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Geraghty's Fixed Point Theorem for Special Multi-Valued Mappings

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Abstract : In this paper, we prove a generalization of Geraghty's fixed point theorem for a type of multi-valued map that called special multi-valued map.

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1 Introduction

Many fixed point theorems have been proved by various authors as generalizations to Banach's contraction principle (see for example [1–7]). One such generalization is due to Geraghty [8] as follows.

Theorem 1.1. *Let (X, d) be a complete metric space, let $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$d(f(x), f(y)) \leq \alpha(d(x, y))\ d(x, y)$$

where $\alpha \in S$, that S is the families of functions from $[0, \infty)$ into $[0, 1)$ which satisfy the simple condition $\alpha(t_n) \to 1 \Longrightarrow t_n \to 0$. Then f has a fixed point $z \in X$, and $\{f^n(x)\}$ converges to z, for each $x \in X$.

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Let (X, d) be a metric space. Let $CB(X)$ denotes the collection of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$ and $x \in X$, define $D(x, A) := \inf\{d(x, a); a \in A\}$ and

$$H_d(A, B) := \max\left\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\right\}.$$

It is easy to see that H_d is a metric on $CB(X)$. H_d is called the Hausdorff metric induced by d. Note that a point $p \in X$ is said to be a fixed point of a multi–valued mapping $T : X \to CB(X)$ if $p \in T(p)$ [9].

The fixed point theory of multi–valued contractions was initiated by Nadler [9] as follows.

Theorem 1.2. *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that for all $x, y \in X$,*

$$H_d(Tx, Ty) \le r\ d(x, y) \tag{1.1}$$

where $0 \le r < 1$. Then T has a fixed point.

This theory was developed in different directions by many authors. See for example [10–15]. In this paper, we prove a version of Geraghty's fixed point theorem for multi–valued mappings.

Throughout this paper, we assume that (X, d) is a complete metric space and H_d is the Hausdorff metric on $CB(X)$ induced by d.

2 Main Results

In this section we have attempted to generalize a fixed point theorem of Geraghty for multi–valued mappings. For this purpose we introduce a notion called special multi-valued map and for this type of multi-valued map we have obtained a fixed point theorem.

Definition 2.1. Let (X, d) be a a metric space, mapping T from X into $CB(X)$ is called *special multi–valued* if

$$\inf_{y\in Tx}\{d(x, y) + d(y, z)\} = D(x, Tx) + D(z, Tx), \tag{2.1}$$

for all $x, z \in X$.

It is clear that every single valued mapping is special multi–valued mapping, also there exist some mappings that are special multi–valued but not single valued.

Example 2.1. *Let $X = \{\frac{1}{2}, \frac{1}{4}, ..., \frac{1}{2^n}, ...\} \bigcup \{0, 1\}$, $d(x, y) = \begin{cases} 1 & x \ne y, \\ 0 & x = y. \end{cases}$*

Define mapping $F : X \to CB(X)$ *as*

$$F(x) = \begin{cases} \{\frac{1}{2^{n+1}}\} & x = \frac{1}{2^n}, \quad n = 1, 2, ..., \\ \{0\} & x = 0, \\ \{0, \frac{1}{2}\} & x = 1. \end{cases}$$

It is clear that above example is special multi- valued but not single valued. Now we prove our main result in this paper.

Theorem 2.2. *Let* T *be special multi–valued mapping that*

$$H_d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + \beta(d(x, y))[D(x, Tx) + D(y, Ty)] + \gamma(d(x, y))[D(x, Ty) + D(y, Tx)]$$

for all $x, y \in X$*, where* α, β, γ *are mappings from* $[0, \infty)$ *into* $[0, 1)$ *such that* $\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} \in S$ *and* $\beta(t) \geq \gamma(t)$ *for all* $t \in [0, \infty)$*. Then* T *has a fixed point.*

Proof. Define a function $\alpha^{'}$ from $[0, \infty)$ into $[0, 1)$ by $\alpha^{'}(t) = \frac{\alpha(t)+1-2\beta(t)-2\gamma(t)}{2}$ for all $t \in [0, \infty)$. Then we have

1) $\alpha(t) < \alpha^{'}(t)$ for all $t \in [0, \infty)$,
2) $\frac{\alpha^{'}+\beta+\gamma}{1-(\beta+\gamma)} \in S$,
3) for $x, y \in X$ and $u \in Tx$, there exists $\nu \in Ty$ such that

$$d(\nu, u) \leq \alpha^{'}(d(x, y))d(x, y) + \beta(d(x, y))[D(x, Tx) + D(y, Ty)] + \gamma(d(x, y))[D(x, Ty) + D(y, Tx)].$$

Putting $u = y$ in 3), we obtain that:

4) For $x \in X$ and $y \in Tx$ there exists $\nu \in Ty$ such that

$$d(\nu, y) \leq \alpha^{'}(d(x, y))d(x, y) + \beta(d(x, y))[D(x, Tx) + D(y, Ty)] + \gamma(d(x, y))[D(x, Ty) + D(y, Tx)].$$

