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Generalized Vector Mixed Variational-Like Inequality Problem Without Monotonicity

Suhel Ahmad Khan[†], Qamrul Haque Khan[‡] and Farhat Suhel[§]

[†]Department of Mathematics, BITS, Pilani-Dubai
P.O. Box 345055, Dubai, U.A.E.
e-mail : khan.math@gmail.com

[‡]Department of Mathematics, Zakir Husain College
University of Delhi, India
e-mail : qhkhan.ssitm@gmail.com

[§]Department of Mathematics, Aligarh Muslim University
Aligarh-202002, India
e-mail : farhat.math@gmail.com

Abstract : In this paper, $L(X,Y)$-diagonally convex mapping and two kinds of η-f-complete semicontinuous mappings are introduced. Further, we studied the solvability for a class of generalized vector mixed variational-like inequality problem in reflexive Banach spaces by using Brouwer's fixed point theorem. The results presented in this paper are extensions and improvements of some earlier and recent results in the literature.

Keywords : Diagonally convex; η-f-complete semicontinuity; η-f-strong semicontinuity; Reflexive Banach space.
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1 Introduction

Let X and Y be two real Banach spaces, $K \subset X$ be a nonempty, closed and convex subset of X and $P \subset Y$ be a nonempty subset of Y. P is called proper cone, if $P \neq Y$ and P is said to be a closed, convex and pointed cone with its apex at the origin, if P is closed and the following conditions hold:

(i) $\lambda P \subset P$, for all $\lambda > 0$;

(ii) $P + P \subset P$;

(iii) $P \cap (-P) = \{0\}$.

The partial order $\leq_P$ in Y, induced by the pointed cone P is defined by declaring $x \leq_P y$ if and only if $y - x \in P$ for all x, y in Y. An ordered Banach space is a pair (Y, P) with the partial order induced by P. The weak order $\not\leq_{\text{int}\,P}$ in an ordered Banach space (Y, P) with $\text{int}\,P \neq \emptyset$ is defined as $x \not\leq_{\text{int}\,P} y$ if and only if $y - x \notin \text{int}\,P$ for all x, y in Y, where $\text{int}\,P$ denotes the interior of P. Let $L(X, Y)$ be the space of all continuous linear mappings from X into Y and $T : K \to L(X, Y)$ be a nonlinear mapping. The vector variational inequality (for short, VVI) consists in finding $x \in K$, such that

$$\langle Tx, y - x\rangle \not\leq_{\text{int}\,P} 0, \quad \forall y \in K.$$

A VVI was first introduced by Giannessi [1] in the setting of finite dimensional Euclidean spaces, later on, a VVI was studied and generalized to infinite dimensional spaces by Chen [2, 3], Chen and Craven [4], Chen and Yang [5]. Since VVI has found many applications in vector optimization, vector complementarity problems, approximate vector optimization, vector equilibria and other areas, therefore it has been widely studied and generalized by many authors. For details we refer [1–13] and references therein.

Recently, Zeng and Yao [13] introduced and considered the concepts of complete semicontinuity and strong semicontinuity for vector set-valued mappings and they proved the solvability for a class of generalized vector variational inequalities without monotonicity assumption by using these concepts and by applying the Brouwer's fixed point theorem. Further, they removed the boundedness assumption of K and extended to the general case of a nonempty closed and convex subset K.

Inspired and motivated by the work of Zeng and Yao [13], in this paper we introduced the concepts of η-f-complete semicontinuous, η-f-strong semicontinuous and $L(X, Y)$-diagonally convex mappings. Further, we studied a class of generalized vector mixed variational-like inequality problem which is an extension of the vector variational-like inequality problems studied by many authors; see for example [6, 9, 10]. Furthermore, utilizing Brouwer's fixed point theorem, we proved the solvability for this class of generalized vector mixed variational-like inequality problem without monotonicity assumption. The results presented in this paper are extension and improvement of the corresponding results of Usman and Khan [12], Zeng and Yao [13], Huang and Fang [8].

