Strong Convergence by a Hybrid Algorithm for Solving Equilibrium Problem and Fixed Point Problem of a Lipschitz Pseudo-Contraction in Hilbert Spaces

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Abstract : In this research, we create and prove a strong convergence theorem by using the hybrid iterative algorithm which was proposed by Yao et al. [Nonlinear Anal. 71 (2009) 4997–5002] for finding the common element of fixed point set of a Lipshitz pseudo-contraction and the set of equilibrium problem in Hilbert spaces. Moreover, the results not only cover the original research but can also be applied for finding the common element of the set of zeroes of a Lipshitz monotone mapping and the set of equilibrium problem in Hilbert spaces.

Keywords : Hybrid algorithm; Pseudo-contractive mapping; Strong convergence; Equilibrium problem; Hilbert space.
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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem (for short, EP) is to find $x \in C$ such that

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$$F(x, y) \geq 0, \forall y \in C. \tag{1.1}$$

The set of solution (1.1) is denote by $EP(F)$. Given a mapping $T : C \to H$ and let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $x \in EP(F)$ if and only if $x \in C$ is a solution of the variational inequality $\langle Tx, y - x \rangle \geq 0$ for all $y \in C$. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP. In other words, the EP is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. There are many papers have appeared in the literature on the existence of solutions of EP (see, for example [1–4] and references therein). Motivated by the work [5–7], Takahashi and Takahashi [8] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the EP (1.1) and the set of fixed point of nonexpansive mapping in the setting of Hilbert space. They also studied the strong convergence of the sequences generated by their algorithm for a solution of the EP which is also a fixed point of a nonexpansive mapping defined on a closed convex subset of a Hilbert space.

Recall, a mapping T with domain $D(T)$ and range $R(T)$ in H is called non-expansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in D(T).$$

The mapping T is said to be a strict pseudo-contraction if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \forall x, y \in D(T). \tag{1.2}$$

In this case, T may be called as κ-strict pseudo-contraction mapping. In the even that $\kappa = 1$, T is said to be a pseudo-contraction, i.e.,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \forall x, y \in D(T). \tag{1.3}$$

It is easy to see that (1.3) is equivalent to

$$\langle x - y, (I - T)x - (I - T)y \rangle \geq 0, \forall x, y \in D(T).$$

By definition, it is clear that

$$\text{nonexpansive} \Rightarrow \text{strict pseudo-contraction} \Rightarrow \text{pseudo-contraction}.$$

However, the following examples show that the converse is not true.

Example 1.1. *Take* $X = \mathbb{R}^2$, $B = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, $B_1 = \{x \in B : \|x\| \leq \frac{1}{2}\}$, $B_2 = \{x \in B : \frac{1}{2} \leq \|x\| \leq 1\}$. *If* $x = (a, b) \in X$ *we define* $x^\perp$ *to be* $(b, -a) \in X$. *Define* $T : B \to B$ *by*

$$Tx = \begin{cases} x + x^\perp, & x \in B_1, \\ \frac{x}{\|x\|} - x + x^\perp, & x \in B_2. \end{cases}$$

Then, T is Lipschitz and pseudo-contraction but not a strict pseudo-contraction.

Example 1.1 is due to Chidume and Mutangadura [9].

Example 1.2. *Take $X = \mathbb{R}^1$ and define $T : X \to X$ by $Tx = 3x$. Then, T is a strict pseudo-contraction but not a nonexpansive mapping.*

Construction of fixed points of nonexpansive mappings via Mann's algorithm [10] has extensively been investigated in the literature; See, for example [10–16] and references therein. However we note that Mann's iterations have only weak convergence even in a Hilbert space (see e.g., [17]). If T is a nonexpansive self-mapping of C, then Mann's algorithm generates, initializing with an arbitrary $x_0 \in C$, a sequence according to the recursive manner