Hence, we can define a sequence $\{x_n\}_{n\in\mathbb{N}}$ which satisfies $x_{n+1} \in Tx_n, x_{n+1} \neq x_n$ and

$$d(x_{n+2}, x_{n+1}) \leq \alpha^{'}(d(x_{n+1}, x_n))d(x_{n+1}, x_n) + \beta(d(x_{n+1}, x_n))[D(x_n, Tx_n) + D(x_{n+1}, Tx_{n+1})] + \gamma(d(x_{n+1}, x_n))[D(x_n, Tx_{n+1}) + D(x_{n+1}, Tx_n)]$$

for all $n \in \mathbb{N}$. It follows that

$$d(x_{n+2}, x_{n+1}) \leq \frac{\alpha^{'}(d(x_{n+1}, x_n)) + \beta(d(x_{n+1}, x_n)) + \gamma(d(x_{n+1}, x_n))}{1 - (\beta(d(x_{n+1}, x_n)) + \gamma(d(x_{n+1}, x_n)))}d(x_{n+1}, x_n)$$

for all $n \in \mathbb{N}$. We show that $\{x_n\}$ is a Cauchy sequence. To this end, we break the argument into two Steps.

Step 1: $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.
Since $\frac{\alpha'(t)+\beta(t)+\gamma(t)}{1-(\beta(t)+\gamma(t))} < 1$ for all t, $\{d(x_n, x_{n+1})\}$ is decreasing and bounded below, so

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r \geq 0.$$

Assume $r > 0$. Then we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \frac{\alpha'(d(x_n, x_{n+1})) + \beta(d(x_n, x_{n+1})) + \gamma(d(x_n, x_{n+1}))}{1 - (\beta(d(x_n, x_{n+1})) + \gamma(d(x_n, x_{n+1})))}, \quad n = 1, 2, ...$$

By letting $n \to \infty$, we see that

$$1 \leq \lim_{n\to\infty} \frac{\alpha'(d(x_n, x_{n+1})) + \beta(d(x_n, x_{n+1})) + \gamma(d(x_n, x_{n+1}))}{1 - (\beta(d(x_n, x_{n+1})) + \gamma(d(x_n, x_{n+1})))}.$$

On the other hand, we have $\frac{\alpha'+\beta+\gamma}{1-(\beta+\gamma)} \in S$. Therefore $r = 0$. This is a contradiction, hence, we prove Step 1.

Step 2: $\{x_n\}$ is a Cauchy sequence.
Assume $\limsup_{n,m\to\infty} d(x_n, x_m) > 0$. By triangle inequality for positive real numbers n, m and for $y \in Tx_m$, we obtain $d(x_n, x_m) \leq d(x_n, y) + d(y, x_m)$. This means that for every positive real numbers m, n, with using of relation (2.1), we have

$$\begin{aligned} d(x_n, x_m) &\leq \inf_{y\in Tx_m} \{d(x_n, y) + d(y, x_m)\} = D(x_m, Tx_m) + D(x_n, Tx_m) \\ &\leq d(x_m, x_{m+1}) + D(x_{m+1}, Tx_m) + d(x_n, x_{n+1}) + D(x_{n+1}, Tx_m) \\ &\leq H_d(Tx_m, Tx_n) + d(x_n, x_{n+1}) + d(x_m, x_{m+1}) \\ &\leq \alpha(d(x_n, x_m))d(x_n, x_m) + \beta(d(x_n, x_m))[D(x_n, Tx_n) + D(x_m, Tx_m)] \\ &+ \gamma(d(x_n, x_m))[D(x_n, Tx_m) + D(x_m, Tx_n)] + d(x_n, x_{n+1}) + d(x_m, x_{m+1}). \end{aligned}$$

Hence,

$$d(x_n, x_m) \leq \frac{(\beta(d(x_n,x_m))+\gamma(d(x_n,x_m)))(d(x_n,x_{n+1})+d(x_m,x_{m+1}))+d(x_n,x_{n+1})+d(x_m,x_{m+1})}{1-(\alpha(d(x_n,x_m))+2\gamma(d(x_n,x_m)))}.$$

Under the assumption $\limsup_{n,m\to\infty} d(x_n, x_m) > 0$, it follows by Step 1, that

$$\limsup_{n,m\to\infty} \frac{1}{1 - (\alpha(d(x_n, x_m)) + 2\gamma(d(x_n, x_m)))} = +\infty$$

for which

$$\limsup_{n,m\to\infty} \alpha(d(x_n, x_m)) + 2\gamma(d(x_n, x_m)) = 1. \tag{2.2}$$

