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A Study on Approximation of Conjugate of Functions Belonging to Lipschitz Class and Generalized Lipschitz Class by Product Summability Means of Conjugate Series of Fourier Series

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Abstract : In this paper, two new theorems on degree of approximation of a function $\bar{f}$, conjugate to a 2π periodic function f, belonging to $Lip\alpha$ class and $W(L_r, \xi(t)$ class by $(C,1)(E,1)$ product summability means of conugate Fourier series have been established.

Keywords : Degree of approximation; Lipα class; $W(L_r, \xi(t))$ class of functions; $(C,1)$ summability; $(E,1)$ summability; $(C,1)(E,1)$ product summability; Conjugate Fourier series; Lebesgue integral.
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1 Introduction

Let f be a $2\pi-$periodic function and Lebesgue integrable. The Fourier series associated with f at a point x is defined by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^{\infty} A_n(x) \tag{1.1}$$

with n^{th} partial sums $s_n(f;x)$. The conjugate series of the Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x) \tag{1.2}$$

with n^{th} partial sums $\overline{s}_n(f;x)$. We call (1.2) as conjugate Fourier series of function f throughout this paper.

$L_\infty-$ norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in R\}.$$

L_r- norm is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx\right)^{\frac{1}{r}}, \ r \geq 1. \tag{1.3}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of degree n under sup norm $\|\cdot\|_\infty$ is defined by

$$\|t_n - f\|_\infty = \sup\{\,|t_n(x) - f(x)| : x \in R\} \ \text{(Zygmund [1])} \tag{1.4}$$

and the degree of approximation $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min \|t_n - f\|_r. \tag{1.5}$$

This method of approximation is called trigonometric Fourier approximation (TFA). A function $f \in Lip\alpha$ if

$$f(x+t) - f(x) = O(\,|t|^\alpha)\, for\ 0 < \alpha \leq 1 \tag{1.6}$$

$f \in Lip(\alpha, r)$, for $0 \leq x \leq 2\pi$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\,|t|^\alpha)\ , 0 < \alpha \leq 1, \text{ and } r \geq 1 \tag{1.7}$$

(Definition 5.38 of Mc Fadden [2])

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$, $f(x) \in Lip(\xi(t), r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t)) \tag{1.8}$$

and that $f(x) \in W(L_r, \xi(t))$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \sin^{\beta r} x \, dx \right)^{\frac{1}{r}} = O(\xi(t)), \quad \beta \geq 0, \ r \geq 1. \tag{1.9}$$

If $\beta = 0$, our newly defined class $W(L_r, \xi(t))$ reduces to the class $Lip(\xi(t), r)$, if $\xi(t) = t^\alpha$ then $Lip(\xi(t), r)$ class reduces to the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ class reduces to the class $Lip\alpha$. We observe that

$$Lip\alpha \subseteq Lip(\alpha, r) \subseteq Lip(\xi(t), r) \subseteq W(L_r, \xi(t)) \text{ for } 0 < \alpha \leq 1, \ r \geq 1.$$

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of its n^{th} partial sums $\{s_n\}$. The $(C,1)$ transform is defined as the n^{th} partial sum of $(C,1)$ summability and is given by

$$\begin{aligned} t_n &= \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} \\ &= \frac{1}{n+1} \sum_{k=0}^{n} s_k \to s \quad \text{as } n \to \infty \end{aligned} \tag{1.10}$$

then the series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by $(C,1)$ method. If

$$(E,1) = E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k \to s \quad \text{as} \quad n \to \infty, \tag{1.11}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable $(E,1)$ to a definite number s ([3]). The $(C,1)$ transform of $(E,1)$ transform defines $(C,1)(E,1)$ product transform and we denote it by $(CE)_n^1$. Thus if

$$\begin{aligned} (CE)_n^1 &= \frac{1}{n+1} \sum_{k=0}^{n} E_k^1 \to s, \ as \ n \to \infty \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{v} \right] \to s \text{ as } n \to \infty \end{aligned} \tag{1.12}$$

where E_n^1 denotes the $(E,1)$ transform of s_n and C_n^1 denotes $(C,1)$ transform of s_n. Then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(C,1)(E,1)$ means or summable $(C,1)(E,1)$ to a definite number s.

