounded in $\mathbb{D}$. Since each k_z is a unit vector in $L^2(\mathbb{D}, dA)$, we have

$$|V\phi(z)| = 3|\langle S_{\overline{\phi}}k_z, k_{\overline{z}}\rangle| \leq 3\|S_{\overline{\phi}}k_z\|.$$

Hence $\|V\phi\|_\infty \leq 3\|S_{\overline{\phi}}\|$. On the other hand, $S_{\overline{\phi}} = S_{\overline{P\phi}} = S_{\overline{PV\phi}} = S_{\overline{V\phi}}$. Thus $V\phi \in L^\infty(\mathbb{D}, dA)$ implies that $S_{\overline{\phi}}$ is bounded with $\|S_{\overline{\phi}}\| \leq \|V\phi\|_\infty$.

Theorem 4.1. *Suppose $2 \leq p < \infty$. Then $S_{\overline{\phi}} \in S_p$ if and only if $V\phi \in L^p(\mathbb{D}, d\lambda)$, where $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$.*

Proof. Suppose $2 \le p < \infty$ and $S_{\overline{\phi}} \in S_p$. Then

$$\begin{aligned}\int_{\mathbb{D}} |(V\phi)(z)|^p d\lambda(z) &\le 3^p \int_{\mathbb{D}} \|S_{\overline{\phi}} k_z\|^p d\lambda(z)\\ &= 3^p \int_{\mathbb{D}} \langle S_{\overline{\phi}} k_z, S_{\overline{\phi}} k_z\rangle^{\frac{p}{2}} d\lambda(z)\\ &= 3^p \int_{\mathbb{D}} \langle S^*_{\overline{\phi}} S_{\overline{\phi}} k_z, k_z\rangle^{\frac{p}{2}} d\lambda(z)\\ &\le 3^p \int_{\mathbb{D}} \langle (S^*_{\overline{\phi}} S_{\overline{\phi}})^{\frac{p}{2}} k_z, k_z\rangle d\lambda(z)\\ &= 3^p \int_{\mathbb{D}} \langle |S_{\overline{\phi}}|^p k_z, k_z\rangle d\lambda(z) < \infty.\end{aligned}$$

Hence $V\phi \in L^p(\mathbb{D}, d\lambda)$.

Conversely, suppose $V\phi \in L^p(\mathbb{D}, d\lambda)$. We shall show that $S_{\overline{\phi}} \in S_p$. Since $S_{\overline{\phi}} = S_{\overline{V\phi}}$, it suffices to show that $S_{\overline{\phi}}$ is in S_p whenever $\phi \in L^p(\mathbb{D}, d\lambda)$. In the following we prove that if $\phi \in L^p(\mathbb{D}, d\lambda)$ then $S_{\overline{\phi}} \in S_p, 1 \le p < \infty$. From Heinz inequality [3], it follows that

$$\begin{aligned}|\langle S_{\overline{\phi}} k_z, k_w\rangle|^2 &\le \langle |S_{\overline{\phi}}| k_z, k_z\rangle \langle |S^*_{\overline{\phi}}| k_w, k_w\rangle\\ &= \langle (S^*_{\overline{\phi}} S_{\overline{\phi}})^{\frac{1}{2}} k_z, k_z\rangle \langle (S_{\overline{\phi}} S^*_{\overline{\phi}})^{\frac{1}{2}} k_w, k_w\rangle\\ &\le \langle (S^*_{\overline{\phi}} S_{\overline{\phi}}) k_z, k_z\rangle^{\frac{1}{2}} \langle (S_{\overline{\phi}} S^*_{\overline{\phi}}) k_w, k_w\rangle^{\frac{1}{2}}\\ &= \|S_{\overline{\phi}} k_z\|_2 \|S_{\overline{\phi}^+} k_w\|_2\\ &= \|PJ(\overline{\phi} k_z)\|_2 \|PJ(\overline{\phi}^+ k_w)\|_2\\ &\le \|\overline{\phi} k_z\|_2 \|\overline{\phi}^+ k_w\|_2\\ &= \left(\int_{\mathbb{D}} |\phi(u)|^2 |k_z(u)|^2 dA(u)\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} |\overline{\phi}^+(v)|^2 |k_w(v)|^2 dA(v)\right)^{\frac{1}{2}}\\ &\le d\langle T_{|\phi|} k_z, k_z\rangle \langle T_{|\phi^+|} k_w, k_w\rangle \quad \text{for some constant } d > 0.\end{aligned}$$