2 Preliminaries

Throughout the paper unless otherwise specified, let X and Y be two real Banach spaces, K be a nonempty, closed and convex subset of X and P be a nonempty subset of Y. Let $P : K \to 2^Y$ be a set-valued mapping such that for each $x \in K$, $P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$ and let $P_- = \bigcap_{x \in K} P(x)$. Now let $C \subseteq Y$ be a nonempty subset of Y. Let $A : L(X,Y) \to L(X,Y)$ be a mapping $\eta : K \times K \to X$ and $f : K \times K \to Y$ be bi-mappings and $V : K \to 2^Y$ and $H : K \times C \to 2^{L(X,Y)}$ be set-valued mappings. In this paper, we consider the following generalized vector mixed variational-like inequality problem (for short, GVMVLIP): Find $x \in K$, $z \in V(x)$ and $\xi \in H(x,z)$ such that

$$\langle A\xi, \eta(y,x)\rangle + f(y,x) \nleq_{\text{int}\, P(x)} 0, \quad \forall y \in K. \tag{2.1}$$

Some Special Cases

(I) If $f \equiv 0$ and $A \equiv I$, the identity mapping of $L(X,Y)$, then GVMVLIP (2.1) reduces to the following generalized vector pre-variational inequality problem of finding $x \in K$ $z \in V(x)$ and $\xi \in H(x,z)$ such that

$$\langle \xi, \eta(y,x)\rangle \nleq_{\text{int}\, P(x)} 0, \quad \forall y \in K,$$

which was introduced and considered by Chadli et al. [6] in 2004.

(II) If $V \equiv 0$, $H \equiv T : K \to 2^{L(X,Y)}$ and $A \equiv I$, the identity mapping of $L(X,Y)$, then GVMVLIP (2.1) reduces to the following generalized vector variational-type inequality problem of finding $x \in K$ such that for all $y \in K$, there exists $s_o \in T(x)$ such that

$$\langle s_0, \eta(y,x)\rangle + f(y,x) \nleq_{\text{int}\, P(x)} 0,$$

which was introduced by Lee et al. [9] in 2000.

(III) If we take $T : K \to L(X,Y)$ and $P(x) = P$, $\forall x \in K$ in (II), then it reduces to the following generalized weak vector variational-like inequality of finding $x \in K$ such that

$$\langle Tx, \eta(y,x)\rangle + f(y,x) \nleq_{\text{int}\, P} 0,$$

which was studied by Lee et al. [10] in 2008.

First, we recall the following concepts and results which are needed in the sequel.

Definition 2.1. A mapping $f : K \to Y$ is said to be

(i) P_--*convex*, if $f(tx + (1-t)y) \leq_{P_-} tf(x) + (1-t)f(y)$, $\forall x, y \in K$, $t \in [0,1]$;

(ii) P_--*concave*, if $-f$ is P_--convex.

Definition 2.2. A mapping $g : X \to Y$ is said to be *completely continuous* if and only if the weak convergence of $\{x_n\}$ in X to x in X implies the strong convergence of $\{g(x_n)\}$ in Y to $g(x)$ in Y.

Definition 2.3 ([14]). Let K be a subset of a topological vector space X and let $F : K \to 2^X$ be a KKM mapping. If for each $x \in K$, $F(x)$ is closed and for at least one $x \in K$, $F(x)$ is compact, then

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

Lemma 2.4 ([11]). *Let X, Y and Z be real topological vector spaces, K and C be nonempty subsets of X and Y, respectively. Let $H : K \times C \to 2^Z$, $V : K \to 2^Y$ be set-valued mappings. If both H, V are upper semicontinuous with compact values, then the set-valued mapping $T : K \to 2^Z$ defined by*

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is upper semicontinuous with compact values.

3 Main Results

In this section, we shall establish the existence results for GVMVLIP (2.1) without monotonicity assumption in the setting of Banach spaces with the help of Brouwer's fixed point theorem. First we recall lemmas due to Brouwer [15], Zeng and Yao [13].

Lemma 3.1 ([15]). *Let X be a nonempty, compact and convex subset of a finite dimensional vector space and $f : X \to X$ be a continuous mapping. Then there exists $x \in X$ such that $f(x) = x$.*

In order to establish the main results, we give the following definitions.

Definition 3.2. Let K be a nonempty, closed and convex subset of a real Banach space X and Y be a real Banach space. Let $P : K \to 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with $\operatorname{int} P(x) \neq \emptyset$. Let $\eta : X \times X \to X$, $f : K \times K \to Y$ and $A : L(X, Y) \to L(X, Y)$ be three mappings and $V : K \to 2^Y$ and $H : K \times C \to 2^{L(X,Y)}$ be two set-valued mappings. Let Ω be a nonempty subset of K.

(i) H and V are said to be *η-f-complete semicontinuous* on Ω, if for each $y \in \Omega$,

$$\{x \in \Omega : \langle A\xi, \eta(y, x)\rangle + f(y, x) \leq_{\operatorname{int} P} 0, \ \ \forall z \in V(x), \ \xi \in H(x, z)\}$$

is open in Ω with respect to the weak topology of X;

(ii) H and V are said to be *η-f-complete strong semicontinuous* on Ω, if for each $y \in \Omega$,

$$\{x \in \Omega : \langle A\xi, \eta(y, x)\rangle + f(y, x) \leq_{\operatorname{int} P} 0, \ \ \forall z \in V(x), \ \xi \in H(x, z)\}$$

is open in Ω with respect to the norm topology of X.