We use the following notations:

$$\psi(t) = f(x+t) + f(x-t),$$

$$\bar{K}_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{\cos\left(\nu + \frac{1}{2}\right) t}{\sin(t/2)} \right],$$

$\tau = \left[\frac{1}{t} \right]$, where τ denotes the greatest integer not greater than $\frac{1}{t}$.

2 Main Theorems

A good amount of work has been done on degree of approximation of functions belonging to $Lip\alpha$, $Lip\,(\alpha, r)$, $Lip\,(\xi\,(t)\,, r)$ and $W\,(L_r, \xi\,(t))$ classes using Cesàro, Nörlund and generalized Nörlund single summability methods by a number of researchers like Alexits [4], Sahney and Goel [5], Qureshi and Neha [6], Quershi [7, 8], Chandra [9], Khan [10], Leindler [11] and Rhoades [12]. But till now nothing seems to have been done so far in the direction of present work. Therefore, in present paper, two theorems on degree of approximation of conjugate of functions $f \in Lip\alpha$ class and $f \in W\,(L_r, \xi\,(t))$ class using $(C,1)(E,1)$ product summability means of conjugate Fourier series have been established in the following form:

Theorem 2.1. *If a function $\overline{f}\,(x)$, conjugate to a 2π- periodic function $f(x)$ belonging to the class $Lip\alpha$, then its degree of approximation by $(C,1)(E,1)$ product means of conjugate Fourier series is given by*

$$\sup_{0<x<2\pi}\left|\overline{(CE)_n^1} - \overline{f}\,(x)\right| = \left\|\overline{(CE)_n^1} - \overline{f}\,(x)\right\|_\infty = \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right) & for\; 0<\alpha<1 \\ O\left(\frac{\log(n+1)\pi e}{(n+1)}\right) & for\; \alpha = 1 \end{cases} \tag{2.1}$$

where $\overline{(CE)_n^1}$ denotes the $\overline{(C,1)(E,1)}$ means of series (1.2) and

$$\bar{f}\,(x) = -\frac{1}{2\pi}\int_0^\pi \psi\,(t)\cot\frac{1}{2}t\;dt$$

provided

$$(2)^\tau \sum_{k=\tau}^{n} (2)^{-k} = O(n+1). \tag{2.2}$$

Theorem 2.2. *If a function $\overline{f}(x)$, conjugate to a 2π- periodic function $f(x)$ belonging to class $W\,(L_r, \xi\,(t))\,, r \geq 1$, then its degree of approximation by $(C,1)(E,1)$ product means of conjugate Fourier series is given by*

$$\left\|\overline{(CE)_n^1} - \overline{f}\,(x)\right\|_r = O\left[(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{(n+1)}\right)\right] \tag{2.3}$$

provided that $\xi\,(t)$ satisfies the condition (2.2),

$$\left(\frac{\xi\,(t)}{t}\right) \text{ is non-increasing in } t, \tag{2.4}$$

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t\,|\psi\,(t)|}{\xi\,(t)}\right)^r \sin^{\beta r}t\;dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \tag{2.5}$$

and

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta} \left| \psi(t) \right|}{\xi(t)} \right)^{r} dt \right\}^{\frac{1}{r}} = O \left\{ (n+1)^{\delta} \right\}, \tag{2.6}$$

where δ *is an arbitrary positive number such that* $s(1-\delta)-1>0, \frac{1}{r}+\frac{1}{s}=1, \ 1 \le r \le \infty$, *conditions (2.5) and (2.6) hold uniformly in* x, $\overline{(CE)_n^1}$ *is* $\overline{(\mathrm{C},1)(\mathrm{E},1)}$ *means of the series (1.2) and*

$$\overline{f}(x) = -\frac{1}{2\pi} \int_{0}^{\pi} \psi(t) \cot \frac{1}{2} t \, dt. \tag{2.7}$$

3 Lemmas

For the proof of our theorems, following lemmas are required.

Lemma 3.1.

$$\overline{K}_n(t) = O\left(\frac{1}{t}\right) \quad for \ \ 0 \le t \le \frac{1}{n+1}.$$

Proof. For $0 \le t \le \frac{1}{n+1}$, $\sin(t/2) \ge (t/\pi)$ and $|\cos nt| \le 1$

$$\begin{aligned}
\left|\overline{K}_n(t)\right| &= \frac{1}{2\pi(n+1)} \left| \sum_{k=0}^{n} \left[\frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)} \right] \right| \\
&\le \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{\left|\cos\left(\nu+\frac{1}{2}\right)t\right|}{\left|\sin(t/2)\right|} \\
&= \frac{1}{2t(n+1)} \sum_{k=0}^{n} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \\
&= \frac{1}{2t(n+1)} \sum_{k=0}^{n} \frac{1}{2^k} 2^k \quad \text{since} \sum_{\nu=0}^{k} \binom{k}{\nu} = 2^k \\
&= \frac{1}{2t(n+1)} \sum_{k=0}^{n} 1 \\
&= O\left(\frac{1}{t}\right).
\end{aligned}$$

□

Lemma 3.2. *For* $0 \le a \le b \le \infty, \ \ 0 \le t \le \pi$ *and any* n, *we have*

$$\overline{K}_n(t) = O\left(\frac{\tau^2}{(n+1)}\right) + O\left(\frac{\tau}{(n+1)} 2^{\tau} \sum_{k=\tau}^{n} 2^{-k}\right).$$

Proof. For $0 \leq \frac{1}{n+1} \leq t \leq \pi,\ \sin(t/2) \geq (t/\pi)$

$$\begin{aligned}
\left|\overline{K}_n(t)\right| &= \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right]\right| \\
&\leq \frac{1}{2t(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}e^{i\left(\nu+\frac{1}{2}\right)t}\right\}\right]\right| \\
&\leq \frac{1}{2t(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}e^{i\nu t}\right\}\right]\right|\left|e^{\frac{it}{2}}\right| \\
&\leq \frac{1}{2t(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}e^{i\nu t}\right\}\right]\right| \\
&\leq \frac{1}{2t(n+1)}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}e^{i\nu t}\right\}\right]\right| \\
&\quad + \frac{1}{2t(n+1)}\left|\sum_{k=\tau}^{n}\left[\frac{1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}e^{i\nu t}\right\}\right]\right|. \qquad (3.1)
\end{aligned}$$

Now considering first term of (3.1)

$$\begin{aligned}
\frac{1}{2t(n+1)}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}e^{i\nu t}\right\}\right]\right| &\leq \frac{1}{2t(n+1)}\left|\sum_{k=0}^{\tau-1}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\right|\left|e^{i\nu t}\right| \\
&\leq \frac{1}{2t(n+1)}\sum_{k=0}^{\tau-1}\left[\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\right] \\
&= \frac{1}{2t(n+1)}\sum_{k=0}^{\tau-1}1 \\
&= \frac{\tau}{2t(n+1)} \\
&= O\left(\frac{\tau^2}{(n+1)}\right). \qquad (3.2)
\end{aligned}$$

Now considering second term of (3.1) and using Abel's Lemma

$$\frac{1}{2t\,(n+1)}\left|\sum_{k=\tau}^{n}\left[\frac{1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}e^{i\nu\,t}\right\}\right]\right|$$

$$\leq \frac{1}{2t\,(n+1)}\sum_{k=\tau}^{n}\frac{1}{2^k}\max_{0\leq\ m\leq\ k}\left|\sum_{\nu=0}^{m}\binom{k}{\nu}e^{i\nu\,t}\right|$$

$$\leq \frac{1}{2t\,(n+1)}2^{\tau}\sum_{k=\tau}^{n}\frac{1}{2^k}$$

$$= O\left[\frac{\tau}{(n+1)}2^{\tau}\sum_{k=\tau}^{n}2^{-k}\right]. \qquad (3.3)$$

Combining (3.1), (3.2) and (3.3), we get

$$\overline{K}_n\,(t) = O\left(\frac{\tau^2}{(n+1)}\right) + O\left(\frac{\tau}{(n+1)}2^{\tau}\sum_{k=\tau}^{n}2^{-k}\right) \qquad (3.4)$$

□

4 Proof of Theorems.

Proof of Theorem 2.1. Let $\overline{s}_n\,(f;x)$ denotes the n^{th} partial sum of the series (1.2), then, following Lal [13], we have

$$\overline{s}_n\,(f;x) - \overline{f}\,(x) = \frac{1}{2\pi}\int_0^{\pi}\psi\,(t)\,\frac{\cos\left(n+\frac{1}{2}\right)\,t}{\sin\frac{t}{2}}dt.$$

Therefore using (1.2), the (E, 1) transform E_n^1 of $\overline{s}_n\,(f;x)$ is given by

$$\overline{E_n^1} - \overline{f}\,(x) = \frac{1}{2\pi\;2^n}\int_0^{\pi}\frac{\psi\,(t)}{\sin\,(t/2)}\;\left\{\sum_{k=0}^{n}\binom{n}{k}\cos\left(k+\frac{1}{2}\right)t\right\}dt.$$

Now denoting $\overline{(\mathrm{C},1)(\mathrm{E},1)}$ transform of $\overline{s}_n$ by $\left(\overline{(CE)_n^1}\right)$, we write

$$\overline{(CE)_n^1} - \overline{f}\,(x) = \frac{1}{2\pi\,(n+1)}\sum_{k=0}^{n}\left[\frac{1}{2^k}\int_0^{\pi}\left(\frac{\phi\,(t)}{\sin\left(\frac{t}{2}\right)}\right)\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\cos\left(\nu+\frac{1}{2}\right)t\right\}dt\right]$$

$$= \int_0^{\pi}\psi\,(t)\,\overline{K}_n\,(t)\;\;dt$$

$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\psi\,(t)\;\overline{K}_n\,(t) = I_{1.1} + I_{1.2}\;(\text{say}). \qquad (4.1)$$

We consider,

$$|I_{1.1}| \leq \int_0^{\frac{1}{n+1}} |\psi(t)| \left|\overline{K}_n(t)\right| dt.$$

Using Lemma 3.1,

$$\begin{aligned} |I_{1.1}| &= \int_0^{\frac{1}{n+1}} \frac{|t^\alpha|}{|t|} dt = \int_0^{\frac{1}{n+1}} t^{\alpha-1}\, dt = \left[\frac{t^\alpha}{\alpha}\right]_0^{\frac{1}{n+1}} \\ &= O\left[\frac{1}{(n+1)^\alpha}\right] \text{ for } \ 0<\alpha<1. \end{aligned} \tag{4.2}$$

Using Lemma 3.2, we have

$$\begin{aligned} |I_{1.2}| &\leq \int_{\frac{1}{n+1}}^{\pi} |\psi(t)| \left|\overline{K}_n(t)\right| dt \\ &= O\left[\int_{\frac{1}{n+1}}^{\pi} \frac{1}{(n+1)\ t^{2-\alpha}} dt\right] + O\left[\int_{\frac{1}{n+1}}^{\pi} \frac{2^\tau}{(n+1)\ t^{1-\alpha}} \sum_{k=\tau}^{n} \frac{1}{2^k} dt\right] \\ &= I_{1.2.1} + I_{1.2.2} \text{ (say).} \end{aligned} \tag{4.3}$$

Now we consider,

$$I_{1.2.1} = O\left[\frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-2} dt\right] = \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right), \ 0<\alpha<1 \\ O\left(\frac{\log \pi(n+1)}{(n+1)}\right), \ \alpha = 1 \end{cases}. \tag{4.4}$$

Using (2.2), we have

$$I_{1.2.2} = O\left\{\int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1} dt\right\} = O\left\{\frac{1}{(n+1)^\alpha}\right\}. \tag{4.5}$$

Combining (4.1) to (4.5) and writing $\log e = 1$,

$$\left\|\overline{(CE)_n^1} - \overline{f}\right\|_\infty = \sup\left\{\ \left|\overline{(CE)_n^1} - \overline{f}\right| : x \in [0, 2\pi]\right\} = \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right) & \text{for } 0<\alpha<1 \\ O\left(\frac{\log(n+1)\,\pi\, e}{(n+1)}\right) & \text{for } \alpha = 1 \end{cases}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Following the proof of Theorem 2.1,

$$\overline{(CE)_n^1} - \overline{f}(x) = \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right] \psi(t)\, \overline{K}_n(t)\, dt = I_{2.1} + I_{2.2} \quad \text{(say)}. \tag{4.6}$$

Now considering,

$$|I_{2.1}| \le \int_0^{\frac{1}{n+1}} |\psi(t)|\, \left|\overline{K}_n(t)\right| dt.$$

Using Hölder's inequality and the fact that $\psi \in W(L_r, \xi(t))$,

$$|I_{2.1}| \le \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{t\,|\,\psi(t)\,|\sin^\beta t}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t)\ \left|\overline{K}_n(t)\right|}{t\,\sin^\beta t} \right\}^s dt \right]^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right) \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t)\ \left|\overline{K}_n(t)\right|}{t\,\sin^\beta t} \right\}^s dt \right]^{\frac{1}{s}} \quad \text{by (2.5).}$$

Since $\sin t \ge \left(\frac{2t}{\pi}\right)$ and using Lemma 3.1,

$$I_{2.1} = O\left(\frac{1}{n+1}\right) \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t)}{t^{2+\beta}} \right\}^s dt \right]^{\frac{1}{s}}.$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$I_{2.1} = O\left\{ \left(\frac{1}{n+1}\right) \xi\left(\frac{1}{n+1}\right) \right\} \left\{ \int_\varepsilon^{\frac{1}{n+1}} \left(\frac{dt}{t^{(2+\beta)s}} \right) \right\}^{\frac{1}{s}} \quad \text{for some } 0 < \varepsilon < \frac{1}{n+1}$$

$$= O\left\{ \left(\frac{1}{n+1}\right) \xi\left(\frac{1}{n+1}\right) \right\} \left[\left\{ \frac{t^{-(2+\beta)s+1}}{-(2+\beta)s+1} \right\}_\varepsilon^{\frac{1}{n+1}} \right]^{\frac{1}{s}}$$

$$= O\left\{ \left(\frac{1}{n+1}\right) \xi\left(\frac{1}{n+1}\right) \right\} \left\{ (n+1)^{2+\beta-\frac{1}{s}} \right\}$$

$$= O\left[\xi\left(\frac{1}{n+1}\right) (n+1)^{\beta+1-\frac{1}{s}} \right]$$

$$= O\left\{ (n+1)^{\beta+\frac{1}{r}}\, \xi\left(\frac{1}{n+1}\right) \right\} \quad \text{since } \frac{1}{r} + \frac{1}{s} = 1,\ 1 \le r \le \infty. \tag{4.7}$$

Now we take,

$$|I_{2.2}| \leq \int_{\frac{1}{n+1}}^{\pi} |\psi(t)| \left|\overline{K}_n(t)\right| dt.$$

Now using Lemma 3.2,

$$\begin{aligned}
|I_{2.2}| &= O\left[\int_{\frac{1}{n+1}}^{\pi} \frac{|\psi(t)|}{t^2(n+1)} dt\right] + O\left[\int_{\frac{1}{n+1}}^{\pi} \frac{|\psi(t)|}{(n+1)t} 2^{\tau} \sum_{k=\tau}^{n} \frac{1}{2^k} dt\right]^{\frac{1}{s}} \\
&= O(I_{2.2.1}) + O(I_{2.2.2}) \quad \text{(say)}. \qquad (4.8)
\end{aligned}$$

Using Hölder's inequality, $|\sin t| \leq 1,\ \sin t \geq (2t/\pi)$, conditions (2.4) and (2.6) and using second mean value theorem for integral,

$$\begin{aligned}
|I_{2.2.1}| &\leq \left(\frac{1}{n+1}\right)\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}|\psi(t)|\sin^{\beta}t}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{-\delta+2}\sin^{\beta}t}\right\}^s dt\right]^{\frac{1}{s}} \\
&= \left(\frac{\pi}{2(n+1)}\right)\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{-\delta+\beta+2}}\right\}^s dt\right]^{\frac{1}{s}} \\
&= O\left\{(n+1)^{\delta-1}\right\}\left[\int_{\frac{1}{\pi}}^{n+1} \left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-2-\beta}}\right\}^s \frac{dy}{y^2}\right]^{\frac{1}{s}} \\
&= O\left\{(n+1)^{\delta-1}\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1} \frac{dy}{y^{s(\delta-2-\beta)+2}}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} \leq \eta \leq n+1 \\
&= O\left\{(n+1)^{\delta-1}\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{1}^{n+1} \frac{dy}{y^{s(\delta-2-\beta)+2}}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} \leq 1 \leq n+1 \\
&= O\left\{(n+1)^{\delta-1}\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{y^{s(2+\beta-\delta)-1}}{s(2+\beta-\delta)-1}\right\}_1^{n+1}\right]^{\frac{1}{s}} \\
&= O\left\{(n+1)^{\delta-1}\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(2+\beta-\delta)-\frac{1}{s}}\right] \\
&= O\left\{\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+1-\frac{1}{s}}\right\} \\
&= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \quad \text{since } \frac{1}{r}+\frac{1}{s}=1. \qquad (4.9)
\end{aligned}$$

Similarly using (2.2), conditions (2.4) and (2.6), $|\sin t| \leq 1$, $\sin t \geq (2t/\pi)$ and second mean value theorem for integrals,

$$|I_{2.2.2}| \leq \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}\,|\,\psi(t)\,|\sin^{\beta}t}{\xi(t)}\right\}^{r} dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{1-\delta}\sin^{\beta}t}\frac{1}{(n+1)}2^{\tau}\sum_{k=\tau}^{n}\frac{1}{2^{k}}\right\}^{s} dt\right]^{\frac{1}{s}}$$

$$= O\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}\,|\,\psi(t)\,|}{\xi(t)}\right\}^{r} dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{1-\delta+\beta}}\right\}^{s} dt\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\right\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{1-\delta+\beta}}\right\}^{s} dt\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\right\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1-\beta}}\right\}^{s} \frac{dy}{y^{2}}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\} \left[\int_{\varepsilon_1}^{n+1} \left\{\frac{dy}{y^{s(\delta-1-\beta)+2}}\right\}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} < \varepsilon_1 < n+1$$

$$= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\} \left[\int_{1}^{n+1} \frac{dy}{y^{s(\delta-1-\beta)+2}}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} \leq 1 \leq n+1$$

$$= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\} \left[\left\{\frac{y^{s(1+\beta-\delta)-1}}{s(1+\beta-\delta)-1}\right\}_{1}^{n+1}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\} \left[(n+1)^{\beta+1-\delta-\frac{1}{s}}\right]$$

$$= O\left\{(n+1)^{\beta+1-\frac{1}{s}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r}+\frac{1}{s}=1. \tag{4.10}$$

Combining (4.8), (4.9) and (4.10),

$$\left|\overline{(CE)^{1}_{n}} - \overline{f}(x)\right| = O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}.$$

Now, using L_r-norm, we get

$$\left\|\overline{(CE)^{1}_{n}} - \overline{f}(x)\right\|_{r} = \left\{\int_{0}^{2\pi} \left|\overline{(CE)^{1}_{n}} - \overline{f}(x)\right|^{r} dx\right\}^{\frac{1}{r}}$$

$$= O\left[\left\{\int_0^{2\pi}\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}\right]$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}\right]$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}.$$

This completes the proof of Theorem 2.2. □

5 Applications

Following corollaries can be derived from our main Theorem 2.2:

Corollary 5.1. *If $\xi(t) = t^{\alpha}$, $0 < \alpha \leq 1$, then the weighted class $W(L_r, \xi(t))$, $r \geq 1$, reduces to the class $Lip(\alpha, r)$ and the degree of approximation of a function $\overline{f}(x)$, conjugate to a 2π - periodic function $f \in Lip(\alpha, r)$, $\frac{1}{r} \leq \alpha < 1$, is given by*

$$|\,\overline{(CE)^1_n} - \overline{f}(x)\,| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right).$$

Proof. The result follows by setting $\beta = 0$ in (2.3). □

Corollary 5.2. *If $\xi(t) = t^{\alpha}$ for $0 < \alpha < 1$ and $r = \infty$ in corollary 1, then $f \in Lip\alpha$ and we have*

$$\left\|\overline{(CE)^1_n} - \overline{f}(x)\right\|_{\infty} = O\left(\frac{1}{(n+1)^{\alpha}}\right).$$

Remark 5.3. *An independent proof of above Corollary 5.1 can be obtained along the same lines of our Theorem 2.2.*

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References

[1] A. Zygmund, Trigonometric series, 2nd rev. ed., Cambridge Univ. Press, Cambridge, 1959.

[2] L. McFadden, Absolute Nörlund summability, Duke Math. J. 9 (1942) 168–207.

[3] G.H. Hardy, Divergent series, first edition, Oxford University Press, 1949.

[4] G. Alexits, Convergence problems of orthogonal series, Pergamon Press, London, 1961.

[5] B.N. Sahney, D.S. Goel, On the degree of continuous functions Ranchi, University Math. J. 4 (1973) 50–53.

[6] K. Qureshi, H.K. Neha, A class of functions and their degree of approximation, Ganita. 41 (1) (1990) 37–42.

[7] K. Qureshi, On the degree of approximation of a periodic function f by almost Nörlund means, Tamkang J. Math. 12 (1) (1981) 35–38.

[8] K. Qureshi, On the degree of approximation of a function belonging to the class $Lip\alpha$, Indian J. pure Appl. Math. 13 (1982) 560–563.

[9] P. Chandra, Trigonometric approximation of functions in L_p - norm, J. Math. Anal. Appl. 275 (1) (2002) 13–26.

[10] H.H. Khan, On degree of approximation of functions belonging to the class $Lip\,(\alpha, p)$, Indian J. Pure Appl. Math. 5 (2) (1974) 132–136.

[11] L. Leindler, Trigonometric approximation in L_p - norm, J. Math. Anal. Appl. 302 (2005) 129–136.

[12] B.E. Rhoades, On the degree of approximation of functions belonging to Lipschitz class by Hausdorff means of its Fourier series, Tamkang J. Math. 34 (2003) 245–247.

[13] S. Lal, On K^λ summability of conjugate series of Fourier series, Bulletin of Calcutta Math. Soc. 89 (1997) 97–104.

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