Thus

$$|\langle S_{\overline{\phi}} K_z, K_w\rangle|^2 \le d\langle T_{|\phi|} K_z, K_z\rangle \langle T_{|\phi^+|} K_w, K_w\rangle.$$

Now $\phi \in L^p(\mathbb{D}, d\lambda)$ implies $|\phi|, |\phi^+| \in L^p(\mathbb{D}, d\lambda)$. Hence $T_{|\phi|}, T_{|\phi^+|} \in S_p$. Hence by Theorem 3.1, $S_{\overline{\phi}} \in S_p$. □

5 $p - C^*$ Summing Operators and m-Berezin Transform

In this section, using the $p - C^*$ summing conditions, we obtain a characterization for bounded linear operators to belong to the class S_p. In fact, we have shown that if $A \in \mathcal{L}(L^2_a(dA_\alpha))$ then $T_{\overline{\phi}}AT_\phi \in S_p$ if $\phi \in H^\infty(\mathbb{D})$ and $|\phi|^2 \in L^p(\mathbb{D}, d\lambda_\alpha)$. Also using the concept of m-Berezin transform, we find conditions on ϕ such that the composition operators defined on $L^2_a(\mathbb{D})$ belong to the Schatten class $S_p, 1 \leq p < \infty$.

A linear map T from a C^* algebra $\mathcal{A}$ into a Banach space X is $p-C^*$ summing (we assume $p \geq 1$) if there is a constant C such that, for any finite sequence $\{w_i\}_{i=1}^N \subset \mathcal{A}^h = \{w \in \mathcal{A} : w^* = w\}$, the following condition holds:

$$\left(\sum_{i=1}^N \|Tw_i\|^p\right)^{\frac{1}{p}} \leq C\|\sum_{i=1}^N |w_i|^p\|^{\frac{1}{p}},$$

where $|w| = (w^*w)^{\frac{1}{2}}$. The least constant C for which this condition is satisfied is denoted by $C_p(T)$. It is shown in [5] that T is p-C^* summing if and only if there is a constant C and state ϕ on $\mathcal{A}$ such that, for all x in $\mathcal{A}^h$, $\|Tx\| \leq C\phi(|x|^p)^{\frac{1}{p}}$. The least of these constants is equal to $C_p(T)$. For example, $\mathcal{L}(L^2_a(dA_\alpha))$ is a C^*- algebra with the unit I. Define $\mathcal{T}_B : \mathcal{L}(L^2_a(dA_\alpha)) \to \mathcal{L}(L^2_a(dA_\alpha))$ by $\mathcal{T}_B(A) = BAB^*$. If $A \geq 0$ then $\mathcal{T}_B(A) \geq 0$. If A=I and $BB^* \in S_p, p \geq 1$, then $\mathcal{T}_B(I) = \mathcal{T}_B(A) = BB^* \in S_p$. In [5] , Nowak has shown that $\mathcal{T}_B$ is $p - C^*$ summing and $C_p(\mathcal{T}_B) \leq \|\mathcal{T}_\mathcal{B}(\mathcal{I})\|_p$ and $\mathcal{T}_B(\mathcal{L}(L^2_a(dA_\alpha))) \subset S_p$ and $\mathcal{T}_B$ is bounded as the map from $\mathcal{L}(L^2_a(dA_\alpha))$ into S_p with the norm $\|\mathcal{T}_B(I)\|_p$.

Theorem 5.1. *Let $p \geq 1$. Let $\phi \in H^\infty(\mathbb{D})$ be such that $|\phi|^2 \in L^p(\mathbb{D}, d\lambda_\alpha)$. Then $T_{\overline{\phi}}AT_\phi \in S_p$ for all $A \in \mathcal{L}(L^2_a(dA_\alpha))$.*