Definition 3.3. Let X and Y be two Banach spaces and let $P : K \to 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let K be a nonempty, convex subset of X and $f : K \times K \to Y$ be a bi-function. A function $\eta : X \times X \to X$ is said to be $L(X,Y)$-*diagonally convex with respect to* f, if for each $\xi \in L(X,Y)$ and finite subset $\{y_1, .., y_n\} \subseteq K$ and $x = \sum_{i=1}^n t_i y_i$ where $\sum_{i=1}^n t_i = 1$, one has

$$\sum_{i=1}^{n} t_i[\langle \xi, \eta(y_i, x)\rangle \ + f(y_i, x)] \not\leq_{\text{int}\, P(x)} 0, \quad \forall x \in K.$$

Remark 3.4. *If $f \equiv 0$, then the above Definition 3.3 reduces to the concept of $L(X,Y)$-diagonal convexity introduced by Chadli et al. [6].*

Example 3.5. *Let X=$\mathbb{R}$, Y=$\mathbb{R}^2$, K=C=$\mathbb{R}_+$ and $P(x)$=$\mathbb{R}^2_+$, $\forall x \in \mathbb{R}_+$. Let $V \equiv I$, identity mapping and the mappings $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as $\eta(y,x) = x(y-x)$, $H : \mathbb{R}_+ \times \mathbb{R}_+ \to 2^{L(\mathbb{R},\mathbb{R}^2)}$, $\xi \in H(x,y)$, where*

$$H(x,y) = \begin{pmatrix} x(y-x)^3 \\ x^2+1 \end{pmatrix}$$

and $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^2$ defined as

$$f(y,x) = \begin{pmatrix} x^2(y-x)^4 \\ x(x-y) \end{pmatrix}, \quad \forall x, y \in \mathbb{R}_+.$$

For any finite subset $\{y_1, .., y_n\} \subseteq \mathbb{R}_+$ and $x = \sum_{i=1}^n t_i y_i$ where $\sum_{i=1}^n t_i = 1$, it follows that

$$\begin{aligned}\sum_{i=1}^{n} t_i[\langle H(x,y_i), \eta(y_i,x)\rangle + f(y_i,x)] &= \sum_{i=1}^{n} t_i \begin{pmatrix} 2x^2(y_i-x)^4 \\ x^3(y_i-x) \end{pmatrix} \\ &= \begin{pmatrix} 2x^2 \sum\limits_{i=1}^{n} t_i(y_i-x)^4 \\ x^3(\sum\limits_{i=1}^{n} t_i y_i - x) \end{pmatrix} \\ &= \begin{pmatrix} 2x^2 \sum\limits_{i=1}^{n} t_i(y_i-x)^4 \\ 0 \end{pmatrix} \not\leq_{int \mathbb{R}^2_+} 0.\end{aligned}$$

This shows that η is $L(X,Y)$-diagonally convex with respect to f.

Next, we give the existence result for GVMVLIP (2.1).

Theorem 3.6. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and Y be a real Banach space. Let $P : K \to 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $\eta : X \times X \to X$, $f : K \times K \to Y$ be two bi-mappings and $A : L(X,Y) \to L(X,Y)$ be a mapping. If $V : K \to 2^Y$ and $H : K \times C \to 2^{L(X,Y)}$ are two set-valued mappings. Suppose the following conditions hold:*

(i) $f : K \times K \to Y$ *is* P_-*-convex in the first argument with the condition* $f(x,y) + f(y,x) = 0,\ \forall x, y \in K$*;*

(ii) for each $x \in K$*, there exists* $z \in V(x)$ *and* $\xi \in H(x,z)$ *such that* $\langle A\xi, \eta(x,x)\rangle = 0$*;*

(iii) for each $(\xi, y) \in L(X,Y) \times K$ *fixed,* $\langle A\xi, \eta(.,y)\rangle : K \to Y$ *is* P_-*-convex;*

(iv) H *and* V *are* η*-*f*-complete semicontinuous mappings on* K*.*

Then there exist $\bar{x} \in K$*,* $\bar{z} \in V(\bar{x})$ *and* $\bar{\xi} \in H(\bar{x}, \bar{z})$ *such that*

$$\langle A\bar{\xi}, \eta(y,\bar{x})\rangle + f(y,\bar{x}) \not\leq_{int\, P(\bar{x})} 0, \quad \forall y \in K.$$

