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Tripartite Ramsey Number $r_t(K_{2,4}, K_{2,4})$

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Abstract : A graph G is $n-partite$, $n \geq 1$, if it is possible to partition the set of points $V(G)$ into n subsets $V_1, V_2, \ldots, V_n$ (called partite sets) such that every element of the set of lines $E(G)$ joins a point of V_i to a point of $V_j, i \neq j$. For $n = 2$, and $n = 3$ such graphs are called *bipartite graph*, and *tripartite graph* respectively. A *complete* $n-partite$ *graph* G is an n-partite graph with the added property that if $u \in V_i$ and $v \in V_j, i \neq j$, then the line $uv \in E(G)$. If $|V_i| = p_i$, then this graph is denoted by $K_{p_1,p_2,\ldots,p_n}$.

For the complete tripartite graph $K_{s,s,s}$ with the number of points $p = 3s$, let each line of the graph has either red or blue colour. The smallest number s such that $K_{s,s,s}$ always contains $K_{m,n}$ with all lines of $K_{m,n}$ have one colour (red or blue) is called *tripartite Ramsey number* and denoted by $r_t(K_{m,n}, K_{m,n})$. In this paper, we show that

$$r_t(K_{2,4}, K_{2,4}) = 7.$$

Keywords : Tripartite Ramsey numbers; Bipartite Ramsey numbers; Ramsey numbers; Tripartite graphs; Bipartite graphs.
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1 Introduction

A graph G is $n-partite$, $n \geq 1$, if it is possible to partition the set of points $V(G)$ into n subsets $V_1, V_2, \ldots, V_n$ (called partite sets) such that every element of the set of lines $E(G)$ joins a point of V_i to a point of $V_j, i \neq j$, see [1]. For

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$n = 2$, such graphs are called *bipartite graphs*. For $n = 3$, such graphs are called *tripartite graphs*. A *complete tripartite graph* G is a tripartite graph with partite sets V_1, V_2, V_3 having the added property that if $u \in V_i$ and $v \in V_j, i \neq j$, then $uv \in E(G)$. When $|V_i| = p_i$, we denote the complete n-partite graph by $K_{p_1,p_2,\ldots,p_n}$.

Consider a complete bipartite graph $K_{s,s}$ of order $p = 2s$. Let each line of $K_{s,s}$ be coloured by using either red or blue colour. We shall call such a $K_{s,s}$ as 2-coloured.

Consider a subgraph $K_{m,n}$ of 2-coloured $K_{s,s}$. If all lines of $K_{m,n}$ have red(blue) colour, we shall say that the $K_{s,s}$ contains a red(blue) $K_{m,n}$. The smallest number s of points such that $K_{s,s}$ always contains red $K_{m,n}$ or blue $K_{m,n}$ is called bipartite Ramsey number and denoted by $r_b(K_{m,n}, K_{m,n})$.

According to the definition of bipartite Ramsey number in this paper, Longani [4], has found that $r_b(K_{1,n}, K_{1,n}) = 2n - 1(n = 1, 2, 3, ...)$, $r_b(K_{2,2}, K_{2,2}) = 5$, and $r_b(K_{2,3}, K_{2,3}) = 9$.

Beineke and Schwenk [2] have also found that $r_b(K_{2,2}, K_{2,2}) = 5$ and $r_b(K_{3,3}, K_{3,3}) = 17$.

In this paper, instead of considering a complete bipartite graph $K_{s,s}$, we shall consider a complete tripartite graph $K_{s,s,s}$ of order $p = 3s$. Let each line of $K_{s,s,s}$ be coloured by using either red or blue colour. The smallest number s of points such that $K_{s,s,s}$ always contains red $K_{m,n}$ or blue $K_{m,n}$ is called tripartite Ramsey number and denoted by $r_t(K_{m,n}, K_{m,n})$.

In [3], Leamyoo have found that $r_t(K_{2,2}, K_{2,2}) = 4$.

2 The Value of $r_t(K_{2,4}, K_{2,4})$

We find the value of $r_t(K_{2,4}, K_{2,4})$ by considering a particular 2-coloured $K_{6,6,6}$ and 2-coloured $K_{7,7,7}$. For a $K_{7,7,7}$, consider all ninety eight lines that are adjacent to all points of a V_i. We call such lines as the lines of the V_i.

Lemma 2.1. *Let $K_{7,7,7}$ be a 2-coloured complete tripartite graph with $p = 21$ and each V_1, V_2, and V_3 be the set of seven non-adjacent points of the $K_{7,7,7}$. There exists at least one V_i of which the number of red lines and blue lines of the V_i are not equal.*

Proof. Consider the three V_i's. Suppose there are forty nine red lines and forty nine blue lines of each V_i.

Since there are totally forty nine red lines of V_1, consider when there are n ($n \geq 0$) red lines which join points of V_1 and V_2, and so there are $49 - n$ red lines which join points of V_1 and V_3. Since for V_2 there are also exactly forty nine red lines of V_2, therefore there are $49 - n$ red lines which join points of V_2 and V_3.

Now we can see that there are $(49 - n) + (49 - n)$ red lines of V_3. Since there are exactly forty nine red lines of V_3, therefore

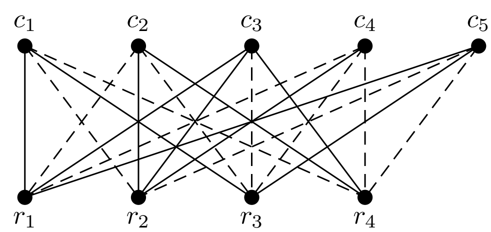
$$\begin{aligned}(49-n)+(49-n) &= 49\\ n &= 24.5.\end{aligned}$$

This is not possible. Therefore, there exist some V_i's of which the number of red lines and blue lines of the V_i's are not equal. □

In order to prove Theorem 2.2 we need to represent 2-colored $K_{m,n}$ with with an $m \times n$ matrix as follows:

Given a 2-colored Km;n with V_1 and V_2 as its partite sets size m and n, respectively. Put $V_1 = \{r_1, r_2, \ldots, r_m\}$ and $V_2 = \{c_1, c_2, \ldots, c_n\}$. Let $B = [b_{ij}]$ be an $m \times n$ matrix where $b_{ij} = 1$ if the line $r_i c_j$ is red, otherwise $b_{ij} = 0$. The following example is to illustrate 2-colored $K_{5,4}$. As in Figure 2.1 (a), we use the dark lines to indicate red lines while dash lines for blue lines.