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real control sequence in the interval $(0, 1)$. If $T : C \to C$ is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm converges weakly to a fixed point of T. Reich [12] showed that the conclusion also holds good in the setting of uniformly convex Banach spaces with a Fréchet differentiable norm. It is well know that Reich's result is on of the fundamental convergence results. Recently, Marino and Xu [18] extended Reich's result [12] to strict pseudo-contraction mapping in the setting of Hilbert spaces. From a practical point of view, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see [19]). Therefore, it is important to develop theory of iterative methods for strict pseudo-contractions. Indeed, Browder and Petryshyn [20] prove that if the sequence $\{x_n\}$ is generated by the following:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.5}$$

for any starting point $x_0 \in C$, $\{\alpha_n\}$ is a constant sequence such that $\kappa < \alpha_n < 1$, $\{x_n\}$ converges weakly to a fixed point of strict pseudo-contraction T. Marino and Xu [18] extended the result of Browder and Petryshyn [20] to Mann's iteration (1.4), they proved $\{x_n\}$ converges weakly to a fixed point of T, provided the control sequence $\{\alpha_n\}$ satisfies the conditions that $\kappa < \alpha_n < 1$, for all n and $\sum_{n=0}^{\infty}(\alpha_n - k)(1 - \alpha_n) = \infty$.

In order to obtain a strong convergence theorem for the Mann iteration method (1.4) to nonexpansive mapping, Nakajo and Takahashi [21] modified (1.4) by employing two closed convex sets that are created in order to form the sequence via metric projection so that strong convergence is guaranteed. Later, it is often referred as the hybrid algorithm or the CQ algorithm. After that, the hybrid algorithm have been studied extensively by many authors (see, for example [18, 22–27]).

A few years ago, Takahashi and Zembayashi [28, 29] proposed some hybrid methods to find the solution of fixed point problem and equilibrium problem in Banach spaces. Subsequently, many authors (see, e.g. [30–34] and references

therein.) have used the hybrid methods to solve fixed point problem and equilibrium problem.

Recently, Yao et al. [35] introduced the hybrid iterative algorithm which just involved one sequence of closed convex set for pseudo-contractive mapping in Hilbert spaces as follows:

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contraction. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:

$$
\begin{cases}
y_n = (1 - \alpha_n)x_n + \alpha_n T z_n, \\
C_{n+1} = \left\{ v \in C_n : \|\alpha_n(I - T)y_n\|^2 \le 2\alpha_n \langle x_n - v, (I - T)y_n \rangle \right\}, \\
x_{n+1} = P_{C_{n+1}}(x_0).
\end{cases}
\tag{1.6}
$$

Theorem 1.3 ([35]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contraction such that $F(T) \ne \varnothing$. Assume the sequence $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{L+1})$. Then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $P_{F(T)}(x_0)$.*

Motivated and inspired by the above research work, in this paper, by employing (1.6) we create a hybrid algorithm to find the common element of fixed point set of a Lipshitz pseudo-contraction and the set of equilibrium problem. More precisely, we also provide some applications of the main theorems for finding the common element of the set of zeroes of a Lipshitz monotone mapping and the set of equilibrium problem in Hilbert spaces.

2   Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a closed convex subset of H. For every point $x \in H$ there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$
\|x - P_C x\| \le \|x - y\| \quad \forall y \in C,
$$

where P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping. It is also known that H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$
\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|
$$

holds for every $y \in H$ with $y \ne x$.

For a given sequence $\{x_n\} \subset C$, let $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denote the weak ω-limit set of $\{x_n\}$.

Now we collect some lemmas which will be used in the proof of the main result in the next section. We note that Lemmas 2.1 and 2.2 are well known.

Lemma 2.1. *Let H be a real Hilbert space. There holds the following identities*

1. $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$ $\forall x, y \in H$.

2. $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$ $\forall x, y \in H$ and $\lambda \in [0, 1]$

Lemma 2.2. *Let C be a closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z \rangle \leq 0 \quad \forall y \in C.$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following condition:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \longmapsto F(x, y)$ is convex and lower semicontinous.

The following lemma appears implicitly in [1].

Lemma 2.3 ([1])**.** *Let C be a nonempty closed convex subset of a Hilbert space H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \ \ for \ all \ \ y \in C.$$

The following lemma was also given in [5].