Proof. Suppose on contrary that the conclusion is not true. Then for each $x \in K$ there exists some $y_0 \in K$ such that

$$\langle A\xi, \eta(y_0,x)\rangle + f(y_0,x) \leq_{\operatorname{int} P(x)} 0, \tag{3.1}$$

for all $z \in V(x)$ and $\xi \in H(x,z)$. For every $y \in K$, define the set N_y as follows:

$$N_y = \{x \in K : \langle A\xi, \eta(y,x)\rangle + f(y,x) \leq_{\operatorname{int} P(x)} 0,\ \forall z \in V(x),\ \xi \in H(x,z)\}. \tag{3.2}$$

Since H and V are η-f-completely semicontinuous on K, the set N_y is open in K with respect to the weak topology of X for every $y \in K$.

Now we assert that $\{N_y : y \in K\}$ is an open cover of K with respect to the weak topology of X. Indeed, first it is easy to see that

$$\bigcup_{y\in K} N_y \subseteq K.$$

Second, for each $x \in K$, by inclusion (3.1) there exists $y_0 \in K$ such that $x \in N_{y_0}$. Hence $x \in \bigcup_{y\in K} N_y$. This shows that $K \subseteq \bigcup_{y\in K} N_y$. Consequently,

$$K = \bigcup_{y\in K} N_y.$$

So the assertion is valid.

Now the weak compactness of K implies that there exists a finite set $\{y_1, ..., y_n\} \subseteq K$ such that

$$K = \bigcup_{i=1}^{n} N_{y_i}.$$

Hence there exists a continuous (with respect to the weak topology of X) partition of unity $\{\beta_1, ..., \beta_n\}$ subordinated to $\{N_{y_1}, ..., N_{y_n}\}$ such that $\beta_j(x) \geq 0,\ \forall x \in K,\ j = 1, ..., n$

$$\sum_{j=1}^{n} \beta_j(x) = 1, \quad \forall x \in K,$$

and

$$\beta_j(x) = \begin{cases} = 0, & \text{whenever} \quad x \notin N_{y_j}, \\ > 0, & \text{whenever} \quad x \in N_{y_j}. \end{cases}$$

Let $p : K \to X$ be defined as

$$p(x) = \sum_{j=1}^{n} \beta_j(x) y_j, \ \forall x \in K. \tag{3.3}$$

Since β_i is continuous with respect to the weak topology of X for each i, p is continuous with respect to the weak topology of X. Let $S = \mathrm{co}\{y_1, ..., y_n\} \subseteq K$. Then S is a simplex of a finite dimensional vector space and p maps S into S. By Brouwer's fixed point theorem [15], there exists some $x_0 \in S$ such that $p(x_0) = x_0$. Now for any given $x \in K$, let

$$k(x) = \{j : x \in N_{y_j}\} = \{j : \beta_j(x) > 0\}.$$

Obviously, $k(x) \neq \emptyset$. Since $x_0 \in S \subseteq K$ is a fixed point of p, we have $x_0 = p(x_0) = \sum_{j=1}^{n} \beta_j(x_0) y_j$. From inclusion (3.2) and the assumptions (i)-(iii), we have

$$\begin{aligned} 0 &= \langle A\xi_0, \eta(x_0, x_0)\rangle + f(x_0, x_0) \\ &= \langle A\xi_0, \eta(p(x_0), x_0)\rangle + f(p(x_0), x_0) \\ &= \left\langle A\xi_0, \eta\left(\sum_{j=1}^{n} \beta_j(x_0) y_j, x_0\right)\right\rangle + f\left(\sum_{j=1}^{n} \beta_j(x_0) y_j, x_0\right) \\ &\leq_{P_-} \sum_{j=1}^{n} \beta_j(x_0)[\langle A\xi_0, \eta(y_j, x_0)\rangle + f(y_j, x_0)] \leq_{\mathrm{int}\, P(x_0)} 0. \end{aligned}$$

which leads to a contradiction since $P_- = \bigcap_{x_0 \in K} P(x_0)$ is a proper cone. Therefore there exist $\bar{x} \in K$, $\bar{z} \in V(\bar{x})$ and $\bar{\xi} \in H(\bar{x}, \bar{z})$ such that

$$\langle A\bar{\xi}, \eta(y, \bar{x})\rangle + f(y, \bar{x}) \not\leq_{\mathrm{int}\, P(\bar{x})} 0, \quad \forall y \in K.$$

This completes the proof. □

If $X = \mathbb{R}^n$, then η-f-complete semicontinuity is equivalent to η-f-strong semicontinuity. Also a bounded and closed subset is equivalent to a compact subset. By Theorem 3.6, we can obtain the following results:

Corollary 3.7. *Let K be a nonempty, compact and convex subset of $\mathbb{R}^n$ and Y be a real Banach space. Let $P : K \to 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $f : K \times K \to Y$ be two bi-mappings and $A : L(\mathbb{R}^n, Y) \to L(\mathbb{R}^n, Y)$ be a mapping. If $V : K \to 2^Y$ and $H : K \times C \to 2^{L(\mathbb{R}^n, Y)}$ are two set-valued mappings. Suppose the following conditions hold:*

(i) $f : K \times K \to Y$ *is* P_-*-convex in the first argument with the condition* $f(x,y) + f(y,x) = 0,\ \forall x, y \in K$*;*

(ii) *for each* $x \in K$*, there exists* $z \in V(x)$ *and* $\xi \in H(x,z)$ *such that* $\langle A\xi, \eta(x,x)\rangle = 0$*;*

(iii) *for each* $(\xi, y) \in L(\mathbb{R}^n, Y) \times K$ *fixed,* $\langle A\xi, \eta(.,y)\rangle : K \to Y$ *is* P_-*-convex;*

(iv) H *and* V *are* η*-*f*-strong semicontinuous mappings on* K*.*

Then there exist $\bar{x} \in K,\ \bar{z} \in V(\bar{x})$ *and* $\bar{\xi} \in H(\bar{x}, \bar{z})$ *such that*

$$\langle A\bar{\xi}, \eta(y, \bar{x})\rangle + f(y, \bar{x}) \nleq_{int\,P(\bar{x})} 0, \quad \forall y \in K.$$

Theorem 3.8. *Let* K *be a nonempty, closed and convex subset of a real reflexive Banach space* X *with* $0 \in K$ *and* Y *be a real Banach space. Let* $P : K \to 2^Y$ *be such that for each* $x \in K, P(x)$ *is a proper, closed, convex cone with int* $P(x) \neq \emptyset$*. Let* $\eta : X \times X \to X,\ f : K \times K \to Y$ *be two bi-mappings and* $A : L(X,Y) \to L(X,Y)$ *be a mapping. If* $V : K \to 2^Y$ *and* $H : K \times C \to 2^{L(X,Y)}$ *are two set-valued mappings. Suppose the following conditions hold:*

(i) $f : K \times K \to Y$ *is* P_-*-convex in the first argument with the condition* $f(x,y) + f(y,x) = 0,\ \forall x, y \in K$*;*

(ii) *for each* $x \in K$*, there exist* $z \in V(x)$ *and* $\xi \in H(x,z)$ *such that* $\langle A\xi, \eta(x,x)\rangle = 0$*;*

(iii) *for each* $(\xi, y) \in L(X,Y) \times K$ *fixed,* $\langle A\xi, \eta(.,y)\rangle : K \to Y$ *is* P_-*-convex;*

(iv) *there exists some* $r > 0$ *such that* **(a)** H *and* V *are* η*-*f*-complete semicontinuous mappings on* K_r*, where* $K_r = \{x \in K : \|x\| \le r\}$*, and*

(b) $\langle A\xi, \eta(0,x)\rangle + f(0,x) \le_{int\,P(x)} 0, \forall z \in V(x),\ \xi \in H(x,z)$ and $x \in K$ with $\|x\| = r$. (3.4)

Then there exists $\bar{x} \in K,\ \bar{z} \in V(\bar{x})$ *and* $\bar{\xi} \in H(\bar{x}, \bar{z})$ *such that*

$$\langle A\bar{\xi}, \eta(y, \bar{x})\rangle + f(y, \bar{x}) \nleq_{int\,P(\bar{x})} 0, \quad \forall y \in K.$$

Proof. One can readily see that all conditions of Theorem 3.6 are fulfilled for nonempty, bounded, closed and convex subset $K_r = K \cap B_r$, where $B_r = \{x \in X : \|x\| \le r\}$. Thus according to Theorem 3.1, there exist $x_r \in K_r, z_r \in V(x_r)$ and $\xi_r \in H(x_r, z_r)$ such that

$$\langle A\xi_r, \eta(v, x_r)\rangle + f(v, x_r) \nleq_{\text{int}\,P(x_r)} 0, \quad \forall v \in K_r. \tag{3.5}$$

Putting $v = 0$ in the above inclusion (3.5), one has

$$\langle A\xi_r, \eta(0, x_r)\rangle + f(0, x_r) \nleq_{\text{int}\,P(x_r)} 0. \tag{3.6}$$

Combining (3.4) with (3.6), we know that $\|x_r\| < r$. For any $y \in K$, choose $t \in (0,1)$ small enough such that $(1-t)x_r + ty \in K_r$. Putting $v = (1-t)x_r + ty$ in (3.5), one has

$$\langle A\xi_r, \eta((1-t)x_r + ty, x_r)\rangle + f((1-t)x_r + ty, x_r) \not\leq_{\operatorname{int} P(x_r)} 0.$$

Since the mappings $f(., x_r)$ and $\langle A\xi_r, \eta(., x_r)\rangle$ are P_--convex in the first argument, we have

$$\langle A\xi_r, \eta((1-t)x_r + ty, x_r)\rangle + f((1-t)x_r + ty, x_r) \leq_{P_-} t[\langle A\xi_r, \eta(y, x_r)\rangle + f(y, x_r)].$$

Consequently, we have

$$\langle A\xi_r, \eta(y, x_r)\rangle + f(y, x_r) \not\leq_{\operatorname{int} P(x_r)} 0, \quad \forall y \in K.$$

This completes the proof. □

Theorem 3.9. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and Y be a real Banach space. Let $P : K \to 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $\eta : X \times X \to X$, $f : K \times K \to Y$ be two bi-mappings and $A : L(X,Y) \to L(X,Y)$ be a mapping. If $V : K \to 2^Y$ and $H : K \times C \to 2^{L(X,Y)}$ are two upper semicontinuous mappings with compact convex values. Suppose the following conditions hold:*

- *(i) for each $y \in K$ fixed, $x \mapsto f(y,x)$ is continuous on the convex hull of each finite subset of K with the condition $f(x,y) + f(y,x) = 0, \ \forall x, y \in K$;*
- *(ii) for each $y \in K$ fixed, $x \mapsto \eta(y,x)$ is continuous on the convex hull of each finite subset of K;*
- *(iii) for each $x \in K$, there exist $z \in V(x)$ and $\xi \in H(x,z)$ such that $\langle A\xi, \eta(x,x)\rangle = 0$;*
- *(iv) η is $L(X,Y)$-diagonally convex with respect to f;*
- *(v) A is continuous mapping;*
- *(vi) H and V are η-f-complete semicontinuous mappings on K.*

Then there exist $\bar{x} \in K, \ \bar{z} \in V(\bar{x})$ and $\bar{\xi} \in H(\bar{x}, \bar{z})$ such that

$$\langle A\bar{\xi}, \eta(y, \bar{x})\rangle + f(y, \bar{x}) \not\leq_{int\, P(\bar{x})} 0, \quad \forall y \in K.$$

Proof. Following the same idea of proof as in Theorem 3.6, suppose on contrary that the conclusion is not true. Then for each $x \in K$, there exists some $y_0 \in K$ such that

$$\langle A\xi, \eta(y_0, x)\rangle + f(y_0, x) \leq_{\operatorname{int} P(x)} 0,$$

for all $z \in V(x)$ and $\xi \in H(x,z)$. For every $y \in K$, define the set N_y as follows:

$$N_y = \{x \in K : \langle A\xi, \eta(y,x)\rangle + f(y,x) \leq_{\operatorname{int} P(x)} 0, \ \ \forall z \in V(x), \ \xi \in H(x,z)\}.$$

Utilizing the η-f-complete semicontinuity of H and V on K, we can show that the set $\{N_y : y \in K\}$ is open cover of K with respect to the weak topology of X. From the weak compactness of K, it follows that there exists a finite set $\{y_1, ..., y_n\} \subseteq K$ such that

$$K = \bigcup_{i=1}^{n} N_{y_i}.$$

Let $B = \mathrm{co}\{y_1, ..., y_n\}$. Then we consider the following set-valued mapping $F : B \to 2^B$ defined as:

$$F_y = \{x \in B : \exists z \in V(x),\ \xi \in H(x,z) \text{ such that } \langle A\xi, \eta(y,x)\rangle + f(y,x) \leq_{\mathrm{int}\, P(x)} 0\},$$

$\forall y \in B$. From assumption (iii), one has $F(y) \neq \emptyset$ since $y \in F(y)$. Now we claim that for each $y \in B, F(y) \subseteq B$ is closed in the norm topology of X. Indeed, suppose $x \in \overline{F(y)}$, the closure of $F(y)$ in the norm topology of X. Since B is compact, there is a sequence $\{x_n\} \subseteq F(y)$ such that $x_n \to x \in B$. Hence, there exists $z_n \in V(x_n)$ and $\xi_n \in H(x_n, z_n)$ such that

$$\langle A\xi_n, \eta(y, x_n)\rangle + f(y, x_n) \not\leq_{\mathrm{int}\, P(x_n)} 0,\ \forall n,$$

which implies that

$$\langle A\xi_n, \eta(y, x_n)\rangle + f(y, x_n) \in Y \backslash\text{-int}\, P(x_n),\ \forall n.$$

Since V is upper semicontinuous with compact values, $V(B)$ is compact. Therefore, without loss of generality one deduces that $z_n \to z \in V(x)$. On the other hand, since H is upper semicontinuous with compact values, $H(B, V(B))$ is compact. It follows without loss of generality that $\xi_n \to \xi \in H(x, z)$. Note that f is continuous on the convex hull of each finite subset of K. Since A is continuous, it follows that $A\xi_n \to A\xi$ as $n \to \infty$. Thus from assumptions (i) and (v), we derive

$$\begin{aligned}
&\|[\langle A\xi_n, \eta(y, x_n)\rangle + f(y, x_n)] - [\langle A\xi, \eta(y, x)\rangle + f(y, x)]\| \\
&\qquad \leq \|\langle A\xi_n, \eta(y, x_n)\rangle - \langle A\xi_n, \eta(y, x)\rangle\| + \|\langle A\xi_n, \eta(y, x)\rangle - \langle A\xi, \eta(y, x)\rangle\| \\
&\qquad\quad + \|f(y, x_n) - f(y, x)\| \\
&\qquad = \|\langle A\xi_n, \eta(y, x_n) - \eta(y, x)\rangle\| + \|\langle A\xi_n - A\xi, \eta(y, x)\rangle\| \\
&\qquad\quad + \|f(y, x_n) - f(y, x)\| \\
&\qquad \leq \|A\xi_n\|\|\eta(y, x_n) - \eta(y, x)\| + \|A\xi_n - A\xi\|\|\eta(y, x)\| \\
&\qquad\quad + \|f(y, x_n) - f(y, x)\| \to 0 \text{ as } n \to \infty,
\end{aligned}$$

that is as $n \to \infty$

$$\langle A\xi_n, \eta(y, x_n)\rangle + f(y, x_n) \to \langle A\xi, \eta(y, x)\rangle + f(y, x).$$

Since $Y \backslash$(-int $P(x_n)$) is closed, we have

$$\langle A\xi, \eta(y, x)\rangle + f(y, x) \in Y \backslash(\text{-int}\, P(x)).$$

Consequently, we get

$$\langle A\xi, \eta(y,x)\rangle + f(y,x) \not\leq_{\mathrm{int}\, P(x)} 0,$$

and so $x \in F(y)$. This shows that $F(y)$ is closed for each $y \in B$. Since B is a compact subset of $X, F(y)$ is also compact for each $y \in B$.

Next we claim that $F : B \to 2^B$ is a KKM mapping. Indeed, let $\{v_1, ..., v_m\} \subseteq B$ and let us verify that $\mathrm{co}\{v_1, ..., v_m\} \subseteq \bigcup_{i=1}^m F(v_i)$. Let $u \in \mathrm{co}\{v_1, ..., v_m\}$, $u = \sum_{i=1}^m t_i v_i$ with $t_i \geq 0$ and $\sum_{i=1}^m t_i = 1$. From assumption (iv) one has

$$\sum_{i=1}^m t_i [\langle \xi, \eta(v_i, u)\rangle + f(v_i, u) \notin \text{-int}\, P(u).$$

Therefore there exists $i \in \{1, ..., m\}$ such that

$$\langle \xi, \eta(v_i, u)\rangle + f(v_i, u) \notin \text{-int}\, P(u).$$

Hence $u \in F(v_i) \subseteq \bigcup_{j=1}^m F(v_j)$. Consequently, from Ky Fan's Lemma 2.4, we conclude that

$$\bigcap_{y \in B} F(y) \neq \emptyset,$$

and hence we may choose

$$\hat{x} \in \bigcap_{y \in B} F(y) \subseteq \bigcup_{i=1}^n F(y_i).$$

This implies that for each $i = 1, ..., n$ there exists $z_i \in V(\hat{x})$ and $\xi_i \in H(\hat{x}, z_i)$ such that

$$\langle A\xi_i, \eta(y_i, \hat{x})\rangle + f(y_i, \hat{x}) \not\leq_{\mathrm{int}\, P(\hat{x})} 0. \tag{3.7}$$

On the other hand, since

$$\hat{x} \in K = \bigcup_{i=1}^n N_{y_i},$$

so there exists some $i = \{1, ..., n\}$ such that $\hat{x} \in N_{y_i}$, that is,

$$\langle A\xi, \eta(y_i, \hat{x})\rangle + f(y_i, \hat{x}) \leq_{\mathrm{int}\, P(\hat{x})} 0,$$

for all $z \in V(\hat{x})$ and $\xi \in H(\hat{x}, z)$, which contradicts (3.7). Thus, the conclusion of the theorem follows. This completes the proof. □

If $X = \mathbb{R}^n$, then η-f-complete semicontinuity is equivalent to η-f-strong semicontinuity. Also a bounded and closed subset is equivalent to a compact subset. By Theorem 3.9, we can obtain the following results:

Corollary 3.10. *Let K be a nonempty, compact and convex subset of $\mathbb{R}^n$ and Y be a real Banach space. Let $P : K \to 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $f : K \times K \to Y$ be two bi-mappings and $A : L(\mathbb{R}^n, Y) \to L(\mathbb{R}^n, Y)$ be a mapping. If $V : K \to 2^Y$ and $H : K \times C \to 2^{L(\mathbb{R}^n,Y)}$ are two upper semicontinuous mappings with compact convex values. Suppose the following conditions hold:*

- *(i) for each $y \in K$ fixed, $x \mapsto f(y,x)$ is continuous on the convex hull of each finite subset of K with the condition $f(x,y) + f(y,x) = 0,\ \forall x, y \in K$;*
- *(ii) for each $y \in K$ fixed, $x \mapsto \eta(y,x)$ is continuous on the convex hull of each finite subset of K;*
- *(iii) for each $x \in K$, there exists $z \in V(x)$ and $\xi \in H(x,z)$ such that $\langle A\xi, \eta(x,x)\rangle = 0$;*
- *(iv) η is $L(\mathbb{R}^n, Y)$-diagonally convex with respect to f;*
- *(v) A is continuous mapping;*
- *(vi) H and V are η-f-strong semicontinuous mappings on K.*

Then there exist $\bar{x} \in K,\ \bar{z} \in V(\bar{x})$ and $\bar{\xi} \in H(\bar{x}, \bar{z})$ such that

$$\langle A\bar{\xi}, \eta(y,\bar{x})\rangle + f(y,\bar{x}) \not\leq_{int\,P(\bar{x})} 0, \quad \forall y \in K.$$

Theorem 3.11. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X with $0 \in K$ and Y be a real Banach space. Let $P : K \to 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $\eta : X \times X \to X$, $f : K \times K \to Y$ be two bi-mappings and $A : L(X,Y) \to L(X,Y)$ be a mapping. If $V : K \to 2^Y$ and $H : K \times C \to 2^{L(X,Y)}$ are two upper semicontinuous mappings with compact convex values. Suppose the following conditions hold:*

- *(i) for each $y \in K$ fixed, $x \mapsto f(y,x)$ is continuous on the convex hull of each finite subset of K with the condition $f(x,y) + f(y,x) = 0,\ \forall x, y \in K$;*
- *(ii) for each $y \in K$ fixed, $x \mapsto \eta(y,x)$ is continuous on the convex hull of each finite subset of K;*
- *(iii) for each $x \in K$, there exists $z \in V(x)$ and $\xi \in H(x,z)$ such that $\langle A\xi, \eta(x,x)\rangle = 0$;*
- *(iv) η is $L(X,Y)$-diagonally convex with respect to f;*
- *(v) A is continuous mapping;*
- *(vi) there exists some $r > 0$ such that* **(a)** *H and V are η-f-complete semicontinuous mappings on K_r, where $K_r = \{x \in K : \|x\| \leq r\}$, and*
- **(b)** $\langle A\xi, \eta(0,x)\rangle + f(0,x) \leq_{int\,P(x)} 0,\ \forall z \in V(x),\ \xi \in H(x,z)$ and $x \in K$ with $\|x\| = r$.

Then there exists $\bar{x} \in K,\ \bar{z} \in V(\bar{x})$ and $\bar{\xi} \in H(\bar{x}, \bar{z})$ such that

$$\langle A\bar{\xi}, \eta(y,\bar{x})\rangle + f(y,\bar{x}) \not\leq_{int\,P(\bar{x})} 0, \quad \forall y \in K.$$

Proof. One can readily see that all conditions of Theorem 3.9 are fulfilled for nonempty, bounded, closed and convex subset $K_r = K \cap B_r$, where $B_r = \{x \in X : \|x\| \leq r\}$. Thus according to Theorem 3.9, there exist $x_r \in K_r, z_r \in V(x_r)$ and $\xi_r \in H(x_r, z_r)$ such that

$$\langle A\xi_r, \eta(v, x_r)\rangle + f(v, x_r) \not\leq_{\mathrm{int}\, P(x_r)} 0, \quad \forall v \in K_r.$$

The remainder of the proof is same as in Theorem 3.8. Thus, we omit it. This completes the proof. □

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