On the other hand, since

$$\frac{\alpha(t) + \beta(t) + \gamma(t)}{1 - (\beta(t) + \gamma(t))} < 1, \tag{2.3}$$

then $\beta(t)+\gamma(t)<\frac{1}{2}$, for all $t\in[0,\infty)$.
Hence, since $\beta(t)\geq\gamma(t)$, for all $t\in[0,\infty)$, by using (2.2) and (2.3)

$$\begin{aligned}\limsup_{n,m\to\infty}&\frac{\alpha(d(x_n,x_m))+\beta(d(x_n,x_m))+\gamma(d(x_n,x_m))}{1-(\beta(d(x_n,x_m))+\gamma(d(x_n,x_m)))}\\&\geq\limsup_{n,m\to\infty}\frac{\alpha(d(x_n,x_m))+2\gamma(d(x_n,x_m))}{1-(\beta(d(x_n,x_m))+\gamma(d(x_n,x_m)))}\\&\geq\limsup_{n,m\to\infty}\alpha(d(x_n,x_m))+2\gamma(d(x_n,x_m))=1.\end{aligned}\tag{2.4}$$

Now since, $\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}\in S$, then by using (2.4), we have

$$\limsup_{n,m\to\infty}\frac{\alpha(d(x_n,x_m))+\beta(d(x_n,x_m))+\gamma(d(x_n,x_m))}{1-(\beta(d(x_n,x_m))+\gamma(d(x_n,x_m)))}=1.$$

It follows that $\limsup_{n,m\to\infty}d(x_n,x_m)=0$ which is a contradiction. Thus, Step 2 is proved.

By completeness of X, there exists $x^*\in X$ such that $\lim_{n\to\infty}x_n=x^*$. Now, we have

$$\begin{aligned}D(x^*,Tx^*)&\leq d(x^*,x_{n+1})+D(x_{n+1},Tx^*)\\&\leq d(x^*,x_{n+1})+H_d(Tx_n,Tx^*)\\&\leq d(x^*,x_{n+1})+\alpha(d(x_n,x^*))d(x_n,x^*)\\&\quad+\beta(d(x_n,x^*))[D(x_n,Tx_n)+D(x^*,Tx^*)]\\&\quad+\gamma(d(x_n,x^*))[D(x_n,Tx^*)+D(x^*,Tx_n)]\end{aligned}$$

for all $n\in\mathbb{N}$. Therefore,

$$\begin{aligned}D(x^*,Tx^*)&\leq d(x^*,x_{n+1})+\alpha(d(x_n,x^*))d(x_n,x^*)\\&\quad+\beta(d(x_n,x^*))[d(x_{n+1},x_n)+D(x^*,Tx^*)]\\&\quad+\gamma(d(x_n,x^*))[D(x_n,Tx^*)+d(x_{n+1},x^*)]\end{aligned}$$

for all $n\in\mathbb{N}$. It follows that

$$\begin{aligned}D(x^*,Tx^*)&\leq\liminf_{n\to\infty}(\beta(d(x_n,x^*))+\gamma(d(x_n,x^*)))D(x^*,Tx^*)\\&=\liminf_{s\to0^+}(\beta(s)+\gamma(s))D(x^*,Tx^*)\\&\leq\limsup_{s\to0^+}(\beta(s)+\gamma(s))D(x^*,Tx^*).\end{aligned}$$

On the other hand, since $\beta(t)+\gamma(t)<\frac{1}{2}$, for all $t\in[0,\infty)$, then we have

$$\limsup_{s\to0^+}(\beta(s)+\gamma(s))<1$$

then $D(x^*,Tx^*)=0$. We know that Tx^* is closed, then $x^*\in Tx^*$. □

Corollary 2.3. *Let T be a mapping from X into X such that*

$$d(Tx,Ty) \leq \alpha(d(x,y))d(x,y) + \beta(d(x,y))[d(x,Tx) + d(y,Ty)] \\ + \gamma(d(x,y))[d(x,Ty) + d(y,Tx)]$$

for all $x, y \in X$, where α, β, γ are mappings from $[0,\infty)$ into $[0,1)$ such that $\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} \in S$ and $\beta(t) \geq \gamma(t)$ for all $t \in [0,\infty)$. Then T has a fixed point.

By Putting $\beta = \gamma = 0$ in Theorem 2.1, since every single valued mapping is special multi–valued mapping, we have the following result, which can be regarded as an extension of Geraghty's fixed point theorem. Indeed, the following corollary is a special multi–valued version of Geraghty's fixed point theorem.

Corollary 2.4. *Let T be special multi–valued mapping, $\alpha \in S$ and let*

$$H_d(Tx,Ty) \leq \alpha(d(x,y))\ d(x,y)$$

for all $x, y \in X$. Then T has a fixed point.

Corollary 2.5. *Let T be special multi–valued mapping and*

$$H_d(Tx,Ty) \leq \beta(d(x,y))[D(x,Tx) + D(y,Ty)]$$

for all $x, y \in X$, where β is a mapping from $[0,\infty)$ into $[0,\frac{1}{2})$ such that $\frac{\beta}{1-\beta} \in S$. Then T has a fixed point.

Corollary 2.6. *Let T be special multi–valued mapping and*

$$H_d(Tx,Ty) \leq \alpha(d(x,y))d(x,y) + \beta(d(x,y))[D(x,Tx) + D(y,Ty)]$$

for all $x, y \in X$, where α, β are mappings from $[0,\infty)$ into $[0,1)$ such that $\frac{\alpha+\beta}{1-\beta} \in S$. Then T has a fixed point.

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