Lemma 2.4 ([5])**.** *Assume that $F : C \times C \to \mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

1. T_r *is single-valued;*

2. T_r *is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$;*

3. $F(T_r) = EP(F)$;

4. $EP(F)$ *is closed and convex.*

The following lemma provides some useful properties of firmly nonexpansive mapping.

Lemma 2.5. *T is firmly nonexpansive if and only if $(I - T)$ is firmly nonexpansive.*

Proof. Suppose first that T is firmly nonexpansive. We want to show that $(I - T)$ is firmly nonexpansive. Consider,

$$
\begin{aligned}
\|(I - T)x - (I - T)y\|^2 &= \|(Tx - Ty) - (x - y)\|^2 \\
&= \|Tx - Ty\|^2 - 2\langle Tx - Ty, x - y \rangle + \|x - y\|^2 \\
&\leq -\langle Tx - Ty, x - y \rangle + \|x - y\|^2 \\
&= \langle (I - T)x - (I - T)y, x - y \rangle.
\end{aligned}
$$

Conversely, assume that $(I - T)$ is firmly nonexpansive. Consider,

$$
\begin{aligned}
\|Tx - Ty\|^2 &= \|((I - T)x - (I - T)y) - (x - y)\|^2 \\
&= \|(I - T)x - (I - T)y\|^2 - 2\langle ((I - T)x - (I - T)y), x - y \rangle + \|x - y\|^2 \\
&\leq -\langle (I - T)x - (I - T)y, x - y \rangle + \|x - y\|^2 \\
&= \langle Tx - Ty, x - y \rangle.
\end{aligned}
$$

$\square$

Lemma 2.6 ([36])**.** *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a continuous pseudo-contractive mapping, then*

1. *$F(T)$ is closed convex subset of C.*

2. *$I - T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \to z$ and $(I - T)x_n \to 0$, then $(I - T)z = 0$.*

Lemma 2.7 ([37])**.** *Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$
\|x_n - u\| \leq \|u - q\| \quad \forall n.
$$

Then $x_n \to q$.

3 Main result

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ be L-Lipschitz pseudo-contraction and F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$ with $\widetilde{F} := F(T) \cap EP(F) \neq \varnothing$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:*

$$
\begin{cases}
y_n = (1 - \alpha_n)x_n + \alpha_n T z_n, \\
z_n = (1 - \beta_n)x_n + \beta_n u_n, \\
u_n \in C \, such \, that \, F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \\
C_{n+1} = \big\{ v \in C_n : \|\alpha_n(I - T)y_n\|^2 + (1 - 2\beta_n)\|x_n - u_n\|^2 \\
\qquad\qquad \leq 2\alpha_n\langle x_n - v, (I - T)y_n \rangle + 2\langle x_n - v, (I - T_{r_n})z_n + (x_n - u_n)\rangle \\
\qquad\qquad + 2\alpha_n\beta_n L\|x_n - u_n\|\|y_n - x_n + \alpha_n(I - T)y_n\|\big\}, \\
x_{n+1} = P_{C_{n+1}}(x_0).
\end{cases}
$$

$$(3.1)$$

Assume the sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ be such that

(1) $0 < a \leq \alpha_n \leq b < \frac{1}{L+1} < 1$ *for all* $n \in \mathbb{N}$,

(2) $0 < \beta_n \leq 1$ *for all* $n \in \mathbb{N}$ *with* $\lim_{n \to \infty} \beta_n = 0$,

(3) $r_n > 0$ *for all* $n \in \mathbb{N}$ *with* $\liminf_{n \to \infty} r_n > 0$.

Then $\{x_n\}$ *converges strongly to* $P_{\widetilde{F}}(x_0)$.

