



## On Sums of Conjugate Secondary Range k-Hermitian Matrices

S. Krishnamoorthy<sup>†</sup>, K. Gunasekaran<sup>†</sup> and B.K.N. Muthugobal<sup>‡,1</sup>

<sup>†</sup>Ramanujan Research Center, Department of Mathematics  
Government Arts College (Autonomous), Kumbakonam- 65000, India

<sup>‡</sup>Research Scholar, Ramanujan Research Center  
Department of Mathematics, Government Arts College (Autonomous)  
Kumbakonam- 65000, India  
e-mail : bkn.math@gmail.com

**Abstract :** The concept of sums of conjugate secondary range k-hermitian matrix (con-s-k-EP) is introduced. The characterizations of con-s-k-EP matrices are obtained. Relations among con-EP, con-s-EP, con-k-EP and con-s-k-EP matrices are discussed. Necessary and sufficient conditions are determined for a matrix to be con-s-k-EP<sub>r</sub> (con-s-k-EP and of rank  $r$ ).

**Keywords :** EP-matrices; Con-EP matrices; Con-s-k-EP matrices; Sums and parallel summable matrices.

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### 1 Introduction

Let  $C_{n \times n}$  be the space of  $n \times n$  complex matrices of order  $n$ . Let  $C_n$  be the space of all complex  $n$ -tuples. For  $A \in C_{n \times n}$ , let  $\bar{A}, A^T, A^*, A^s, \bar{A}^s, A^\dagger, R(A), N(A)$  and  $\rho(A)$  denote the conjugate, transpose, conjugate transpose, secondary transpose, conjugate secondary transpose, Moore-Penrose inverse, range spaces, null spaces and rank of  $A$ , respectively. A solution  $X$  of the equation  $AXA = A$  is denoted by  $A^-$  (Generalized Inverses of  $A$ ). For  $A \in C_{n \times n}$ , the Moore-Penrose inverse  $A^\dagger$  of  $A$  is the unique solution of the equations  $AXA = A, XAX = X, [AX]^* =$

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<sup>1</sup>Corresponding author email: bkn.math@gmail.com (B.K.N. Muthugobal)

$$AX, [XA]^* = XA, [1].$$

Lee [2] has initiated the study of secondary symmetric matrices that is matrices whose entries are symmetric about the secondary diagonal. Antono and Paul [3] have studied per-symmetric matrices that is matrices are symmetric about both the diagonals and their applications to communication theory. In [2], Lee has shown that for a complex matrix  $A$ , the usual transpose  $A^T$  and  $A^s$  are related as  $A^s = VA^T V$  where 'V' is the permutation matrix with units in its secondary diagonal. Also the conjugate transpose  $A^*$  and the secondary conjugate transpose  $\overline{A^s}$  are related as  $\overline{A^s} = VA^*V$ . Throughout let 'k' be fixed product of disjoint transpositions in  $S_n = \{1, 2, \dots, n\}$  and 'K' be the associated permutation matrix.

In [4], Meenakshi has developed a theory for sums of EP matrices. She proved that sums of EP matrices is EP and found that under which conditions sums of matrices will be EP. Also proved that the parallel sum of parallel summable EP matrices will be EP. This paper motivated us to generalize the concept of EP matrices to con-s-k-EP matrices.

In [5], we defined con-s-k-EP as a generalization of EP matrix as, a matrix  $A \in C_{n \times n}$  is said to be con-s-k-EP if it satisfies the condition.  $A_x = 0 \Leftrightarrow A^s k(x) = 0$ . From the above definition, it can be verified that A is con-s-k-EP  $\Leftrightarrow N(A) = N(KVA^T VK) \Leftrightarrow N(A) = N(A^T VK)$ . Moreover, A is said to be con-s-k-EP<sub>r</sub>, if A is of con-s-k-EP and rank r.

## 2 Sums of con-s-k-EP matrices

In this section we give necessary and sufficient conditions for sums of con-s-k-EP matrices to be con-s-k-EP. As an application it is shown that sum and parallel sum of parallel assumable con-s-k-EP matrices are con-s-k-EP.

**Theorem 2.1.** *Let  $A_i$  ( $i = 1, 2, \dots, m$ ) be con-s-k-EP matrices then  $A = \sum_{i=1}^m A_i$  is con-s-k-EP if any one of the following equivalent conditions hold.*

$$(i) \quad N(A) \subseteq N(A_i) \text{ for } i = 1, 2, \dots, m.$$

$$(ii) \quad N(A) = \bigcap_{i=1}^m N(A_i).$$

$$(iii) \quad \rho \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix} = \rho(A).$$

*Proof.* (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii)

$N(A) \subseteq N(A_i)$  for each  $i$ .  $\Rightarrow N(A) \subseteq \bigcap_{i=1}^m N(A_i)$ . Since  $N(A) \subseteq N(\sum A_i) \supseteq N(A_1) \cap N(A_2) \cap \dots \cap N(A_m)$ , it follows that  $N(A) \supseteq \bigcap_{i=1}^m N(A_i)$ .

Always  $\bigcap_{i=1}^m N(A_i) \subseteq N(A)$ . Hence  $N(A) = \bigcap_{i=1}^m N(A_i)$ . Thus (ii) holds.

Now,  $N(A) = \bigcap_{i=1}^m N(A_i) = N \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}$ . Therefore,  $\rho(A) = \rho \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}$  and (iii)

holds.

Conversely, since  $\rho(A) = \rho \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}$  and  $N \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix} = \bigcap_{i=1}^m N(A_i) \subseteq N(A) \Rightarrow$

$N(A) = \bigcap_{i=1}^m N(A_i)$ , and (ii) holds. Hence,  $N(A) \subseteq N(A_i)$  for each  $i$  and (i) holds. Since, each  $A_i$  is con-s-k-EP,  $N(A_i) = N(A_i^T V K)$ , for each  $i$ . Now,  $N(A) = N(A_i)$  for each  $i$ ,  $\Rightarrow N(A) \subseteq \bigcap_{i=1}^m N(A_i) = \bigcap_{i=1}^m N(A_i V K) \subseteq N(A) = N(A^T V K)$  and  $\rho(A) = \rho(A^T V K)$ . Hence  $N(A) = N(A^T) = N(A^T V K)$ . Thus  $A$  is con-s-k-EP.  $\square$

**Remark 2.2.** In particular, if  $A$  is non singular the conditions automatically hold and  $A$  is con-s-k-EP. Theorem 2.1 fails if we relax the conditions on the  $A_i$ 's.

**Example 2.3.** Consider  $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . For  $k =$

(1,2)(3) the associated permutation matrix  $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  and  $V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ ,

$KVA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  is con-EP  $\Rightarrow A$  is con-s-k-EP.

$KVB = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$  is not con-EP.  $\Rightarrow B$  is not con-s-k-EP.

$A + B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$  and  $KV(A + B) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$  is not con-EP.

Therefore,  $A + B$  is not con-s-k-EP.

**Remark 2.4.** If rank is additive, that is  $\rho(A) = \sum \rho(A_i)$  then by [5],  $R(A_i) \cap R(A_j) = 0, i \neq j$  which implies that  $N(A) \subseteq N(A_i)$  for each  $i$ , this implies  $N(A) \subseteq N(A_i^T V K)$  for each  $i$ , Hence  $A$  is con-s-k-EP. The conditions given in Theorem 2.1 are weaker then the conditions of rank additively and can be seen by the following example.

**Example 2.5.** Let  $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ . For  $k = (2, 3)$ ,

the associated permutation matrix

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A + B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Hence  $A, B$  and  $A + B$  are con-s-k-EP. Conditions (i) and (ii) of Theorem 2.1 hold, but  $\rho(A + B) \neq \rho(A) + \rho(B)$ .

**Theorem 2.6.** Let  $A_i, (i = 1, 2, \dots, m)$  be con-s-k-EP matrices such that  $\sum_{i \neq j} A_i^T A_j = 0$  then  $A = \sum A_i$  is con-s-k-EP.

*Proof.* Since,  $\sum_{i \neq j} (A_i^T A_j) = 0, A^T A = (\sum A_i^T)(\sum A_i) = \sum A_i^T A_i,$

$$\begin{aligned} N(A) &= N(A^T A) = N(\sum A_i^T A_i) = N \left[ \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \right] = N \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \\ &= N(A_1) \cap N(A_2) \cap \dots \cap N(A_m) \\ &= N(A_1^T V K) \cap N(A_2^T V K) \cap \dots \cap N(A_m^T V K) \\ &= N(A) \subseteq N(A_i^T V K) \text{ for each } i \\ &= N(A_i) \text{ for each } i. \end{aligned}$$

Now  $A$  is con-s-k-EP follows from Theorem 2.1. □

**Remark 2.7.** Theorem 2.6 fails if we relax the conditions that  $A_i$ 's are con-s-k-EP. Let  $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ . For  $k = (1, 2)(3)$ , the

associated permutation matrix be  $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ . Now,

$KVA = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, KV(A + B) = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}$  is not con-EP. Therefore  $(A + B)$  is not con-s-k-EP, but  $A^T B + B^T A = 0$ .

**Remark 2.8.** The conditions given in Theorem 2.6 imply those in Theorem 2.1 but not conversely. This can be seen by the following example.

**Example 2.9.** Let  $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ . For  $k = (1)(2, 3)$ , the

associated permutation Matrix  $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ . Both  $A$  and

$B$  are con-s-k-EP matrices. By [5, Theorem 2.11],  $N(A + B) = N(KV(A + B)) = N(KVA + KVB) \subseteq N(KVA) = N(A)$ . Therefore,  $N(A + B) \subseteq N(A)$  also  $N(A + B) \subseteq N(B), A^T B + B^T A \neq 0$ .

**Remark 2.10.** *The conditions given in Theorem 2.1 and Theorem 2.6 are only sufficient for the sum of con-s-k-EP, matrices to be con-s-k-EP, but not necessary as illustrated by the following example.*

**Example 2.11.** Let  $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ . For  $k = (2, 3)(1)$ ,

the associated permutation matrices  $K = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ ,  $V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ .  $A$  and

$B$  are con-s-k-EP matrices neither the conditions in Theorem 2.1 or in Theorem 2.6 hold, however  $A + B$  is con-s-k-EP.

**Remark 2.12.** *If  $A$  and  $B$  are con-s-k-EP matrices, then by [5, Theorem 2.11]  $A^T = H_1(KVAKV)$  and  $B^T = H_2(KVBKV)$ , where  $H_1, H_2$  are non singular  $n \times n$  matrices. If  $H_1 = H_2$  then  $A^T + B^T = H_1(KV(A+B)KV) \Rightarrow (A+B)^T = H_1(KV(A+B)KV) \Rightarrow (A+B)$  is con-s-k-EP.*

If  $H_1 - H_2$  is non singular, then the above conditions are also necessary for the sum con-s-k-EP matrices to be con-s-k-EP is given by the following theorem.

**Theorem 2.13.** *Let  $A^T = H_1(KVAKV)$  and  $B^T = H_2(KVBKV)$  such that  $H_1 - H_2$  is non singular,  $K$  be the permutation matrix associated with  $k$  and  $V$  be the permutation matrix with units in the secondary diagonal then  $A + B$  is con-s-k-EP  $\Leftrightarrow N(A+B) \subseteq N(A)$  and  $N(A+B) \subseteq N(B)$ .*

*Proof.* Since  $A^T = H_1(KVAKV)$  and  $B^T = H_2(KVBKV)$  by [5, Theorem 2.11]  $A$  and  $B$  are con-s-k-EP matrices. Since,  $N(A+B) \subseteq N(B)$  by Theorem 2.1,  $A+B$  is con-s-k-EP conversely, let us assume that  $A+B$  is con-s-k-EP. By [5, Theorem 2.11], there exists a non singular matrix  $G$  such that

$$\begin{aligned} (A+B)^T &= G(KV(A+B)KV) \\ &\Rightarrow A^T + B^T = G(KV(A+B)KV) \\ &\Rightarrow H_1(KVAKV) + H_2(KVBKV) = G(KV(A+B)KV) \\ &\Rightarrow (H_1KVA + H_2KVB)KV = G(KV(A+B)KV) \\ &\Rightarrow H_1(KVA) + H_2(KVB) = GKV(A+B) \\ &\Rightarrow H_1(KVA) + H_2(KVB) = G(KVA) + G(KVB) \\ &\Rightarrow (H_1 - G)KVA = (G - H_2)KVB \\ &\Rightarrow L(KV)A = M(KV)B, \end{aligned}$$

where  $L = (H_1 - G)$ ,  $M = (G - H_2)$ . Now,  $(L + M)KVA = LKVA + MKVA = MKVB + MKVA = M(KV)(A+B)$  and  $(L + M)KVB = L(KV)(A+B)$ . By hypothesis  $(L + M) = H_1 - G + G - H_2 = H_1 - H_2$  is non singular, therefore,  $N(A+B) \subseteq N(M(KV)(A+B)) = N((L + M)KVA) = N(KVA) = N(A)$ . That is,  $N(A+B) \subseteq N(A)$ . Also,  $N(A+B) \subseteq N(L(KV)(A+B)) = N((L + M)KVB) = N(KVB) = N(B)$ . Therefore,  $N(A+B) \subseteq N(B)$ . Thus,  $A+B$  is con-s-k-EP  $\Rightarrow N(A+B) \subseteq N(A)$  and  $N(A+B) \subseteq N(B)$ .  $\square$

**Remark 2.14.** The conditions  $H_1 - H_2$  is nonsingular is essential for Theorem 2.12 as illustrated in the following example.

**Example 2.15.** Let  $A = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}$  and  $B = \begin{pmatrix} 3 & 5 & 3 \\ 3 & 3 & 5 \\ 5 & 3 & 3 \end{pmatrix}$  are both con-s-k-EP

matrices for  $k = (1)(2,3)$ , the associated permutation matrix be  $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

and  $V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ . Further,  $A = A^T = (KV)A(KV)$  and  $B = B^T =$

$(KV)B(KV) \Rightarrow H_1 - H_2 = I$ . Thus,  $KVA KV = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 2 \end{pmatrix}$ ,  $A+B = \begin{pmatrix} 5 & 8 & 6 \\ 6 & 5 & 8 \\ 7 & 6 & 5 \end{pmatrix}$

is also con-s-k-EP, but  $N(A+B) \not\subseteq N(A)$  or  $N(A+B) \not\subseteq N(B)$ .

Thus, Theorem 2.6 fails. We note that when  $k(i) = i$  for  $i = 1, 2, \dots, n$ . The permutation matrix  $K$  reduces to  $I$  and Theorem 2.2, 2.6 and 2.13 reduces to results found for con-s-k-EP matrices.

### 3 Parallel Summable Con-s-k-EP Matrices

Here it is shown that sum and parallel sums of parallel summable con-s-k-EP matrices are con-s-k-EP. First we shall quote the definition and some properties of parallel summable matrices which are used in this section.

**Definition 3.1.** Two matrices  $A$  and  $B$  are said to be *parallel summable (PS)* if  $N(A+B) \subseteq N(B)$  and  $N(A+B)^T \subseteq N(A^T)$  or equivalently  $N(A+B) \subseteq N(A)$ ,  $N(A+B)^T \subseteq N(A^T)$ .

**Definition 3.2.** If  $A$  and  $B$  are parallel summable then the parallel sum of  $A$  and  $B$  denoted by  $(A \dot{+} B)$  is defined as  $A \dot{+} B = A(A+B)^- B$ . The product  $A(A+B)^- B$  is invariant for all choices of generalized inverses  $(A+B)^-$  of  $(A+B)$  under the conditions that  $A$  and  $B$  are parallel summable [6, p. 188].

**Properties 3.3.** Let  $A$  and  $B$  be a pair of parallel summable matrices, then the following hold.

P1:  $A \dot{+} B = B \dot{+} A$ .

P2:  $A^T$  and  $B^T$  are PS and  $(A \dot{+} B)^T = A^T \dot{+} B^T$ .

P3: If  $U$  is nonsingular then  $UA$  and  $UB$  are PS and  $UA \dot{+} UB = U(A \dot{+} B)$ .

P4:  $R(A \dot{+} B) = R(A) \dot{+} R(B)$  and  $N(A \dot{+} B) = N(A) \dot{+} N(B)$ .

P5:  $(A \dot{+} B) \dot{+} C = A \dot{+} (B \dot{+} C)$  if all the parallel sum operations are defined.

**Lemma 3.4.** *Let  $A$  and  $B$  be con- $s$ - $k$ -EP matrices then  $A$  and  $B$  are parallel summable matrices  $\Leftrightarrow N(A + B) \subseteq N(A), N(A + B) \subseteq N(B)$ .*

*Proof.*  $A$  and  $B$  are parallel summable  $\Rightarrow N(A + B) \subseteq N(A)$  follows from Definition 3.2. Conversely, if  $N(A + B) \subseteq N(A)$  then  $N(KVA + KVB) \subseteq N(KVA)$  also  $N(KVA + KVB) \subseteq N(KVB)$ . Since  $KVA$  and  $KVB$  are con-EP matrices and  $N(KVA + KVB) \subseteq N(KVA)$  and  $N(KVA + KVB) \subseteq N(KVA)$ . By Theorem 2.1,  $KVA + KVB$  is con-EP. Hence,

$$\begin{aligned} N(KVA + KVB)^T &= N(KVA + KVB) = N(KVA) \cap N(KVB) \\ &= N(KVA)^T \cap N(KVB)^T. \end{aligned}$$

Therefore,  $N(KVA + KVB)^T \subseteq N(KVA)^T$  and  $N(KVA + KVB)^T \subseteq N(KVB)^T$ . Also,  $N(KVA + KVB) \subseteq N(KVA)$ . By hypothesis, hence by Definition 3.2,  $AVK$  and  $BVK$  are parallel summable.  $N(KVA + KVB) \subseteq N(KVA)$ ,  $N(KV(A + B)) \subseteq N(KVA)$ ,  $N(A + B) \subseteq N(A)$ . Also,  $N(KVA + KVB)^T \subseteq N(KVA)^T$ ,  $N(KV(A + B))^T \subseteq N(KVA)^T$ ,  $N(A + B)^T \subseteq N(A)^T$ . Therefore,  $A$  and  $B$  are parallel summable.  $\square$

**Remark 3.5.** *Lemma 3.4 fails if we relax the conditions that  $A$  and  $B$  are con- $s$ - $k$ -EP. Let  $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . For  $k = (1, 2)(3)$  the associated*

*permutation matrix be  $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$  and  $V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ .  $A$  is con- $s$ - $k$ -EP,*

*$B$  is not con- $s$ - $k$ -EP,  $N(A + B) \subseteq N(A)$  and  $N(A + B) \subseteq N(B)$  but  $N(A + B)^T \not\subseteq N(A)^T, N(A + B)^T \not\subseteq N(B)^T$ . Hence  $A$  and  $B$  are not parallel summable.*

**Theorem 3.6.** *Let  $A$  and  $B$  are parallel summable con- $s$ - $k$ -EP matrices, then  $A \dot{\pm} B$  and  $A + B$  are con- $s$ - $k$ -EP.*

*Proof.* Since  $A$  and  $B$  are parallel summable con- $s$ - $k$ -EP matrices, by Lemma 3.4,  $N(A + B) \subseteq N(A)$  and  $N(A + B) \subseteq N(B)$ ,  $N(KVA + KVB)^T \subseteq N(KVA)$  and  $N(KVA + KVB)^T \subseteq N(KVA)^T$ . Therefore,  $KV(A + B) = KVA + KVB$  is con- $s$ - $k$ -EP then  $A + B$  is con- $s$ - $k$ -EP follows from Theorem 2.1. Since  $A$  and  $B$  are parallel summable con- $s$ - $k$ -EP matrices,  $KVA$  and  $KVB$  are parallel summable con-EP matrices. Therefore,  $R(KVA)^T = R(KVA)$  and  $R(KVB)^T = R(KVB)$ .  $R(KVA \dot{\pm} KVB)^T = R(KVA)^T \dot{\pm} R(KVB)^T = R(KVA)^T \cap R(KVB)^T = R(KVA) \cap R(KVB) = R(KVA \dot{\pm} KVB)$ . Thus,  $KVA \dot{\pm} KVB$  is con- $s$ - $k$ -EP when ever  $A$  and  $B$  are con- $s$ - $k$ -EP matrices.  $\square$

**Remark 3.7.** *The sum and parallel sum of parallel summable con- $s$ - $k$ -EP matrices are con- $s$ - $k$ -EP.*

**Corollary 3.8.** *Let  $A$  and  $B$  are con- $s$ - $k$ -EP matrices with  $N(A + B) \subseteq N(A)$ . If  $C$  is con- $s$ - $k$ -EP commuting with both  $A$  and  $B$ , then  $C(A + B)$  and  $C(A \dot{\pm} B) = CA \dot{\pm} CB$  are con- $s$ - $k$ -EP.*

*Proof.* If  $A$  and  $B$  are con-s-k-EP matrices with  $N(A + B) \subseteq N(A)$  then by Theorem 2.1,  $(A + B)$  is con-s-k-EP. Now  $KVA$ ,  $KVB$  and  $KV(A + B)$  are con-EP. Since  $C$  commutes with both  $A$ ,  $B$  and  $(A + B)$ ,  $KVC$  commutes with  $KVA$ ,  $KVB$  and  $KV(A + B)$  and by [7, Theorem 1.3],  $KVCA$ ,  $KVCB$  and  $KVC(A + B)$  are con-EP. Therefore,  $CA$ ,  $CB$  and  $C(A + B)$  are con-s-k-EP. Now, by Theorem 3.6,  $CA + CB$  is con-s-k-EP by  $P3$   $KVC = KV(CA \dot{\pm} CB)$ . Since  $CA \dot{\pm} CB$  is con-s-k-EP,  $KV(CA \dot{\pm} CB)$  is con-EP  $\Rightarrow KV(CA \dot{\pm} CB)$  is con-EP  $\Rightarrow KVC(A \dot{\pm} B)$  is con-EP  $\Rightarrow C(A \dot{\pm} B)$  is con-s-k-EP.  $\square$

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