(a)



(b)

$$\begin{array}{c} \\ r_1\\ r_2\\ r_3\\ r_4\end{array}\begin{array}{c}\begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5\end{array}\\ \begin{bmatrix}1 & 0 & 1 & 0 & 1\\ 0 & 1 & 1 & 1 & 0\\ 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 1 & 0 & 0\end{bmatrix}\end{array}$$

Figure 2.1.

Theorem 2.2. *$r_t(K_{2,4}, K_{2,4}) = 7$.*

Proof. Consider the 2-coloured $K_{6,6,6}$ graph illustrated in Figure 2.2.

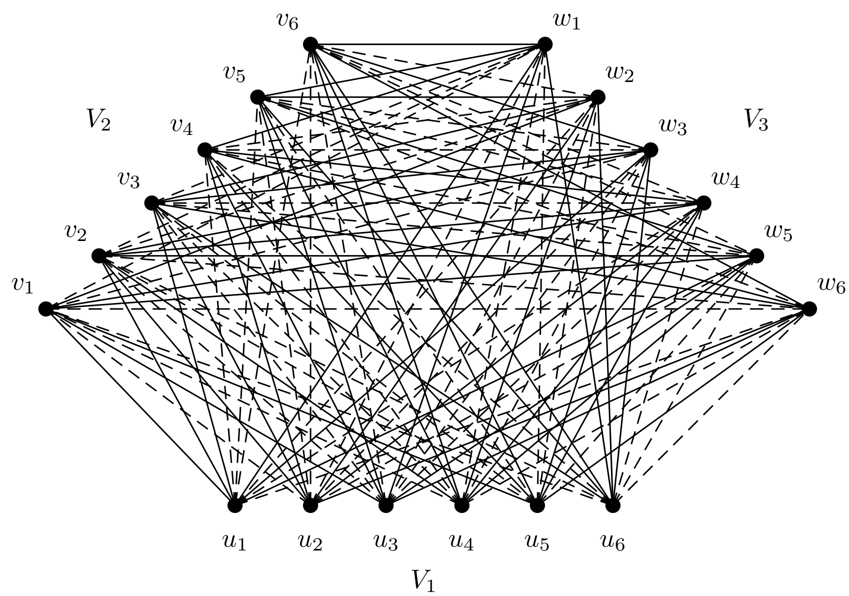


Figure 2.2.

It can be verified that the $K_{6,6,6}$ contains neither red $K_{2,4}$ nor blue $K_{2,4}$. Therefore $r_t(K_{2,4}, K_{2,4}) > 6$. That is

$$r_t(K_{2,4}, K_{2,4}) \geq 7. \tag{2.1}$$

Let $K_{7,7,7}$ be a 2-coloured complete tripartite graph. Consider the set V_1, V_2 and V_3 of seven non-adjacent points of the $K_{7,7,7}$:

$$V_1 = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\},$$
$$V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\},$$
$$V_3 = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}.$$

From Lemma 2.1, we can assume that from V_1, the number of red lines are greater than the number of blue lines, that is the number of red lines are equal to fifty or greater. We only need to consider the case when the number of red lines from V_1 is fifty and show that in such case the $K_{7,7,7}$ always contains red $K_{2,4}$. For the cases when the number of red lines is greater than fifty, the results follow immediately.

Let $V(G_1) = V_1$ and $V(G_2) = V_2 \cup V_3$. For $V(G_1)$, let u_1, u_2, u_3, u_4, u_5, u_6, u_7 be respectively replaced by r_1, r_2, r_3, r_4, r_5, r_6, r_7. Also for $V(G_2)$, let v_1, v_2, v_3, v_4, v_5, v_6, v_7, w_1, w_2, w_3, w_4, w_5, w_6, w_7 be respectively replaced by c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}. That is,

$$V(G_1) = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\},$$
$$V(G_2) = \{c_1, c_2, \ldots, c_{14}\}.$$

By ignoring the lines between V_2 and V_3 and consider the defined $V(G_1)$ and $V(G_2)$ the $K_{7,7,7}$ is now reduced to 2-coloured $K_{7,14}$. In order to prove the theorem we only need to show that this $K_{7,14}$ always contains red $K_{2,4}$.

We find the value of $r_t(K_{2,4}, K_{2,4})$ by considering the 2-coloured $K_{7,14}$.

If there are m, n, s, t, u, v ($1 \leq m, n \leq 7$ and $1 \leq s, t, u, v \leq 14$) such that some submatrices

$$\begin{bmatrix} b_{ms} & b_{mt} & b_{mu} & b_{mv} \\ b_{ns} & b_{nt} & b_{nu} & b_{nv} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{2.2}$$

then the $K_{7,14}$ contains red $K_{2,4}$.

Let d_1, d_2, d_3, d_4, d_5, d_6, d_7 be degrees of red lines of r_1, r_2, r_3, r_4, r_5, r_6, r_7 respectively. We can choose r_i's such that $d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5 \geq d_6 \geq d_7$. Here we have the conditions that

$$d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 = 50$$

and $0 \leq d_i \leq 14$, $i = 1, 2, 3, 4, 5, 6, 7$.

Next, we consider two main cases.

Case 1. $d_1 + d_2 \geq 18$.

Here, the possible $d_1 \geq d_2$ are $9 \geq 9, 10 \geq 8, 10 \geq 9, 10 \geq 10, 11 \geq 7, 11 \geq 8, 11 \geq 9, 11 \geq 10, 11 \geq 11, 12 \geq 7, 12 \geq 8, 12 \geq 9, 12 \geq 10, 12 \geq 11, 12 \geq 12, 13 \geq 7, 13 \geq 8, 13 \geq 9, 13 \geq 10, 13 \geq 11, 13 \geq 12, 13 \geq 13, 14 \geq 7, 14 \geq 8, 14 \geq 9, 14 \geq 10, 14 \geq 11, 14 \geq 12, 14 \geq 13, 14 \geq 14$.

It is easy to show that for all of these cases the $K_{7,14}$ always contains red $K_{2,4}$. For example, consider cases $d_1 = 9$, and $d_2 = 9$. For a case in Table 2.1, parts of the matrix involving r_1 and r_2 would be

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	**1**	**1**	**1**	**1**					
r_2						**1**	**1**	**1**	**1**	1	1	1	1	1

Table 2.1.

from which submatrix of the form (2.2) appears, that is the $K_{7,7,7}$ contains red $K_{2,4}$.

Case 2. $d_1 + d_2 < 18$.

With the conditions for d_i, there are five subcases to consider.

Subcase 2.1. $d_1 = 10, d_2 = 7, d_3 = 7, d_4 = 7, d_5 = 7, d_6 = 6, d_7 = 6$.

In this case we consider three points r_1, r_2, r_3 of $V(G_1)$. When there are four or more points c_i's each of which is joined to both of r_1 and r_2 by red lines, then we can see that the $K_{7,14}$ contains red $K_{2,4}$.

For other cases, suppose that r_1 is joined by ten red lines to $c_1, c_2, c_3, c_4, c_5, c_6$, c_7, c_8, c_9, c_{10} and r_2 is joined by seven red lines to $c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$, we consider seven red lines joining to r_3. Either at least four of seven red lines are joined from r_3 to some points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ or at least four of these seven red lines are joined to some points among $c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$. In either case, we can see that red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.2 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	**1**	**1**	1	1	1	1	1	1				
r_2								1	1	1	1	1	1	1
r_3	**1**	**1**	**1**	**1**								1	1	1

Table 2.2.

Subcase 2.2. $d_1 = 9, d_2 = 8, d_3 = 7, d_4 = 7, d_5 = 7, d_6 = 6, d_7 = 6.$

In this case, as in subcase 2.1, we also need to consider three points r_1, r_2, r_3 of $V(G_1)$. The method in showing that the $K_{7,14}$ always contain red $K_{2,4}$ is almost exactly the same as in the subcase 2.1 above.

Subcase 2.3. $d_1 = 9, d_2 = 7, d_3 = 7, d_4 = 7, d_5 = 7, d_6 = 7, d_7 = 6.$

In this case we consider five points r_1, r_2, r_3, r_4, r_5 of $V(G_1)$. When there are four or more points c_i's each of which is joined to both of r_1 and r_2 by red lines, then we can see that the $K_{7,14}$ contains red $K_{2,4}$.

2.3.1. Two c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by nine red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7$, c_8, c_9 and r_2 is joined by seven red lines to $c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$.

Consider seven red lines joining to r_3. Either at least four of seven red lines are joined from r_3 to some points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least four of these seven red lines are joined to some points among $c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$. In either case, we see that red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.3 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	**1**	**1**	1	1	1	1	1					
r_2								1	1	1	1	1	1	1
r_3	**1**	**1**	**1**	**1**								1	1	1

Table 2.3.

2.3.2. Three c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by nine red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7$, c_8, c_9 and r_2 is joined by seven red lines to $c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}$.

Consider when c_{14} is not joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose c_{14} is not joined to r_3 for example, then either at least four red lines from r_3 are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least four red lines from r_3 are joined to points among $c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}$. In either case, we see that red $K_{2,4}$ is contained in $K_{7,14}$, see Table 2.4 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	**1**	**1**	1	1	1	1	1					
r_2							1	1	1	1	1	1	1	
r_3	**1**	**1**	**1**	**1**							1	1	1	

Table 2.4.

Consider when the point c_{14} is joined to all r_i's $(i = 3, 4, 5, 6, 7)$ by red lines. Here we consider two subcases.

(1) Some $r_i, i = 3, 4, 5, 6, 7$ are joined by red lines to some of c_7, c_8, c_9.

Suppose r_3 is joined to c_7 by red line, for example, then there are five other red lines joining r_3. Either at least three of the red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6$ or at least three of red lines are joined to points among c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}. In either case, red $K_{2,4}$ is formed. For example, see Table 2.5.

Similarly, if r_3 is joined to c_8 or c_9 by red line, we can show that red $K_{2,4}$ is also formed.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	**1**	1	1	1	**1**	1	1					
r_2							1	1	1	1	1	1	1	
r_3	**1**	**1**	**1**				**1**					1	1	1

Table 2.5

(2) None of $r_i, i = 3, 4, 5, 6, 7$ are joined by red lines to c_7, c_8 and c_9.

Since each of $r_i, i = 3, 4, 5, 6, 7$ is joined by red lines to c_{14}, there are six other red lines joining r_i. The six red lines from each r_3 and r_4 will join to points among c_1, c_2, c_3, c_4, c_5, c_6, c_{10}, c_{11}, c_{12}, c_{13}. Therefore, there are at least two points among c_1, c_2, c_3, c_4, c_5, c_6, c_{10}, c_{11}, c_{12}, c_{13} which join r_3 and r_4. First, consider the case when there are at least three points among $c_1, c_2, c_3, c_4, c_5, c_6, c_{10}, c_{11}, c_{12}$, c_{13} which join r_3 and r_4. Also, since c_{14} is joined by red lines to r_3, r_4, we can see that red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.6 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1	1					
r_2							1	1	1	1	1	1	1	
r_3	1	1	**1**								1	**1**	**1**	**1**
r_4			**1**	1	1					1		**1**	**1**	**1**

Table 2.6.

Next, we consider the case when there are two points among c_1, c_2, c_3, c_4, c_5, c_6, c_{10}, c_{11}, c_{12}, c_{13} which join r_3 and r_4. Suppose that these two points are c_{12} and c_{13}. When we consider only r_1, r_2, r_3, r_4 there are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.7 for example. For such cases, we shall consider the seven red lines joining r_5.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1	1					
r_2							1	1	1	1	1	1	1	
r_3	1	1	1								1	1	1	1
r_4				1	1	1				1		1	1	1

Table 2.7.

From Table 2.7, r_3 is joined to $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}, c_{14}$ and r_4 is joined to c_4, c_5, c_6, c_{10}, c_{12}, c_{13}, c_{14} and since r_5 is joined by one of the red lines to c_{14}, there are six other red lines joining r_5. Either at least three red lines are joined to points among c_1, c_2, c_3, c_{11} or at least three red lines are joined to points among $c_4, c_5, c_6, c_{10}, c_{12}, c_{13}$. Assume that there are three red lines which are joined to points among c_1, c_2, c_3, c_{11}, say c_2, c_3, c_{11} or there are three red lines which are joined to points among $c_4, c_5, c_6, c_{10}, c_{12}, c_{13}$, say c_4, c_{10}, c_{12}, since all r_3, r_4, and r_5 are joined to c_{14}. We can verify that in either case the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.8 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1	1					
r_2							1	1	1	1	1	1	1	
r_3	1	1	1								1	1	1	1
r_4				**1**	1	1				**1**		**1**	1	**1**
r_5		1	1	**1**						**1**	1	**1**		**1**

Table 2.8.

Subcase 2.4. $d_1 = 8, d_2 = 8, d_3 = 7, d_4 = 7, d_5 = 7, d_6 = 7, d_7 = 6$.

In this case, as in Subcase 2.3, we also need to consider five points r_1, r_2, r_3, r_4, r_5 of $V(G_1)$. The method in showing that the $K_{7,14}$ always contain red $K_{2,4}$ is almost exactly the same as in the Subcase 2.3 above.

Subcase 2.5. $d_1 = 8, d_2 = 7, d_3 = 7, d_4 = 7, d_5 = 7, d_6 = 7, d_7 = 7$.

When there are four or more c_i's each of which is joined by red lines to both of r_1 and r_2, then we see that the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.9 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	**1**	**1**	**1**	**1**						
r_2					**1**	**1**	**1**	**1**	1	1	1			

Table 2.9.

We consider three more cases.

2.5.1. One of c_i's is joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by eight red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ and r_2 is joined by seven red lines to $c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$.

Consider the seven red lines joining r_3. Either at least four of the seven red lines are joined, from r_3, to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ or at least four red lines are joined to points among $c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$. In either case, we see that $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.10 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	**1**	**1**	1	1	1	1						
r_2								1	1	1	1	1	1	1
r_3	**1**	**1**	**1**	**1**					1	1	1			

Table 2.10.

2.5.2. Two of c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by eight red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ and r_2 is joined by seven red lines to $c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}$. Consider two subcases (1) and (2).

(1) c_{14} is not joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

If c_{14} is not joined to r_3 for example, then either at least four red lines from r_3 are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least four red lines from r_3 are joined to points among $c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}$. In either case, we see that red $K_{2,4}$ is contained in $K_{7,14}$, see Table 2.11 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	**1**	**1**	1	1	1	1						
r_2							1	1	1	1	1	1	1	
r_3	**1**	**1**	**1**	**1**					1	1	1			

Table 2.11.

(2) c_{14} is joined to all r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Here we consider two possibilities (a) and (b).

(a) Some r_i's $(i = 3, 4, 5, 6, 7)$ are joined to c_7 or c_8 or both by red lines.

For example, suppose r_3 is joined to c_7 by red line, then there are five other red lines joining r_3. From these five red lines, either at least three of the red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6$ or at least three red lines are joined to points among $c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}$. In either case, red $K_{2,4}$ is formed. For example, see Table 2.12.

Similarly, if r_3 is joined to c_8 by red line, we can show that red $K_{2,4}$ is also formed, and if r_3 is joined to c_7 and c_8 by red lines, then there are four other red lines joining r_3. For these four red lines, either at least two of the red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6$ or at least two red lines are joined to points among $c_9, c_{10}, c_{11}, c_{12}, c_{13}$. In either case, we can see that red $K_{2,4}$ is contained in $K_{7,14}$.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	**1**	1	1	1	**1**	1						
r_2							1	1	1	1	1	1	1	
r_3	**1**	**1**	**1**				**1**		1	1				1

Table 2.12.

(b) None of r_i's $(i = 3, 4, 5, 6, 7)$ are joined to c_7 and c_8 by red lines.

We consider r_4. Since each of $r_i (i = 3, 4, 5, 6, 7)$ is joined by red lines to c_{14}, there are six other red lines joining each r_i. The six red lines joining each r_3 and r_4 will join to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}, c_{11}, c_{12}, c_{13}$. Therefore, there are at least one point among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}, c_{11}, c_{12}, c_{13}$ which join r_3 and r_4. First, consider the case when there are at least three points among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}, c_{11}, c_{12}, c_{13}$ each of which joins r_3 and r_4. Also, since c_{14} is joined by red lines to r_3, r_4, we can see that red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.13 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2							1	1	1	1	1	1	1	
r_3	1	1	**1**								1	**1**	**1**	**1**
r_4			**1**	1	1					1		**1**	**1**	**1**

Table 2.13.

Thus we shall consider the case when there are one and two points among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}, c_{11}, c_{12}, c_{13}$ which join r_3 and r_4.

Consider when there is one point among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}, c_{11}, c_{12}, c_{13}$ which join r_3 and r_4. Suppose this point is c_{13}, see Table 2.14 for example. When we consider only r_1, r_2, r_3, r_4 there are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.14 for example. For such cases, we shall consider the seven red lines joining r_5.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2							1	1	1	1	1	1	1	
r_3	1	1	1								1	1	1	1
r_4				1	1	1			1	1			1	1

Table 2.14.

From Table 2.14, r_3 is joined to $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}$ and r_4 is joined to $c_4, c_5, c_6, c_9, c_{10}, c_{13}$ and since r_5 is joined by one of the red lines to c_{14}, there are six other red lines joining r_5. From these six red lines, either at least three red lines are joined to points among $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}$ or at least three red lines are joined to points among $c_4, c_5, c_6, c_9, c_{10}, c_{13}$. Assume that there are three red lines that are joined to points among $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}$, say c_2, c_3, c_{11} or there are three red lines that are joined to points among $c_4, c_5, c_6, c_9, c_{10}, c_{13}$, say c_9, c_{10}, c_{13}. Also, since c_{14} is joined by red lines to r_3, r_4 and r_5, we can verify that in either case the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.15 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2							1	1	1	1	1	1	1	
r_3	1	**1**	**1**								1	1	**1**	**1**
r_4				1	1	1			1	1			1	1
r_5		**1**	**1**						1	1	1		**1**	**1**

Table 2.15.

Consider when there are two points among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}$, c_{11}, c_{12}, c_{13} which join r_3 and r_4.

Suppose these points are c_{12} and c_{13}. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.16 for example. So, we consider the seven red lines joining r_5.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2							1	1	1	1	1	1	1	
r_3	1	1	1								1	1	1	1
r_4				1	1	1				1		1	1	1

Table 2.16.

From Table 2.16, r_3 is joined to c_1, c_2, c_3, c_{11}, c_{12}, c_{13} and r_4 is joined to c_4, c_5, c_6, c_{10}, c_{12}, c_{13} and since r_5 is joined by one of the red lines to c_{14}. If r_5 is not joined to c_9, there are six other red lines joining r_5. From these six red lines, either at least three red lines are joined to points among $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}$ or at least three red lines are joined to points among $c_4, c_5, c_6, c_{10}, c_{12}, c_{13}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$. If r_5 is joined to c_9, there are five other red lines joining r_5. From these five red lines, either at least three red lines joining to points among $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}$, say c_1, c_2, c_{11} or at least three red lines are joined to points among $c_4, c_5, c_6, c_{10}, c_{12}, c_{13}$, say c_4, c_{10}. Also, since c_{14} is joined by red lines to r_3, r_4 and r_5. We can verify that in either case the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.17 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2							1	1	1	1	1	1	1	
r_3	1	1	1								**1**	1	1	**1**
r_4				1	1	1				1		1	1	1
r_5	**1**	**1**		1					1	1	**1**			**1**

Table 2.17.

2.5.3. Three of c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by eight red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ and r_2 is joined by seven red lines to $c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}$. We consider two subcases.

(1) Some r_i's $(i = 3, 4, 5, 6, 7)$ are not joined to c_{13} and c_{14} by red lines.

If both c_{13} and c_{14} are not joined to r_3 for example, then either at least four red lines from r_3 are joined to points among $c_1, c_2, \ldots, c_7$ or at least four red lines are joined to points among $c_8, c_9, \ldots, c_{12}$. In either case, we see that red $K_{2,4}$ is contained in $K_{7,14}$, see Table 2.18 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	**1**	**1**	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	**1**	**1**	**1**	**1**					1	1	1			

Table 2.18.

(2) Each of all r_i's $(i = 3, 4, 5, 6, 7)$ is joined to c_{13} or c_{14} or both by red lines.

Here we consider two possibilities (2.1) and (2.2).

(2.1) Three or more r_i's $(i = 3, 4, 5, 6, 7)$ are joined to one of c_{13} and c_{14}.

Suppose r_3, r_4, r_5 are joined to one of c_{13} and c_{14}, then at least two of r_3, r_4, r_5 are joined to c_{13} or at least two of r_3, r_4, r_5 are joined to c_{14}. Let r_3, r_4 be joined to c_{13}. Consider 4 subcases.

(2.1.1) r_3 or r_4 are joined to c_6, c_7, and c_8.

Suppose that r_3 is joined to c_6, c_7, c_8. There are three other red lines joining r_3. From these three red lines joining r_3, either at least two of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least two red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$. In either case, red $K_{2,4}$ is formed. For this subcase, see Table 2.19 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1			1	1	**1**	**1**	**1**	**1**						
r_2						1	1	1	1	1	1	1		
r_3				1	**1**	**1**	**1**	**1**	1				1	

Table 2.19.

(2.1.2) r_3 or r_4 are joined to two of c_6, c_7, and c_8.

Suppose that r_3 is joined to c_7 and c_8, there are four other red lines joining r_3. From these four red lines, either at least two of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least two red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$. In either case, we see that red $K_{2,4}$ is contained in the $K_{7,14}$, see

Table 2.20 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	**1**	**1**	**1**	**1**	1	1		
r_3	1	1					**1**	**1**	**1**	**1**			1	

Table 2.20.

(2.1.3) r_3 or r_4 are joined to one of c_6, c_7, and c_8.

Suppose that r_3 is joined to c_8, there are five other red lines joining r_3. From these five red lines joining r_3, either at least three of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least three red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$. In either case, we can see that red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.21 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	**1**	**1**	**1**	**1**	1		
r_3	1	1						**1**	**1**	**1**	**1**		1	

Table 2.21.

(2.1.4) r_3 and r_4 are not joined to c_6, c_7, and c_8.

Since r_3 is joined to c_{13} and not joined to c_6, c_7 and c_8. There are six other red lines joining r_3. From these six red lines joining r_3, either at least three of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least three red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$. Consider the case when there are at least four points among $c_1, c_2, \ldots, c_5$ or at least four points among $c_9, c_{10}, \ldots, c_{12}$ that are joined to r_3. In either case, we can see that red $K_{2,4}$ is contained in the $K_{7,14}$.

Thus we consider the case when there are three points among $c_1, c_2, \ldots, c_5$, say c_1, c_2, c_3 and three points among $c_9, c_{10}, \ldots, c_{12}$, say c_9, c_{10}, c_{11} that are joined to r_3 by red lines. There are cases, see Table 2.22 for example, when the $K_{7,14}$ does not contain red $K_{2,4}$, so we consider r_4.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1	1							1	1	1	1	

Table 2.22.

Since r_4 is joined to c_{13} and not joined to c_6, c_7, c_8, there are six other red lines joining r_4. From these six red lines, we can see that there are at least three points among $c_1, c_2, \ldots, c_5, c_9$, c_{10}, $\ldots$, c_{12} that are joined to both r_3 and r_4. In either case, red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.23 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1	**1**							1	**1**	**1**	**1**	
r_4			**1**	1	1				1		**1**	**1**	**1**	

Table 2.23.

From subcases (2,1,1), (2.1.2), and (2.1.3), if there is at least one of r_i's ($i = 3, 4, 5, 6, 7$) that is joined to some c_6, c_7, c_8, then we can see that red $K_{2,4}$ is contained in the $K_{7,14}$. Later, if we consider the cases when r_i's ($i = 3, 4, 5, 6, 7$) are joined to one of c_{13} and c_{14}, we shall consider only the case when r_i's ($i = 3, 4, 5, 6, 7$) is not joined to c_6, c_7, c_8.

(2.2) Less than three r_i's ($i = 3, 4, 5, 6, 7$) that are joined to one of c_{13} and c_{14}. Consider three subcases.

(2.2.1) One of r_i's ($i = 3, 4, 5, 6, 7$), say r_7 is joined to one of c_{13} and c_{14}. In these cases there are four of r_i's ($i = 3, 4, 5, 6, 7$), say r_3, r_4, r_5 and r_6 that are joined to both c_{13} and c_{14}. Consider two possibilities (a) and (b).

(a) Some r_i's ($i = 3, 4, 5, 6$) are joined to at least two of c_6, c_7 and c_8.

Suppose that r_3 is joined to c_7 and c_8. There are three other red lines joining r_3. From these three red lines, either at least two of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least two red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$. In either case, red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.24 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1	**1**	**1**						
r_2						1	1	1	1	1	1	1		
r_3	**1**	**1**					**1**	**1**	1				1	1

Table 2.24.

(b) Each of r_i's ($i = 3, 4, 5, 6$) is joined to one or none of c_6, c_7 and c_8. There are three possibilities.

(i) Each of all r_i's ($i = 3, 4, 5, 6$) is joined to one of c_6, c_7 and c_8.

Since all r_i's ($i = 3, 4, 5, 6$) are joined to one of c_6, c_7 and c_8, then at least two of r_i are joined to c_6 or at least two of r_i are joined to c_7 or at least two of r_i are joined to c_8. Let r_3, r_4 are joined to c_8. For this case we consider only three of r_3, r_4, r_5, r_6 then we can see that red $K_{2,4}$ is contained in the $K_{7,14}$.

Since r_3 is joined by red lines to c_8, c_{13}, and c_{14}. There are four other red lines joining r_3. From these four red lines, either at least two of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least two red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$. For the cases when there are three or more red lines joining r_3 and points among $c_1, c_2, \ldots, c_5$ or $c_9, c_{10}, \ldots, c_{12}$, we can see that red $K_{2,4}$ is contained in the $K_{7,14}$ see Table 2.25 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	**1**	1	1	1	1	**1**						
r_2						1	1	1	1	1	1	1		
r_3	**1**	**1**	**1**					**1**	1				1	1

Table 2.25.

Thus we shall consider when each r_i $(i = 3, 4, 5, 6)$ has two red lines that are joined to points among $c_1, c_2, \ldots, c_5$ and two red lines that are joined to points among $c_9, c_{10}, \ldots, c_{12}$.

Suppose that r_3 is joined by red lines to two points among $c_1, c_2, \ldots, c_5$, say c_1, c_2, and joined by red lines to two points among $c_9, c_{10}, \ldots, c_{12}$, say c_9, c_{10}. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.26 for example. We consider r_4.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1						1	1	1			1	1

Table 2.26.

Since r_4 is joined to c_8, c_{13}, and c_{14}, there are four other red lines joining r_4. From these four red lines, either at least two of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ and at least two red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$. For the case when there is at least one of c_1, c_2, c_9, c_{10} that is joined to both r_3 and r_4, since c_8, c_{13}, and c_{14} are joined to both r_3 and r_4, then red $K_{2,4}$ is contained in the $K_{7,14}$ see Table 2.27 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1						**1**	1	**1**			**1**	**1**
r_4			1	1				**1**		**1**	1		**1**	**1**

Table 2.27.

Thus we consider the case when none of c_1, c_2, c_9, c_{10} is joined to both r_3 and r_4, there are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.28 for example, so we consider r_5.

Since r_5 is joined to one of c_6, c_7 and c_8, say c_7, if r_5 is not joined to c_5, there are four red lines joining r_5. From these four red lines, either two of the red lines are joined to points among c_1, c_2, c_9, c_{10} or two of the red lines are joined to points among c_3, c_4, c_{11}, c_{12}, and since c_{13} and c_{14} are joined to both r_3 and r_4, then we can see that red $K_{2,4}$ is contained in the $K_{7,14}$. If r_5 is joined to c_5, there are three red lines joining r_5. From these three red lines, either two of the red lines are joined to points among c_1, c_2, c_9, c_{10} or two red lines are joined to points among c_3, c_4, c_{11}, c_{12}. Also, since c_{13} and c_{14} are joined to r_3, r_4, and r_5, so we can see that red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.28 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	**1**						1	1	**1**			**1**	**1**
r_4			1	1				1			1	1	1	1
r_5		**1**			1		1			**1**	1		**1**	**1**

Table 2.28.

(ii) All r_i's ($i = 3, 4, 5, 6$) are not joined to c_6, c_7, and c_8.

For this case we consider only three of r_3, r_4, r_5, r_6 and we can see that red $K_{2,4}$ is contained in the $K_{7,14}$. Since r_3 is joined to c_{13} and c_{14}, there are five other red lines joining r_3. For the cases when r_3 is joined by red lines to at least four points among $c_1, c_2, \ldots, c_5$ or at least four points among $c_9, c_{10}, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$.

Thus we shall consider the cases when each of r_i's ($i = 3, 4, 5, 6$) is joined by red lines to three points among $c_1, c_2, \ldots, c_5$ or three points among $c_9, c_{10}, c_{11}, c_{12}$. Consider when r_3 is joined by red lines to three points among $c_1, c_2, \ldots, c_5$ or three points among $c_9, c_{10}, c_{11}, c_{12}$. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.29 for example, so we consider r_4.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1	1						1	1			1	1

Table 2.29.

Since r_4 is joined to c_{13} and c_{14}, there are other five red lines joining r_4. We can see that there is at least one of $c_1, c_2, c_3, c_9, c_{10}$ that is joined by red lines to both r_3 and r_4. For the cases when r_4 is joined by red lines to at least two points among $c_1, c_2, c_3, c_9, c_{10}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$. Thus we consider the case when one point among $c_1, c_2, c_3, c_9, c_{10}$, say c_3, is joined by red line to both r_3 and r_4. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.30 for example, so we consider r_5.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1	1						1	1			1	1
r_4			1	1	1						1	1	1	1

Table 2.30.

Since r_5 is joined to c_{13} and c_{14}, there are other five red lines joining r_5. Either there are at least three red lines that are joined to points among $c_1, c_2, c_3, c_9, c_{10}$ or at least three red lines that are joined to points among $c_3, c_4, c_5, c_{11}, c_{12}$. In either case, the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.31 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1	1						1	1			1	1
r_4			1	1	1						**1**	**1**	**1**	**1**
r_5	1	1		1							**1**	**1**	**1**	**1**

Table 2.31.

From subcases (a) of (2.2.1), if there is at least one of r_i's $(i = 3, 4, 5, 6, 7)$ that is joined to at least two of c_6, c_7, c_8, then we can see that red $K_{2,4}$ is contained in the $K_{7,14}$. Later, if we consider the cases when r_i's $(i = 3, 4, 5, 6, 7)$ are joined to c_{13} and c_{14}, we shall consider only the case when r_i's $(i = 3, 4, 5, 6, 7)$ is joined to one or none of c_6, c_7, c_8.

(iii) Some r_i's $(i = 3, 4, 5, 6)$ are joined by red line to one point among c_6, c_7, c_8 and some r_i's $(i = 3, 4, 5, 6)$ are not joined to c_6, c_7, and c_8.

We need to consider only two key cases 1) and 2). It will become clear later that the required results, for all subcases with condition (iii), will follow from the results of 1) or 2).

1) Two of r_i's $(i = 3, 4, 5, 6)$ are joined to one of c_6, c_7, c_8 and one of the remaining r_i's is joined to none of c_6, c_7, c_8.

Suppose two of r_i's $(i = 3, 4, 5, 6)$ are joined to one of c_6, c_7, c_8 are r_3, r_5 and one of r_i is joined to none of c_6, c_7, c_8 is r_4.

Suppose that r_3 is joined to one of c_6, c_7 and c_8, say c_8, and since r_3 is joined to c_{13}, c_{14}, there are four other red lines joining r_3. From these four red lines, either at least two of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least two red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$.

For the cases when three or more red lines joining r_3 and points among $c_1, c_2, \ldots, c_5$ or $c_9, c_{10}, \ldots, c_{12}$, we can see that red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.25.

Thus we consider when each of r_3, r_5 has two red lines that are joined to points among $c_1, c_2, \ldots, c_5$, and two red lines that are joined to points among $c_9, c_{10}, \ldots, c_{12}$.

Suppose that r_3 is joined by red lines to two points among $c_1, c_2, \ldots, c_5$, say c_1, c_2, and joined by red lines to two points among $c_9, c_{10}, \ldots, c_{12}$, say c_9, c_{10}. For this case there are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.26, so we consider r_4.

Since r_4 is joined to c_{13} and c_{14}, there are five other red lines joining r_4. For the cases when at least four of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or when at least four of the red lines are joined to points among $c_9, c_{10}, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$. Thus we shall consider the case when there are three red lines that are joined to points among $c_1, c_2, \ldots, c_5$ or there are three red lines that are joined to points among $c_9, c_{10}, c_{11}, c_{12}$. When there are at least two points among c_1, c_2, c_9, c_{10} that are joined by red line to both r_3 and r_4, we can see that the $K_{7,14}$ contains red $K_{2,4}$. Thus we consider when there is at most one point among c_1, c_2, c_9, c_{10} that is joined by red line to both r_3 and r_4.

Consider the case when there is one point among c_1, c_2, c_9, c_{10} that is joined by red line to both r_3 and r_4, say c_{10}. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$ see Table 2.32 for example, so we consider r_5.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1						1	1	1			1	1
r_4			1	1						1	1	1	1	1

Table 2.32.

Since r_5 is joined to c_{13} and c_{14} and is joined to one of c_6, c_7, c_8, say c_7. If r_5 is not joined by red lined to c_5, there are four other red lines joining r_5. Either there are at least two red lines that are joined to points among c_1, c_2, c_9, c_{10} or at least two red lines that are joined to points among $c_3, c_4, c_{10}, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$. If r_5 is joined to c_5, there are three other red lines joining r_5. Either there are at least two red lines that are joined to points among c_1, c_2, c_9, c_{10} or at least two red lines that are joined to points among $c_3, c_4, c_{10}, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.33 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1						1	1	1			1	1
r_4			1	1						1	**1**	**1**	**1**	**1**
r_5		1			1		1				**1**	**1**	**1**	**1**

Table 2.33.

Consider the case when all of c_1, c_2, c_9, c_{10} are not joined to r_4. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.34 for example, so we consider r_5.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1						1	1	1			1	1
r_4			1	1	1						1	1	1	1

Table 2.34.

Since r_5 is joined to c_{13} and c_{14} and is joined by red line to one point among c_6, c_7, c_8, say c_7, there are four other red lines joining r_5. Either there are at least two red lines that are joined to points among c_1, c_2, c_9, c_{10} or at least two red lines that are joined to points among $c_3, c_4, c_5, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.35 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1						1	1	1			1	1
r_4			1	1	**1**						1	**1**	**1**	**1**
r_5	1				**1**		1			1		**1**	**1**	**1**

Table 2.35.

2) One of r_i's $(i = 3, 4, 5, 6)$ is joined to one of c_6, c_7, c_8 and two of the remaining r_i's are joined to none of c_6, c_7, c_8.

Suppose that one of r_i's $(i = 3, 4, 5, 6)$ that is joined to one of c_6, c_7, c_8 is r_3 and two of r_i that are joined to none c_6, c_7, c_8 are r_4, r_5.

From Table 2.34, we can see that there are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, so we consider r_5. Since r_5 is joined to c_{13} and c_{14}, there are five other red lines joining r_5. Either there are at least three red lines that are joined to points among c_1, c_2, c_9, c_{10} or at least three red lines that are joined to points among $c_3, c_4, c_5, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.36 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1						1	1	1			1	1
r_4			1	1	1						**1**	**1**	**1**	**1**
r_5	1				1					1	**1**	**1**	**1**	**1**

Table 2.36.

(2.2.2) Two of r_i's $(i = 3, 4, 5, 6, 7)$, say r_3 and r_4, are joined to one of c_{13} and c_{14}.

In these cases there are three of r_i's ($i = 3, 4, 5, 6, 7$), say r_5, r_6 and r_7, that are joined to both c_{13} and c_{14}.

From what we have mentioned at the end of (2.1), here we need to consider only the cases when r_3 and r_4 are not joined to c_6, c_7, and c_8.

(a) r_3 and r_4 are joined by red lines to c_{13}.

From Table 2.23, we can see that the $K_{7,14}$ contains red $K_{2,4}$. Similarly, if r_3 and r_4 are joined by red lines to c_{14}, we can see that red $K_{2,4}$ is also formed.

(b) r_3 is joined by red lines to c_{13} and r_4 is joined by red lines to c_{14}.

Since r_3 is joined to c_{13} and is not joined to c_6, c_7, and c_8, there are six other red lines joining r_3. From these six red lines joining r_3, either at least three of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least three of the red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$. Consider the cases when there are at least four points among $c_1, c_2, \ldots, c_5$ or at least four points among $c_9, c_{10}, \ldots, c_{12}$ that are joined to r_3. In either case, we can see that red $K_{2,4}$ is contained in the $K_{7,14}$.

Thus we consider the cases when there are three points among $c_1, c_2, \ldots, c_5$ and three points among $c_9, c_{10}, \ldots, c_{12}$ that are joined to r_3 by red lines. Suppose these six points are $c_1, c_2, c_3, c_{10}, c_{11}, c_{12}$, we can see that there are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, so we consider r_4.

Since r_4 is joined to c_{14}, there are six other red lines joining r_4. From these six red lines, we can see that there are at least three lines are joined to points among $c_1, c_2, \ldots, c_5, c_9$, c_{10}, $\ldots$, c_{12} that are joined to both r_3 and r_4. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$ see Table 2.37 for example, so we consider r_i's ($i = 5, 6, 7$). We consider three subcases.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1	1							1	1	1	1	
r_4			1	1	1				1		1	1		1

Table 2.37.

From what we have mentioned at the end of (ii) of (b) in (2.2.1), here we need to consider only the cases when r_5, r_6, and r_7 are joined to one and none of c_6, c_7, and c_8.

(i) All r_i's ($i = 5, 6, 7$) are not joined to c_6, c_7, c_8.

From Table 2.37, we can see that r_3 is joined to $c_1, c_2, c_3, c_{10}, c_{11}, c_{12}$ and r_4 is joined to $c_3, c_4, c_5, c_9, c_{11}, c_{12}$. We consider r_5. Since r_5 is joined to c_{13}, c_{14}, then there are five other red lines joining r_5. From these five red lines, either at least three lines are joined to points among $c_1, c_2, c_3, c_{10}, c_{11}, c_{12}$ or at least three lines are joined to points among $c_3, c_4, c_5, c_9, c_{11}, c_{12}$. Suppose that these five points are $c_1, c_2, c_4, c_9, c_{10}$. Since r_3 and r_5 are joined to c_{13}, we can see that the $K_{7,14}$

contains red $K_{2,4}$ see Table 2.38 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	**1**	**1**	1							**1**	1	1	**1**	
r_4			1	1	1				1		1	1		1
r_5	**1**	**1**		1					1	**1**			**1**	1

Table 2.38.

(ii) Each of all r_i's ($i = 5, 6, 7$) is joined to one of c_6, c_7, c_8.

Since r_5 is joined to c_{13}, c_{14} and is joined to one of c_6, c_7, c_8, say c_8, then there are four other red lines joining r_5. From these four red lines, either at least two lines are joined to points among $c_1, c_2, c_3, c_{10}, c_{11}, c_{12}$ or at least two lines are joined to points among $c_3, c_4, c_5, c_9, c_{11}, c_{12}$. For this case there are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.39 for example, so we consider r_6.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	1	1	1							1	1	1	1	
r_4			1	1	1				1		1	1		1
r_5		1		1				1	1	1			1	1

Table 2.39.

Since r_6 is joined to c_{13}, c_{14} and is joined by red line to one point among c_6, c_7, c_8, say c_7, there are four other red lines joining r_6. From these four red lines, either there are at least three red lines that are joined to points among $c_1, c_2, c_3, c_{10}, c_{11}, c_{12}$ or there are at least three red lines that are joined to points among $c_3, c_4, c_5, c_9, c_{11}, c_{12}$ or there are at least two red lines that are joined to points among c_2, c_4, c_9, c_{10}. We can see that the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.40 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1	1	1						
r_2						1	1	1	1	1	1	1		
r_3	**1**	1	1							**1**	**1**	1	**1**	
r_4			1	1	1				1		1	1		1
r_5		1		1				1	1	1			1	1
r_6	**1**				1		1			**1**	**1**		**1**	1

Table 2.40.

(iii) Some r_i's ($i = 5, 6, 7$) are joined to one of c_6, c_7, c_8 and some r_i's ($i = 5, 6, 7$) are not joined to c_6, c_7, c_8.

For this case we can see that there is at least one of r_5, r_6, and r_7 that is not joined to c_6, c_7, c_8. From subcase (i) and Table 2.38, we can see that the $K_{7,14}$ contains red $K_{2,4}$.

(2.2.3) None r_i's $(i = 3, 4, 5, 6, 7)$ are joined to one of c_{13} and c_{14}.

In these cases all r_i's $(i = 3, 4, 5, 6, 7)$ are joined to both c_{13} and c_{14}.

From the case (2.2.1), since if four of r_i's $(i = 3, 4, 5, 6, 7)$ are joined to c_{13} and c_{14} then the $K_{7,14}$ contains red $K_{2,4}$. Therefore in the case (2.2.3) the $K_{7,14}$ contains red $K_{2,4}$.

Hence,

$$r_t(K(2, 4), K(2, 4)) \leq 7. \tag{2.3}$$

Therefore, from the inequalities (2.1) and (2.3), we have $r_t(K(2, 4), K(2, 4)) = 7$ as required. □

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