Proof. By Lemma 2.6 (i) and Lemma 2.4 (iv), we see that $F(T)$ and $EP(F)$ are closed and convex respectively, then $\widetilde{F}$ is also. Hence $P_{\widetilde{F}}$ is well defined. Next, we will prove by induction that $\widetilde{F} \subset C_n$ for all $n \in \mathbb{N}$. Note that $\widetilde{F} \subset C = C_1$. Assume that $\widetilde{F} \subset C_k$ holds for some $k \geq 1$. Let $p \in \widetilde{F}$, thus $p \in C_k$. We observe that

$$
\begin{aligned}
\|x_k - p &- \alpha_k(I-T)y_k\|^2 \\
&= \|x_k - p\|^2 - \|\alpha_k(I-T)y_k\|^2 - 2\alpha_k\langle(I-T)y_k, x_k - p - \alpha_k(I-T)y_k\rangle \\
&= \|x_k - p\|^2 - \|\alpha_k(I-T)y_k\|^2 - 2\alpha_k\langle(I-T)y_k - (I-T)p, y_k - p\rangle \\
&\quad - 2\alpha_k\langle(I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\rangle \\
&\leq \|x_k - p\|^2 - \|\alpha_k(I-T)y_k\|^2 - 2\alpha_k\langle(I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\rangle \\
&= \|x_k - p\|^2 - \|(x_k - y_k) + (y_k - x_k + \alpha_k(I-T)y_k)\|^2 \\
&\quad - 2\alpha_k\langle(I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\rangle \\
&= \|x_k - p\|^2 - \|x_k - y_k\|^2 - \|y_k - x_k + \alpha_k(I-T)y_k\|^2 \\
&\quad - 2\langle x_k - y_k, y_k - x_k + \alpha_k(I-T)y_k\rangle \\
&\quad - 2\alpha_k\langle(I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\rangle \\
&\leq \|x_k - p\|^2 - \|x_k - y_k\|^2 - \|y_k - x_k + \alpha_k(I-T)y_k\|^2 \\
&\quad + 2|\langle x_k - y_k - \alpha_k(I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\rangle|. \tag{3.2}
\end{aligned}
$$

Consider the last term of (3.2) we obtain

$$
\begin{aligned}
|\langle x_k - y_k &- \alpha_k(I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\rangle| \\
&= \alpha_k|\langle x_k - Tz_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\rangle| \\
&= \alpha_k|\langle x_k - Tx_k + Tx_k - Tz_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\rangle| \\
&= \alpha_k|\langle(I-T)x_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\rangle \\
&\qquad\quad + \langle Tx_k - Tz_k, y_k - x_k + \alpha_k(I-T)y_k\rangle| \\
&\leq \alpha_k(L+1)\|x_k - y_k\|\|y_k - x_k + \alpha_k(I-T)y_k\| \\
&\quad + \alpha_k L\|x_k - z_k\|\|y_k - x_k + \alpha_k(I-T)y_k\| \\
&\leq \frac{\alpha_k(L+1)}{2}(\|x_k - y_k\|^2 + \|y_k - x_k + \alpha_k(I-T)y_k\|^2) \\
&\quad + \alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k + \alpha_k(I-T)y_k\|. \tag{3.3}
\end{aligned}
$$

Substituting (3.3) into (3.2), we obtain

$$\|x_k - p - \alpha_k(I-T)y_k\|^2$$
$$\leq \|x_k - p\|^2 - \|x_k - y_k\|^2 - \|y_k - x_k + \alpha_k(I-T)y_k\|^2$$
$$+ \alpha_k(L+1)(\|x_k - y_k\|^2 + \|y_k - x_k + \alpha_k(I-T)y_k\|^2)$$
$$+ 2\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k + \alpha_k(I-T)y_k\|$$
$$\leq \|x_k - p\|^2 + 2\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k + \alpha_k(I-T)y_k\|. \quad (3.4)$$

Notice that

$$\|x_k - p - \alpha_k(I-T)y_k\|^2 = \|x_k - p\|^2 - 2\alpha_k\langle x_k - p, (I-T)y_k\rangle + \|\alpha_k(I-T)y_k\|^2. \quad (3.5)$$

Therefore, from (3.4) and (3.5), we get

$$\|\alpha_k(I-T)y_k\|^2 \leq 2\alpha_k\langle x_k - p, (I-T)y_k\rangle + 2\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k + \alpha_k(I-T)y_k\|. \quad (3.6)$$

On the other hand, by using Lemma 2.5 we obtain

$$\|x_k - p - \beta_k(I-T_{r_k})z_k\|^2$$
$$= \|x_k - p\|^2 - \|\beta_k(I-T_{r_k})z_k\|^2 - 2\beta_k\langle(I-T_{r_k})z_k, x_k - p - \beta_k(I-T_{r_k})z_k\rangle$$
$$= \|x_k - p\|^2 - \|\beta_k(I-T_{r_k})z_k\|^2 - 2\beta_k\langle(I-T_{r_k})z_k - (I-T_{r_k})p, z_k - p\rangle$$
$$- 2\beta_k\langle(I-T_{r_k})z_k, x_k - z_k - \beta_k(I-T_{r_k})z_k\rangle$$
$$\leq \|x_k - p\|^2 - \|\beta_k(I-T_{r_k})z_k\|^2 - 2\langle\beta_k(I-T_{r_k})z_k, x_k - z_k - \beta_k(I-T_{r_k})z_k\rangle$$
$$= \|x_k - p\|^2 - \|\beta_k(I-T_{r_k})z_k\|^2$$
$$+ \left(\|\beta_k(I-T_{r_k})z_k\|^2 - \|x_k - z_k\|^2 + \|x_k - z_k - \beta_k(I-T_{r_k})z_k\|^2\right)$$
$$= \|(x_k - z_k) + (z_k - p)\|^2 - \|x_k - z_k\|^2 + \|\beta_k(I-T_{r_k})x_k - \beta_k(I-T_{r_k})z_k\|^2$$
$$= \|x_k - z_k\|^2 + 2\langle x_k - z_k, z_k - p\rangle + \|z_k - p\|^2 - \|x_k - z_k\|^2$$
$$+ \|\beta_k(I-T_{r_k})x_k - \beta_k(I-T_{r_k})z_k\|^2$$
$$= 2\langle x_k - z_k, (z_k - x_k) + (x_k - p)\rangle + \|(1-\beta_k)(x_k - p) + \beta_k(T_{r_k}x_k - p)\|^2$$
$$+ \|\beta_k(I-T_{r_k})x_k - \beta_k(I-T_{r_k})z_k\|^2$$
$$\leq 2\langle x_k - p, \beta_k(I-T_{r_k})x_k\rangle - 2\|x_k - z_k\|^2 + \beta_k\|T_{r_k}x_k - p\|^2$$
$$- \beta_k(1-\beta_k)\|x_k - T_{r_k}x_k\|^2 + (1-\beta_k)\|x_k - p\|^2 + \beta_k^2\|x_k - z_k\|^2$$
$$\leq 2\langle x_k - p, \beta_k(I-T_{r_k})x_k\rangle + \beta_k\|x_k - p\|^2 + (1-\beta_k)\|x_k - p\|^2$$
$$- \beta_k(1-\beta_k)\|x_k - T_{r_k}x_k\|^2 + \beta_k^2\|(I-T_{r_k})x_k\|^2$$
$$= 2\langle x_k - p, \beta_k(I-T_{r_k})x_k\rangle + \|x_k - p\|^2 - \beta_k(1-\beta_k)\|x_k - u_k\|^2$$
$$+ \beta_k^2\|x_k - u_k\|^2. \quad (3.7)$$

Notice that

$$\|x_k - p - \beta_k(I-T_{r_k})z_k\|^2 = \|x_k - p\|^2 - 2\beta_k\langle x_k - p, (I-T_{r_k})z_k\rangle + \beta_k^2\|(I-T_{r_k})z_k\|^2. \quad (3.8)$$

Combining (3.7) and (3.8) and then it implies that

$$\beta_k(1-\beta_k)\|x_k - u_k\|^2 \le \beta_k\langle x_k - p, (I - T_{r_k})z_k + (x_k - u_k)\rangle + \beta_k^2\|x_k - u_k\|^2$$
$$- \beta_k^2\|(I - T_{r_k})z_k\|^2$$
$$\le 2\beta_k\langle x_k - p, (I - T_{r_k})z_k + (x_k - u_k)\rangle + \beta_k^2\|x_k - u_k\|^2.$$

Since $\beta_n > 0$ for all n, so we get

$$(1 - 2\beta_k)\|x_k - u_k\|^2 \le 2\langle x_k - p, (I - T_{r_k})z_k + (x_k - u_k)\rangle. \tag{3.9}$$

It follows form (3.6) and (3.9) we obtain

$$\|\alpha_k(I - T)y_k\|^2 + (1 - 2\beta_k)\|x_k - u_k\|^2$$
$$\le 2\alpha_k\langle x_k - p, (I - T)y_k\rangle + 2\langle x_k - p, (I - T_{r_k})z_k + (x_k - u_k)\rangle$$
$$+ 2\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k + \alpha_k(I - T)y_k\|.$$

Therefore, $p \in C_{k+1}$. By mathematical induction, we have $\widetilde{F} \subset C_n$ for all $n \in \mathbb{N}$. It is easy to check that C_n is closed and convex and then $\{x_n\}$ is well define. From $x_n = P_{C_n}(x_0)$, we have $\langle x_0 - x_n, x_n - y\rangle \ge 0$ for all $y \in C_n$. Using $\widetilde{F} \subset C_n$, we also have $\langle x_0 - x_n, x_n - u\rangle \ge 0$ for all $u \in \widetilde{F}$. So, for $u \in \widetilde{F}$, we have

$$0 \le \langle x_0 - x_n, x_n - u\rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle$$
$$= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - u\rangle$$
$$\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|.$$

Hence,

$$\|x_0 - x_n\| \le \|x_0 - u\| \quad \text{for all} \quad u \in \widetilde{F}. \tag{3.10}$$

This implies that $\{x_n\}$ is bounded and then $\{y_n\}$, $\{Ty_n\}$, $\{z_n\}$, $\{T_{r_n}z_n\}$ and $\{u_n\}$ are also.

From $x_n = P_{C_n}(x_0)$ and $x_{n+1} = P_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0. \tag{3.11}$$

Hence,

$$0 \le \langle x_0 - x_n, x_n - x_{n+1}\rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle$$
$$= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle$$
$$\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|,$$

and therefore $\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|$, which implies that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. From Lemma 2.1 and (3.11), we obtain

$$\|x_{n+1} - x_n\|^2 = \|(x_{n+1} - x_0) - (x_n - x_0)\|^2$$
$$= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle$$
$$\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\|\alpha_n(I - T)y_n\|^2 + (1 - 2\beta_n)\|x_n - u_n\|^2$$
$$\leq 2\alpha_n \langle x_n - x_{n+1}, (I - T)y_n \rangle + 2 \langle x_n - x_{n+1}, (I - T_{r_n})z_n + (x_n - u_n) \rangle$$
$$+ 2\alpha_n\beta_n L\|x_n - u_n\|\|y_n - x_n + \alpha_n(I - T)y_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, we obtian

$$\|y_n - Ty_n\| \to 0 \quad \text{and} \quad \|x_n - u_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

We note that

$$\|x_n - Tx_n\| \leq \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\|$$
$$\leq (L + 1)\|x_n - y_n\| + \|y_n - Ty_n\|$$
$$\leq \alpha_n(L + 1)\|x_n - Tz_n\| + \|y_n - Ty_n\| \tag{3.13}$$
$$\leq \alpha_n(L + 1)\|x_n - Tx_n\| + \alpha_n(L + 1)\|Tx_n - Tz_n\| + \|y_n - Ty_n\|$$
$$\leq \alpha_n(L + 1)\|x_n - Tx_n\| + \alpha_n\beta_n L(L + 1)\|x_n - u_n\| + \|y_n - Ty_n\|,$$

that is,

$$\|x_n - Tx_n\| \leq \frac{\alpha_n\beta_n L(L + 1)}{1 - \alpha_n(L + 1)}\|x_n - u_n\| + \frac{1}{1 - \alpha_n(L + 1)}\|y_n - Ty_n\| \to 0 \text{ as } n \to \infty.$$

Next, we will show that

$$\omega_w(x_n) \subset \widetilde{F}. \tag{3.14}$$

Since $\{x_n\}$ is bounded, the reflexivity of H guarantees that $\omega_w(x_n) \neq \varnothing$. Let $p \in \omega_w(x_n)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$ and by Lemma 2.6 (ii) we have $p \in F(T)$. On the other hand, since $\|x_n - u_n\| \to 0$ and $x_{n_i} \rightharpoonup p$, so we have $u_{n_i} \rightharpoonup p$. It follows from $u_n = T_{r_n}x_n$ and (A2) that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq F(y, u_n) \quad \text{for all} \quad y \in C.$$

Replacing n by n_i, we have

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}).$$

By using (A4) and (3), we obtain $0 \geq F(y, p)$ for all $y \in C$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) = F(y_t, ty + (1 - t)p) \leq tF(y_t, y) + (1 - t)F(y_t, p) \leq tF(y_t, y).$$

Dividing by t, we have

$$F(y_t, y) \geq 0 \quad \text{for all} \quad y \in C.$$

From (A3) we have $0 \leq \lim_{t \to 0} F(y_t, y) = \lim_{t \to 0} F(ty + (1 - t)p, y) \leq F(p, y)$ for all $y \in C$, and hence $p \in EP(F)$. So, $p \in F(T) \cap EP(F) = \widetilde{F}$ and then we have (3.14). Therefore, by inequality (3.10) and Lamma 2.7, we obtain $\{x_n\}$ converges strongly to $P_{\widetilde{F}}(x_0)$. This completes the proof. $\qquad \square$

Corollary 3.2 (Yao et al. [35, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be L-Lipschitz pseudo-contraction such that $F(T) \neq \varnothing$. Assume the sequence $\{\alpha_n\}$ be such that $0 < a \leq \alpha_n \leq b < \frac{1}{L+1} < 1$ for all n. Then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $P_{F(T)}(x_0)$.*

Proof. Put $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \geq 1$ in Theorem 3.1. Then, $T_{r_n} = P_C$ for all $n \geq 1$. So, $u_n = P_C x_n$ for all $n \geq 1$(Note that $x_1 = P_C x_0$). Since $x_n = P_{C_n} x_0 \in C_n \subset C$ for all $n \geq 1$, so we have $u_n = x_n$ and then $z_n = x_n$ for all $n \geq 1$. Thus $(I - T_{r_n})z_n = x_n - P_C x_n = 0$ for all $n \geq 1$. For this reason, (1.6) is a special case of (3.1). Applying Theorem 3.1, we have the desired result. $\qquad\square$

Recall that a mapping A is said to be *monotone*, if $\langle x - y, Ax - Ay \rangle \geq 0$ for all $x, y \in H$ and *inverse strongly monotone* if there exists a real number $\gamma > 0$ such that $\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2$ for all $x, y \in H$. For the second case A is said to be γ-inverse strongly monotone. It follows immediately that if A is γ-inverse strongly monotone, then A is monotone and *Lipschitz continuous*, that is, $\|Ax - Ay\| \leq \frac{1}{\gamma}\|x - y\|$. The pseudo-contractive mapping and strictly pseudo-contractive mapping are strongly related to the monotone mapping and inverse strongly monotone mapping, respectively. It is well known that

1. A is monotone $\iff T := (I - A)$ is pseudo-contractive.

2. A is inverse strongly monotone $\iff T := (I - A)$ is strictly pseudo-contractive.

Indeed, for (ii), we notice that the following equality always holds in a real Hilbert space

$$\|(I-A)x-(I-A)y\|^2 = \|x-y\|^2 + \|Ax-Ay\|^2 - 2\langle x-y, Ax-Ay \rangle \ \forall x, y \in H, \ (3.15)$$

with out loss of generality we can assume that $\gamma \in (0, \frac{1}{2}]$ and then it yields

$$\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2$$
$$\iff -2\langle x - y, Ax - Ay \rangle \leq -2\gamma \|Ax - Ay\|^2$$
$$\iff \|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + (1 - 2\gamma)\|Ax - Ay\|^2 \ \text{(via (3.15))}$$
$$\iff \|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2$$
$$\text{(where } T := (I - A) \text{ and } \kappa := 1 - 2\gamma).$$

Corollary 3.3. *Let $A : H \to H$ be L-Lipschitz monotone mapping and F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$ which $A^{-1}(0) \cap EP(F) \neq \varnothing$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:*

$$
\begin{cases}
y_n = x_n - \alpha_n(x_n - z_n) - \alpha_n A z_n, \\
z_n = (1 - \beta_n)x_n + \beta_n u_n, \\
u_n \in C \, such \, that \, F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \\
C_{n+1} = \big\{ v \in C_n : \|\alpha_n A y_n\|^2 + (1 - 2\beta_n)\|x_n - u_n\|^2 \\
\qquad \leq 2\alpha_n \langle x_n - v, A y_n \rangle + 2\langle x_n - v, (I - T_{r_n})z_n + (x_n - u_n) \rangle \\
\qquad + 2\alpha_n \beta_n L \|x_n - u_n\| \|y_n - x_n + \alpha_n A y_n\| \}, \\
x_{n+1} = P_{C_{n+1}}(x_0).
\end{cases}
$$

$$(3.16)$$

Assume $0 < a \leq \alpha_n \leq b < \frac{1}{L+2} < 1$ *for all* $n \in \mathbb{N}$, $\{\beta_n\}$ *and* $\{r_n\}$ *be as in Theorem 3.1. Then* $\{x_n\}$ *converges strongly to* $P_{A^{-1}(0) \cap EP(F)}(x_0)$.

Proof. Let $T := (I - A)$. Then T is pseudo-contractive and $(L + 2)$-Lipschitz. Hence, it follows from Theorem 3.1, we have the desired result. $\square$

References

[1] E. Blum, W. Oettli, From optimization and variational inequilities to equilibrium problem. Math. Student. 63 (1) (1994) 123–145.

[2] F. Flores-Bazan, Existence theory for finite-dimensional pseudomonotone equilibrium problems, Acta Appl. Math. 77 (2003) 249–297.

[3] N. Hadjisavvas, S. Komlósi, S. Schaible, Handbook of Generalized Convexity and Generalized Monotonicity, Springer-Verlag, Berlin, Heidelberg, New York, 2005.

[4] N. Hadjisavvas, S. Schaible, From scalar to vector equilibrium problems in the quasimonotone case, J. Optim. Theory Appl. 96 (1998) 297–309.

[5] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117–136.

[6] A. Moudafi, Viscosity approximation methods for fixed-point problems, J. Math. Anal. Appl. 241 (2000) 46–55.

[7] A. Tada, W. Takahashi, Strong convergence theorem for an equilibrium problem and a nonexpansive mapping. In: W. Takahashi, T. Tanaka, (Eds.), Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, 2006, pp. 609–617.

[8] S. Takahashi, W. Takahashi, Viscosity approximationmethods for equilibriumproblems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–515.

[9] C.E. Chidume, S.A. Mutangadura, An example on the Mann iteration method for Lipschitz pseudocontractions, Proc. Amer. Math. Soc. 129 (8) (2001) 2359–2363.

[10] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506–510.

[11] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Problems 20 (2004) 103–120.

[12] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 67 (1979) 274–276.

[13] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process. J. Math. Anal. Appl. 178 (1993) 301–308.

[14] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 58 (1992) 486–491.

[15] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. 66 (2002) 240–256.

[16] L.C. Zeng, A note on approximating fixed points of nonexpansive mapping by the Ishikawa iterative processes, J. Math. Anal. Appl. 226 (1998) 245–250.

[17] A. Genel, J. Lindenstrass, An example concerning fixed points, Israel J. Math. 22 (1975) 81–86.

[18] G. Marino, H.K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007) 336–346.

[19] O. Scherzer, Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems, J. Math. Anal. Appl. 194 (1991) 911–933.

[20] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967) 197–228.

[21] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003) 372–379.

[22] M.O. Osilike, