



Cone D-Metric Spaces and Some Fixed Point Theorems

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Abstract : In this paper, we introduce and study the concept of cone D-metric spaces, in which the cone does not need to be normal. Using this new notion we prove several fixed point theorems, which are generalizations of fixed point theorems by Ume [J.S. Ume, Remarks on nonconvex minimization theorems and fixed point theorems in complete D-metric spaces, Indian J. Pure Appl. Math. 32 (1) (2001) 25–36], Rhoades [B.E. Rhoades, A fixed point theorem for generalized metric spaces, Pacific J. Math. 10 (1960) 637–675] and Dhage [B.C. Dhage, Generalized metric spaces and mapping with fixed point, Bull. Calcutta Math. Soc. 84 (1992) 329–336].

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1 Introduction and Preliminaries

In applied mathematics, Ordered normed spaces and cones have many applications, such as Newton's approximation method [1, 2] and in optimization theory

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[3]. Huang and Zhang [4] used the concept of cone metric spaces as a generalization of metric spaces. They have replaced the real numbers (as the co-domain of a “metric”) by an ordered Banach space. The authors described the convergence in cone metric spaces and introduced their completeness. Then they proved some fixed point theorems for contractive mappings on cone metric spaces. In their theorems, cone is normal. Rezapour and Hambarani [5] proved these theorems by omitting normality of cone. Afterward, many results about fixed point theory in cone metric spaces were investigated by several authors; see also [3, 6–28] for more details and references therein.

Recently, Du [3] used the scalarization function and investigated the equivalence of vectorial versions of fixed point theorems in cone metric spaces and scalar versions of fixed point theorems in metric spaces. He showed that many of the fixed point theorems for mappings satisfying contractive conditions of a linear type in metric spaces can be considered as corollaries of corresponding theorems in metric spaces. These investigations, by Kadelburg et al. [24], even easier than that of Du [3], is completed by Minkowski functionals in topological vector space.

Afterward, Haghi et al. [11] showed that some generalizations in fixed point theory are not real generalizations. They proved that some recent generalizations in common fixed point theory such as [7, 29, 30] could easily be obtained from the corresponding fixed point theorems.

Nevertheless, the fixed point theory in cone metric spaces proceeds to be actual, since the method of scalarization function or Minkowski functional can not be applied for a wide class of weakly contractive mapping, satisfying nonlinear contractive conditions.

The concept of D-metric was introduced by Dhage [31]. He proved some fixed point theorems in this space. This result was further improved by Rhoades [32] using a contractive mapping from X into itself. The idea of D-metric apparently seems to be akin to the notion of 2-metric introduced by Gähler [33–35]. The aim of this paper is to generalize and to unify fixed point theorems of Dhage [31], Rhoades [32] and Ume [36] on T -orbitally complete cone D-metric spaces.

The following definitions and results will be needed in the sequel, on the basis of [4, 7, 10, 26]. Let E always be a real Banach space and P a subset of E . P is called a *cone* if:

- (i) P is closed, non-empty and $P \neq \{0\}$;
- (ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b ;
- (iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering \leq with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P .

There exist two kinds of cones: **normal** and **non-normal** ones. The cone P in a real Banach space E is called *normal* if

$$\inf\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0,$$

or, equivalently, if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K \|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P . It is clear that $K \geq 1$.

Example 1.1. Let $E = C_R^1[0, 1]$ with $\|f\| = \|f\|_\infty + \|f'\|_\infty$ on $P = \{f \in E \mid f \geq 0\}$. This cone is not normal cone [6].

Definition 1.2. Let X be a non-empty set. A function $D : X \times X \times X \rightarrow E$ is said to be *cone D-metric* if for all $x, y, z, a \in X$, the following conditions are satisfied:

- (a) $0 \leq D(x, y, z)$ and $D(x, y, z) = 0$ if and only if $x = y = z$;
- (b) $D(x, y, z) = D(p\{x, y, z\})$, where p is a permutation of x, y, z ;
- (c) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$.

Definition 1.3.

- (a) A sequence $\{x_n\}$ in X is called a *D-Cauchy sequence* if for each $0 \ll \epsilon$ with $\epsilon \in E$, there exists a positive integer n_0 such that, for all $m > n, p > n_0$, $D(x_m, x_n, x_p) \ll \epsilon$.
- (b) A sequence $\{x_n\}$ in X is called a *D-convergent* to a point $x \in X$ if for each $0 \ll \epsilon$ with $\epsilon \in E$, there exists a positive integer n_0 such that, for all $m, n > n_0$, $D(x_m, x_n, x) \ll \epsilon$.

Let (X, D) be a cone D-metric space. Then the following properties are often used (particular when dealing with cone D-metric spaces in which the cone need not be normal (see Example 1.1)); The proof of following assertions lies on the lines of the proof in [7] and therefore, we omit these:

- (p₁) If $u \leq v$ and $v \ll w$ then $u \ll w$.
- (p₂) If $0 \leq u \ll \epsilon$ for each $\epsilon \in \text{int } P$ then $u = 0$.
- (p₃) If $a \leq b + c$ for each $c \in \text{int } P$ then $a \leq b$.
- (p₄) If $0 \leq x \leq y$, and $a \geq 0$, then $0 \leq ax \leq ay$.
- (p₅) If $0 \leq x_n \leq y_n$ for each $n \in \mathbb{N}$, and x_n is D-convergent to x , y_n is D-convergent to y , then $0 \leq x \leq y$.
- (p₆) If $0 \leq D(x_m, x_n, x) \leq b_n$ and $b_n \rightarrow 0$, then the sequence $\{x_n\}$ is D-convergent to x .
- (p₇) If E is a real Banach space with a cone P and if $a \leq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.
- (p₈) If $\epsilon \in \text{int } P$, $0 \leq a_n$ and $a_n \rightarrow 0$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll \epsilon$.

From (\mathbf{p}_8) it follows that the sequence $\{x_n\}$ D-converges to $x \in X$ if $D(x_m, x_n, x) \rightarrow 0$ as $n, m \rightarrow \infty$ and $\{x_n\}$ is a D-Cauchy sequence if $D(x_m, x_n, x_p) \rightarrow 0$ as $n, m, p \rightarrow \infty$.

Remark 1.4. Let (X, D) be a cone D-metric space. Let us remark that the family $\{B(x, e) : x \in X, 0 \ll e\}$, where $B(x, e) = \{y \in X : D(x, y, y) \ll e \text{ if } y = x \text{ and } D(x, y, z) \ll e + c \text{ if } y \neq x \text{ where } c \in \text{int}P, c \neq 0\}$, is a sub-basis for topology on X . We denote this cone topology by τ . Throughout this paper we assume that the D-metric space X is equipped with the topology τ .

Definition 1.5. Let T be a mapping of a D-metric space X into itself. For $A \subset X$, for each $x \in X$, let

$$O(x, n) = \{x, Tx, \dots, T^n x\}, \quad O(x, \infty) = \{x, Tx, T^2 x, \dots\}.$$

(X, D) is said to be *T-orbitally complete* (resp. *complete*) if every D-Cauchy sequence contained in $\{x, Tx, T^2 x, \dots\}$ (resp. D-Cauchy sequence) is D-converges. M is said to be *D-bound* of X if for any $x, y, z \in X$ we have $D(x, y, z) \leq M$; in this case X is called a *D-bounded space*.

Example 1.6. Let $X = [0, 1] \subset \mathbb{R}$. In Example 1.1, we defined cone D-metric by

$$D(x, y, z) = (|x - y| + |y - z| + |x - z|)\varphi$$

for all $x, y, z \in X$, where $\varphi : [0, 1] \rightarrow \mathbb{R}$ such that $\varphi(t) = e^t$. Then (X, D) is a complete cone D-metric space.

This example shows that the category of cone D-metric spaces is larger than category of D-metric spaces.

Note that any complete cone D-metric space is *T-orbitally complete*, but the converse is not valid, for example:

Example 1.7. In Example 1.6, if set $X = [0, 1)$, be a subset of \mathbb{R} equipped with the same cone D-metric and $T : [0, 1) \rightarrow [0, 1)$ be a mapping $Tx = \frac{1}{2}x^2, x \in X$. Then X is *T-orbitally complete space* but it is not complete space.

Definition 1.8. Let (X, D) be a cone D-metric space and $T : X \rightarrow X$

- (1) T is continuous at $x \in X$ if x_n is a sequence in X and x_n is D-convergent to x implies $T(x_n)$ is D-convergent to $T(x)$.
- (2) $G : X \rightarrow P$ is lower semicontinuous at $x \in X$ if for any ϵ in E with $0 \ll \epsilon$, there is n_0 in \mathbb{N} such that

$$G(x) \leq G(x_n) + \epsilon, \quad \text{for all } n \geq n_0,$$

whenever $x_n \in X$ and x_n is D-convergent to x .

(3) For $x \in X$, $O(x; \infty) = \{x, Tx, T^2x, \dots\}$ is called the orbit of x . $G : X \rightarrow P$ is T -orbitally lower semicontinuous at x if for any ϵ in E with $0 \ll \epsilon$, there is n_0 in \mathbb{N} such that

$$G(u) \leq G(x_n) + \epsilon, \quad \text{for all } n \geq n_0,$$

whenever $x_n \in O(x; \infty)$ and x_n is D -convergent to u .

Remark 1.9. Let us remark that in Definition 1.8, by setting $E = \mathbb{R}$, $P = [0, \infty)$, $\|x\| = |x|$, $x \in E$, we get the well-know definitions of continuity, lower and T -orbitally lower semicontinuity.

Using the definition of a cone D-metric for X and topology τ on X , we have the following Lemma.

Lemma 1.10. The cone D-metric D is a continuous function from $X \times X \times X$ into E in the topology τ on X .

Proof. Let $x, y, z, a, b, c \in X$ and from definition of cone D-metric, we obtain

$$\begin{aligned} D(x, y, z) &\leq D(a, y, z) + D(x, a, z) + D(x, y, a) \\ &\leq D(x, y, z) + D(x, y, z) + D(a, y, b) + D(x, a, z) + D(x, y, a) \\ &\leq D(b, y, z) + D(c, b, z) + D(a, c, z) + D(a, b, c) + D(a, y, b) + D(x, a, z) \\ &\quad + D(x, y, a). \end{aligned}$$

So

$$\begin{aligned} D(x, y, z) - D(a, b, c) &\leq D(a, x, z) + D(a, x, y) + D(b, y, z) + D(b, y, a) \\ &\quad + D(c, z, a) + D(c, z, b), \end{aligned}$$

and

$$\begin{aligned} D(a, b, c) - D(x, y, z) &\leq D(a, x, c) + D(a, x, b) + D(b, y, x) + D(b, y, c) \\ &\quad + D(c, z, x) + D(c, z, y). \end{aligned}$$

Since, for all $0 \ll \epsilon$ with $\epsilon \in E$ there is $0 \ll \delta$, $\delta \in E$ with $\delta \ll \frac{\epsilon}{12}$ such that

$$x \in B(a, \delta), \quad y \in B(b, \delta), \quad z \in B(c, \delta)$$

imply for any $u \in X$, we have

$$D(a, x, u) \ll \delta + \gamma, \quad D(b, y, u) \ll \delta + \gamma, \quad \text{and} \quad D(c, z, u) \ll \delta + \gamma$$

respectively, where $\gamma \in \text{int}P$ with $\gamma \neq 0$ and $\gamma \ll \delta$. Hence for any $0 \ll \epsilon$, there is a $0 \ll \delta$ with $\frac{\epsilon}{9} \ll \delta$ such that

$$x \in B(a, \delta), \quad y \in B(b, \delta), \quad z \in B(c, \delta)$$

imply

$$\begin{aligned} |D(x, y, z) - D(a, b, c)| &\leq \delta + \gamma + \delta + \gamma + \delta + \gamma + \delta + \gamma + \delta + \gamma + \delta + \gamma \\ &= 6\delta + 6\gamma \ll 12\delta \ll \epsilon. \end{aligned}$$

Therefore, cone D-metric is uniformly continuous. \square

2 Fixed Point Theorems

In this section we generalize some fixed point theorems due to Ume [36], Rhoades [17] and Dhage [31]. We note that the methods of Du [3], Kadelburg et al. [24] and Haghi et al. [11] for cone contraction mappings in cone metric spaces can not be applied for contraction mappings in cone D-metric spaces.

The following result generalizes Ume's theorem:

Theorem 2.1. *Let (X, D) be T -orbitally complete cone D -metric space and let $g : X \times X \rightarrow X$ be a continuous function satisfying*

$$v \leq D(x, y, g(x, y)) + D(y, z, g(y, z)), \quad (2.1)$$

$v \in \{D(x, z, g(x, z)), D(x, y, g(x, z)), D(y, z, g(x, z))\}$, for all $x, y, z \in X$. Let for each $x \in X$, $D(x, y, g(x, y))$ is lower semicontinuous at y in X . Let T be a self-map of X satisfying

$$D(Tx, T^2x, g(Tx, T^2x)) \leq rD(x, Tx, g(x, Tx)) \quad (2.2)$$

for all $x \in X$ and $0 \leq r < 1$. Let for every $y \in X$ with $y \neq Ty$, there exists $c \in \text{int}(P)$, $c \neq 0$, such that

$$c \ll D(x, y, g(x, y)) + D(x, Ty, g(x, Ty)) \quad \text{for all } x \in X. \quad (2.3)$$

Then, there exists $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $D(v, v, g(v, Tv)) = 0$.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}_{n=0}^{\infty}$ satisfying the following: $x_0 = x$ and $x_n = T^n x$ for any $n \in \mathbb{N}$. Then we have, for any $n \in \mathbb{N}$,

$$\begin{aligned} D(x_n, x_{n+1}, g(x_n, x_{n+1})) &\leq rD(x_{n-1}, x_n, g(x_{n-1}, x_n)) \\ &\leq r^2D(x_{n-2}, x_{n-1}, g(x_{n-2}, x_{n-1})) \\ &\leq \dots \leq r^n D(x, x_1, g(x, x_1)). \end{aligned}$$

From the hypotheses, we have

$$\begin{aligned} D(x_n, x_{n+1}, g(x_n, x_{n+1})) &\leq \sum_{j=0}^{p-1} D(x_{n+j}, x_{n+j+1}, g(x_{n+j}, x_{n+j+1})) \\ &\leq \frac{r^n(1-r^p)}{1-r} D(x, x_1, g(x, x_1)) \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} D(x_n, x_{n+p}, x_{n+p+t}) &\leq D(x_n, x_{n+p}, g(x_n, x_{n+p+t})) + D(x_n, x_{n+p+t}, g(x_n, x_{n+p+t})) \\ &\quad + D(x_{n+p}, x_{n+p+t}, g(x_n, x_{n+p+t})) \\ &\leq 2\{D(x_n, x_{n+p}, g(x_n, x_{n+p})) + D(x_{n+p}, x_{n+p+t}, g(x_{n+p}, x_{n+p+t}))\} \\ &\quad + D(x_n, x_{n+p+t}, g(x_n, x_{n+p+t})). \end{aligned}$$

Thus

$$D(x_n, x_{n+p}, x_{n+p+t}) \leq \frac{5r^n}{1-r} D(x, x_1, g(x, x_1)). \tag{2.5}$$

Since X is T -orbitally complete cone D-metric space, $\{x_n\}$ D-converge to some point $z \in X$.

Assume that $z \neq Tz$. Then, by hypothesis there exists $c \in \text{int}(P), c \neq 0$, such that

$$c \ll D(x, z, g(x, z)) + D(x, Tx, g(x, Tx)) \text{ for all } x \in X. \tag{2.6}$$

By $(\mathbf{p}_1), (\mathbf{p}_8), (2.4)$ and (2.5) , there exists $n_0 \in \mathbb{N}$ such that

$$D(x_{n_0}, x_{n_0+1}, g(x_{n_0}, x_{n_0+1})) \ll \frac{c}{6} \tag{2.7}$$

and

$$D(x_{n_0}, x_m, g(x_{n_0}, x_m)) \ll \frac{c}{6} \tag{2.8}$$

for all $m > n_0$. Now, by Definition 1.8(2) and (2.8), there exists $m_0 > n_0$, such that

$$D(x_{n_0}, z, g(x_{n_0}, z)) \leq D(x_{n_0}, x_m, g(x_{n_0}, x_m)) + \frac{c}{6} \ll \frac{c}{3}. \tag{2.9}$$

for all $m > m_0$.

Finally, by $(\mathbf{p}_1), (\mathbf{p}_8), (2.7)$ and (2.9) , we have

$$D(x_{n_0}, z, g(x_{n_0}, z)) + D(x_{n_0}, x_{n_0+1}, g(x_{n_0}, x_{n_0+1})) \ll \frac{c}{2} \tag{2.10}$$

and by $(\mathbf{p}_1), (2.6)$ and (2.10) , we have $c = 0$. This is a contradiction. Therefore $z = Tz$. If $v = Tv$ we have,

$$D(v, v, g(v, v)) = D(Tv, T^2v, g(Tv, T^2v)) \leq rD(v, Tv, g(v, Tv)) = rD(v, v, g(v, v)),$$

and by (\mathbf{p}_7) we have $D(v, v, g(v, v)) = 0$. □

Since any cone D-metric space is D-metric space, hence by Theorem 2.1, we can achieve Ume's theorem [36] as follows:

Corollary 2.2 ([36]). *Let (X, D) be complete and let $g : X \times X \rightarrow X$ be a continuous function satisfying*

$$\begin{aligned} \max\{D(x, z, g(x, z)), D(x, y, g(x, z)), D(y, z, g(x, z))\} \\ \leq D(x, y, g(x, y)) + D(y, z, g(y, z)), \end{aligned}$$

for all $x, y, z \in X$. Let for each $x \in X, D(x, y, g(x, y))$ is lower semicontinuous at y in X . Let T be a self-map of X satisfying

$$D(Tx, T^2x, g(Tx, T^2x)) \leq r.D(x, Tx, g(x, Tx))$$

for all $x \in X$ and $0 \leq r < 1$. Let for every $y \in X$ with $y \neq Ty$,

$$\inf\{D(x, y, g(x, y)) + D(x, Tx, g(x, Tx)) : x \in X\} > 0.$$

Then there exists a $z \in X$ such that $z = Tz$. Moreover, if $v = Tv$, then $D(v, v, g(v, Tv)) = 0$.

Theorem 2.3. *Let (X, D) be a T -orbitally complete cone D -metric space and X be a D -bounded. Let $T : X \rightarrow X$ be a mapping such that for every $x, w \in X$*

$$D(Tx, T^2x, Tw) \leq rD(x, Tx, w) \quad (2.11)$$

where $r \in]0, 1[$. Then there exists $z \in X$ such that $z = Tz$.

Proof. We prove this theorem in two cases.

First case: we prove for all $u \in X$ with $u \neq Tu$, there exists $c \in \text{int}P$ with $c \neq 0$ such that

$$c \ll D(x, Tx, u) + D(x, Tx, T^2x) + D(x, T^2x, u), \quad (2.12)$$

for all $x \in X$. If there exists $u \in X$ with $u \neq Tu$ such that for all $c \in \text{int}P$ with $c \neq 0$ there exists $x \in X$ such that

$$D(x, Tx, u) + D(x, Tx, T^2x) + D(x, T^2x, u) = 0. \quad (2.13)$$

Then, there exists a sequence $\{x_n\}$ in X such that for all $0 \ll \varepsilon$ there exists N such that for all $n \geq N$, we have

$$D(x_n, Tx_n, u) + D(x_n, Tx_n, T^2x_n) + D(x_n, T^2x_n, u) \ll \varepsilon. \quad (2.14)$$

Thus for all $n \geq N$, we have

$$D(x_n, Tx_n, u) \ll \varepsilon, D(x_n, Tx_n, T^2x_n) \ll \varepsilon \text{ and } D(x_n, T^2x_n, u) \ll \varepsilon, \quad (2.15)$$

and so, by Definition 1.2, for all $n \geq N$, we have $D(Tx_n, T^2x_n, u) \ll \varepsilon$.

By Lemma 1.10, cone D -metrics are continuous and so

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} T^2x_n = u. \quad (2.16)$$

Hence, from (2.11) and (2.15) for all $n \geq N$, we have

$$D(Tx_n, T^2x_n, Tu) \leq rD(x_n, Tx_n, u) \ll \varepsilon,$$

by Definition 1.8, (2.11) and (2.16),

$$D(Tx_n, u, Tu) \leq D(Tx_n, T^2x_n, Tu) + \varepsilon \leq rD(x_n, Tx_n, u) + \varepsilon$$

and in the same way, by Definition 1.8, (2.11) and (2.16),

$$\begin{aligned} D(Tx_n, T^2x_n, u) &\leq D(Tx_n, T^2x_n, T^2x_n) + \varepsilon \\ &\leq rD(x_n, Tx_n, Tx_n) + \varepsilon \\ &\leq rD(x_n, Tx_n, T^2x_n) + \varepsilon. \end{aligned}$$

which from (2.15) implies that, for all $n \geq N$,

$$D(Tx_n, T^2x_n, Tu) \ll \varepsilon, D(Tx_n, u, Tu) \ll \varepsilon \text{ and } D(Tx_n, T^2x_n, u) \ll \varepsilon.$$

Hence by Definition 1.2, for all $n \geq N$, we have $D(T^2x_n, u, Tu) \ll \varepsilon$ and hence $u = Tu$. This is a contradiction.

Second case: Fix $u \in X$. Define $u_n = T^n u$ for each integer $n \in \mathbb{N}$. From (2.11), we have

$$D(u_n, u_{n+1}, u_{n+2}) \leq rD(u_{n-1}, u_n, u_{n+1}) \leq \dots \leq r^n D(u_0, u_1, u_2).$$

So, for all $0 \ll \varepsilon$ there exists N such that for all $n \geq N$, we have

$$D(u_n, u_{n+1}, u_{n+2}) \ll \varepsilon. \tag{2.17}$$

Thus, for any $p > m > n$ for which $m = n + k$ and $p = m + t$ ($k, t \in \mathbb{N}$), we have

$$\begin{aligned} D(u_n, u_m, u_p) &\leq D(u_n, u_{n+1}, u_{n+2}) + \dots + D(u_{p-2}, u_{p-1}, u_p) \\ &\ll \sum_{j=n}^p 2M \frac{\varepsilon}{2^j} \\ &\leq \frac{1}{2^{n-1}} M\varepsilon, \end{aligned}$$

where M is D -bound of X . Hence, $\{u_n\}$ is a D -Cauchy sequence. Since (X, D) is a T -orbitally complete metric space, there exists a point $u \in X$ such that $u_n \rightarrow u$; further u is a fixed point. Let $n \in \mathbb{N}$ be fixed. Then, by Lemma 1.10, D is continuous and so by Definition 1.8 and above relation, there exists N such that for all $n \geq N$ we have

$$D(u_n, u_m, u) \leq D(u_n, u_m, u_p) + \varepsilon \ll \frac{1}{2^{n-1}} M\varepsilon. \tag{2.18}$$

Assume that $u \neq Tu$. Then, by (2.12) there exists $c \in \text{int}P$ with $c \neq 0$ such that,

$$c \ll D(x, Tx, u) + D(x, Tx, T^2x) + D(x, T^2x, u) \tag{2.19}$$

By (\mathbf{p}_1) , (\mathbf{p}_8) , (2.17) and (2.18), there exists $n_0 \in \mathbb{N}$ such that

$$D(x_{n_0}, x_{n_0+1}, x_{n_0+2}) \ll \frac{c}{6}, \tag{2.20}$$

$$D(x_{n_0}, x_{n_0+2}, x_m) \ll \frac{c}{6} \tag{2.21}$$

and

$$D(x_{n_0}, x_{n_0+1}, x_m) \ll \frac{c}{6} \tag{2.22}$$

for all $m > n_0$. Now, by Definition 1.8(2) and (2.19), there exists $m_0 > n_0$, such that

$$D(x_{n_0}, x_{n_0+1}, u) \leq D(x_{n_0}, x_{n_0+1}, x_m) + \frac{c}{6} \ll \frac{c}{3}, \tag{2.23}$$

and

$$D(x_{n_0}, x_{n_0+2}, u) \leq D(x_{n_0}, x_{n_0+2}, x_m) + \frac{c}{6} \ll \frac{c}{3} \tag{2.24}$$

for all $m > m_0$. Finally, by (\mathbf{p}_1) , (\mathbf{p}_8) , (2.21), (2.24) and (2.25), we have

$$D(x_{n_0}, x_{n_0+1}, x_{n_0+2}) + D(x_{n_0}, x_{n_0+1}, u) + D(x_{n_0}, x_{n_0+2}, u) \ll \frac{5c}{6} \quad (2.25)$$

and by (\mathbf{p}_1) , (2.20) and (2.26), we have $c = 0$. This is a contradiction. Therefore $z = Tz$. \square

The concept of quasi contraction on cone metric spaces is defined by Ilić and Rakočević [37]. In the next theorem by using of quasi contraction on cone D-metric spaces, we called D-quasi contraction, we show that any self map has a unique fixed point.

Theorem 2.4. *Let X be T -orbitally complete and bounded cone D-metric space, T be a self-map of X satisfying in the following D-quasi contraction:*

$$D(Tx, Ty, Tz) \leq ru, \quad (2.26)$$

where $u \in \{D(x, y, z), D(x, Tx, z), D(y, Ty, Tz), D(x, Ty, z), D(y, Tx, z)\}$, for all $x, y, z \in X$ and $0 \leq r < 1$. Then T has a unique fixed point u in X , and T is continuous at u .

Proof. Let $x_0 \in X$ and define $x_{n+1} = Tx_n$. If $x_{n+1} = x_n$ for some n , Then T has fixed point. Assume that $x_{n+1} \neq x_n$ for each n . In (2.19), setting $x = x_{n-1}$, $y = x_n$, $z = x_{n+p-1}$, we have

$$D(x_n, x_{n+1}, x_{n+p}) \leq r.u,$$

where $u \in \{D(x_{n-1}, x_n, x_{n+p-1}), D(x_n, x_{n+1}, x_{n+p-1}), D(x_{n-1}, x_{n+1}, x_{n+p-1}), D(x_n, x_n, x_{n+p-1})\}$. Hence,

$$D(x_n, x_{n+1}, x_{n+p}) \leq r^n u,$$

where $u \in \{D(x_a, x_b, x_c) : 0 \leq a \leq n, 1 \leq b \leq n+1, c = p\}$. Let M be D-bound of X . So, we have

$$D(x_n, x_{n+1}, x_{n+p}) \leq r^n M. \quad (2.27)$$

Using Definition 1.2(c) and (2.28),

$$\begin{aligned} D(x_n, x_{n+p}, x_{n+p+t}) &\leq D(x_n, x_{n+p}, x_{n+1}) + D(x_n, x_{n+1}, x_{n+p+t}) \\ &\quad + D(x_{n+1}, x_{n+p}, x_{n+p+t}) \\ &\leq 2r^n M + D(x_{n+1}, x_{n+p}, x_{n+p+t}) \\ &\leq 2r^n M + D(x_{n+1}, x_{n+p}, x_{n+2}) + D(x_{n+1}, x_{n+2}, x_{n+p+t}) \\ &\quad + D(x_{n+2}, x_{n+p}, x_{n+p+t}) \\ &\leq 2(r^n + r^{n+1})M + D(x_{n+2}, x_{n+p}, x_{n+p+1}) \\ &\quad \vdots \\ &\leq 2(r^n + r^{n+1} + \dots + r^{n+p-1})M + D(x_{n+p-1}, x_{n+p}, x_{n+p+t}) \\ &\leq 2M \sum_{k=n}^{n+p} r^k \leq \frac{2Mr^n}{1-r}. \end{aligned}$$

Therefore $\{x_n\}$ is D-Cauchy. Since X is T-orbitally complete, $\{x_n\}$ d-converges. Call the limit v in X . From (2.27),

$$D(x_n, x_{n+1}, Tv) \leq ru,$$

where $u \in \{D(x_{n-1}, x_n, v), D(x_n, x_{n+1}, v), D(x_{n-1}, x_{n+1}, v), D(x_n, x_n, v)\}$. From Lemma 1.10 and (\mathbf{p}_5) , we can conclude that $D(v, v, Tv) \leq 0$, which implies that $v = Tv$.

To prove uniqueness, assume that $w \neq v$ is also fixed point of T . From (2.27), $D(v, w, v) = D(Tv, Tw, Tv) \leq ru$ where $u \in \{D(v, w, v), D(v, Tv, v), D(w, Tw, v), D(v, Tw, v), D(w, Tv, v)\}$, and so $D(v, w, v) \leq rD(w, w, v)$. But, at the same way, $D(w, w, v) \leq rD(v, w, v)$, and Hence

$$D(v, w, v) \leq r^2D(v, w, v),$$

a contradiction. Therefore $v = w$. T is continuous at v because if $\{y_n\} \subseteq X$ with $\lim_{n \rightarrow \infty} y_n = v$, then, substituting in (2.27), with $x = z = v$, $y = y_n$, we obtain

$$D(Tv, Ty_n, Tv) \leq ru \tag{2.28}$$

where $u \in \{D(v, y_n, v), D(v, Tv, v), D(y_n, Ty_n, v), D(v, Ty_n, v), D(y_n, Tv, v)\}$. Now, for any $c \in \text{int}P$ there exists n_0 such that for all $n \geq n_0$ we have

$$D(v, y_n, v) \ll c, D(y_n, Ty_n, v) \ll c, \text{ and } D(y_n, Tv, v) \ll c. \tag{2.29}$$

Hence $D(v, Ty_n, v) \leq ru$ where $u \in \{c, D(v, Ty_n, v)\}$, which by Lemma 1.10, (\mathbf{p}_5) , (\mathbf{p}_7) and (\mathbf{p}_8) implies that $\lim_{n \rightarrow \infty} Ty_n = v = Tv$, and T is continuous at v . \square

From the previous theorem we can obtain Rhoades's theorem [32]. He proved this results by using a contractive mapping from X into itself.