Proof. Let $\phi \in H^\infty(\mathbb{D})$ be such that $|\phi|^2 \in L^p(\mathbb{D}, d\lambda_\alpha)$ for some $p \in [1, \infty)$. Then from [1], it follows that

$$T^*_\phi T_\phi = T_{\overline{\phi}}T_\phi = T_{|\phi|^2} \in S_p.$$

Define $\mathcal{T} : \mathcal{L}(L^2_a(dA_\alpha)) \longrightarrow \mathcal{L}(L^2_a(dA_\alpha))$ as $\mathcal{T}(A) = T_\phi AT^*_\phi = T_\phi AT_{\overline{\phi}}$. Then $\mathcal{T}(A) \geq 0$ if $A \geq 0$ and since $T^*_\phi T_\phi \in S_p$ we have $\mathcal{T}(I) = T_\phi T^*_\phi \in S_p$. Hence $\mathcal{T}$ is $p - C^*$ summing and $\mathcal{T}(A) = T_\phi AT^*_\phi \in S_p$ for all $A \in \mathcal{L}(L^2_a(dA_\alpha))$. □

Lemma 5.2. *Let $p \geq 1, T \in \mathcal{L}(L^2_a(dA_\alpha))$ and $T_n \in S_p$ for all $n \in \mathbb{N}$. If $T_n \to T$ in weak operator topology and $\|T_n\|_p \leq C < \infty$ for all $n \in \mathbb{N}$ and for some constant $C > 0$ then $T \in S_p$ and $\|T\|_p \leq C$.*

Proof. For each $n \in \mathbb{N}$, define

$$\zeta_n(K) = tr(T_nK).$$

Then $\zeta_n \in S^*_q$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\|\zeta_n\| = \|T_n\|_p \leq C < \infty$. By Banach-Alaoglu's theorem [6], there exists a subsequence $\{\zeta_{n_k}\}$ such that $\zeta_{n_k} \longrightarrow \zeta$ in w^*-topology and $\zeta \in S^*_q$. Therefore $tr(T_{n_k}K) = \zeta_{n_k}(K) \longrightarrow \zeta(K)$, for all $K \in S_q$ and $|\zeta(K)| \leq M\|K\|_q$, for some constant $M > 0$. On the other hand, since

$T_n \longrightarrow T$ in weak operator topology, $tr(T_nK) \longrightarrow tr(TK)$ for all operators K of finite rank. The lemma follows since

$$\|T\|_p = \sup\{|tr(TK)| : rank(K) < \infty \ \text{ and } \ \|K\|_q \leq 1\} < \infty.$$

□

Theorem 5.3. *Let $T \in \mathcal{L}(L^2_a(dA_\alpha))$ and let $T = V|T|$ be the polar decomposition of T. If $T \in S_p$, then $V \in S_p$, if $1 \leq p < \infty$.*

Proof. Let $T_n = T(|T| + \frac{1}{n})^{-1}$. We shall first prove that $T_n \longrightarrow V$ strongly as $n \to \infty$. Let $\{E_\lambda\}$ be the spectral family for $|T|$. Then T_n strongly converges to $I - E_0$ as $n \to \infty$. The reason is as follows: Notice that $|T| = \int_0^\infty \lambda dE_\lambda$ is the spectral decomposition of $|T|$. Let $S_n = |T|(|T| + \frac{1}{n})^{-1}$. Then $S_nE_0f = (|T| + \frac{1}{n})^{-1}|T|E_0f = 0$ for $f \in L^2_a(dA_\alpha)$ and

$$\begin{aligned}\|S_nf - (I - E_0)f\|^2 &= \|(S_n - I)(I - E_0)f\|^2 \\ &= \int_0^\infty \left|\frac{\lambda}{\lambda + \frac{1}{n}} - 1\right|^2 d\|E_\lambda(I - E_0)f\|^2 \\ &= \int_0^\infty \left|\frac{\frac{1}{n}}{\lambda + \frac{1}{n}}\right|^2 d\|E_\lambda(I - E_0)f\|^2.\end{aligned}$$

From Lebesgue's dominated convergence theorem, it follows that S_n strongly converges to $I - E_0$ as $n \to \infty$. Thus we have $T_n \to V(I - E_0)$ strongly as $n \to \infty$. Since E_0 is the projection onto the eigenspace $\{f \in L^2_a(dA_\alpha) : Tf = 0\}$, we get $VE_0 = 0$. Consequently, $T_n \longrightarrow V$ strongly as $n \to \infty$. Now suppose $T \in S_p$. Then $T_n \in S_p, \|T_n\|_p \leq C < \infty$ for some constant $C > 0$ and $T_n \longrightarrow V$ strongly as $n \to \infty$. By Lemma 5.2, $V \in S_p$. □

If m is a nonnegative integer and $z \in \mathbb{D}$, the function $K_z^{(m)}(w) = \frac{1}{(1-\overline{z}w)^{2+m}}, w \in \mathbb{D}$ is the reproducing kernel of z in the weighted Bergman space $L^2_a(dA_m)$, where

$$dA_m(w) = (m + 1)(1 - |w|^2)^m dA(w).$$

The m-Berezin transform of an operator $S \in \mathcal{L}(L^2_a(\mathbb{D}))$ is defined as

$$(B_mS)(z) = (m + 1)(1 - |z|^2)^{2+m} \sum_{j=0}^{m} \binom{m}{j}(-1)^j \left\langle S(w^jK_z^{(m)}), w^jK_z^{(m)}\right\rangle.$$

It is clear that $B_mS \in L^\infty(\mathbb{D})$ for every $S \in \mathcal{L}(L^2_a(\mathbb{D}))$. Using the fact that

$$\sum_{j=0}^{m} \binom{m}{j}(-1)^j|w|^{2j} = (1 - |w|^2)^m,$$

we see that if $S = T_\phi$ with $\phi \in L^\infty(\mathbb{D})$, then

$$\begin{aligned}(B_m\phi)(z) &= (B_mT_\phi)(z) \\ &= (m + 1)(1 - |z|^2)^{2+m} \sum_{j=0}^{m} \binom{m}{j}(-1)^j \int_{\mathbb{D}} \frac{\phi(w)|w|^{2j}}{|1 - \overline{z}w|^{2(2+m)}} dA(w)\end{aligned}$$

$$= \int_{\mathbb{D}} \phi(w) \frac{(1-|z|^2)^{2+m}}{|1-\bar{z}w|^{2(2+m)}} (m+1)(1-|w|^2)^m dA(w)$$
$$= \int_{\mathbb{D}} \phi(\phi_z(\rho))(m+1)(1-|\rho|^2)^m dA(\rho),$$

where the last equality comes from the change of variables $w = \phi_z(\rho)$. Notice that $\|B_m(\phi)\|_\infty \leq \|\phi\|_\infty$ for all $\phi \in L^\infty(\mathbb{D})$. The 0-Berezin transform of an operator is the usual Berezin transform. The m-Berezin transforms of functions (not necessarily bounded) were introduced by Berezin in [7]. It is not difficult to verify that for $S \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $m \geq 0$;

$$(m+2)(1-|z|^2)B_m\left(S - T_{\bar{w}}ST_w\right)(z) = (m+1)B_{m+1}\left(T_{1-\bar{w}z}ST_{1-w\bar{z}}\right)(z)$$

for every $z \in \mathbb{D}$ and $\|B_m S\|_\infty \leq (m+2)2^m\|S\|$.

Corollary 5.4. *Let $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic. Suppose there is $p > 3$ such that*

$$\sup_{z\in\mathbb{D}} \|T_{(B_m C_\phi)o\phi_z} 1\|_p < C \quad \textit{and} \quad \sup_{z\in\mathbb{D}} \|T^*_{(B_m C_\phi)o\phi_z} 1\|_p < C \tag{5.1}$$

where $C > 0$ is independent of m and $B_m C_\phi \in L^p(\mathbb{D}, d\lambda)$ for all nonnegative integer m. If further $\|T_{B_m C_\phi}\|_p < K$ for some constant $K > 0$ independent of m then $C_\phi \in S_p$.

Proof. Let $C_\phi \in \mathcal{L}(L^2_a(\mathbb{D}))$ and satisfies the condition (5.1). It follows from [8] that $T_{B_m C_\phi} \longrightarrow C_\phi$ in $\mathcal{L}(L^2_a(\mathbb{D}))$ norm as $m \longrightarrow \infty$. Hence $T_{B_m C_\phi} \xrightarrow{w} C_\phi$. By Lemma 5.2 and Theorem 2.1, $C_\phi \in S_p$ as $B_m C_\phi \in L^p(\mathbb{D}, d\lambda)$. □

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