Corollary 2.5 ([32]). *Let X be complete and bounded D-metric space, T be a self-map of X satisfying*

$$D(Tx, Ty, Tz) \leq r \max\{D(x, y, z), D(x, Tx, z), D(y, Ty, Tz), D(x, Ty, z), D(y, Tx, z)\}$$

for all $x, y, z \in X$ and $0 \leq r < 1$. Then T has a unique fixed point u in X , and T is continuous at u .

In the following theorem we generalize Dhage's theorem [31].

Theorem 2.6. *Let T be a self-mapping of a T-orbitally complete and D-bounded cone D-metric space X satisfying*

$$D(Tx, Ty, Tz) \leq rD(x, y, z)$$

for all $x, y, z \in X$ and for some $0 \leq r < 1$. Then T has a unique fixed point u in X , and T is continuous at u .

Proof. From Theorem 2.4, it is clear. \square

In 1992, Dhage [31] proved a fixed point theorem in D-metric space as follows:

Corollary 2.7 ([31]). *Let T be a self-mapping of a complete and bounded D-metric satisfying*

$$D(Tx, Ty, Tz) \leq rD(x, y, z)$$

for all $x, y, z \in X$ and $0 \leq r < 1$. Then T has a unique fixed point u in X , and T is continuous at u .

Proof. It can be obtained from Theorem 2.6, because any cone D-metric space is a D-metric space. \square

Corollary 2.8. *Let T be a selfmap of a T -orbitally complete and D -bounded cone D-metric space X satisfying the condition that there exists a positive integer q such that*

$$D(T^q x, T^q y, T^q z) \leq rD(x, y, z) \quad (2.30)$$

for all $x, y, z \in X$, for some $0 \leq r < 1$. Then T has a unique fixed point u , and T is T -orbitally continuous at u .

The following example shows that our generalizations are useful.

Example 2.9. *Let $E = C_R^1[0, 1] \times C_R^1[0, 1]$ with $\|\mathbf{f}\| = \|\mathbf{f}\|_\infty + \|\mathbf{f}'\|_\infty$ on $P = \{\mathbf{f} = (f, g) \in E \mid f, g \geq 0\}$. This cone is non-normal cone. Let $X = \{(x, 0, 0) \in R^3 \mid 0 \leq x \leq 1\} \cup \{(0, 0, x) \in R^3 \mid 0 \leq x \leq 1\}$. The mapping $D : X \times X \times X \rightarrow E$ is defined by*

$$\begin{aligned} D((x, 0, 0), (y, 0, 0), (z, 0, 0)) \\ = \left(\frac{4}{3}(|x - y| + |y - z| + |z - x|)\varphi, (|x - y| + |y - z| + |z - x|)\varphi \right), \end{aligned}$$

$$\begin{aligned} D((0, 0, x), (0, 0, y), (0, 0, z)) \\ = \left((|x - y| + |y - z| + |z - x|)\varphi, \frac{2}{3}(|x - y| + |y - z| + |z - x|)\varphi \right), \end{aligned}$$

$$\begin{aligned} D((x, 0, 0), (0, 0, y), (0, 0, z)) = \dots = D((0, 0, z), (0, 0, y), (x, 0, 0)) \\ = \left(\left(\frac{4}{3}x + y + z \right)\varphi, \left(x + \frac{2}{3}(y + z) \right)\varphi \right) \end{aligned}$$

where $\varphi : [0, 1] \rightarrow R$ such that $\varphi(t) = e^t$. Then (X, D) is a complete cone D-metric space. Let mapping $T : X \rightarrow X$ with $T((x, 0, 0)) = (0, 0, x)$ and $T((0, 0, x)) = (\frac{1}{2}x, 0, 0)$. Then T satisfies the contractive condition with constant $k = \frac{3}{4} \in [0, 1)$. It is obvious that T has a unique fixed point $(0, 0, 0) \in X$. On the other hand, we see that T is not a contractive mapping in the Euclidean D-metric on X .

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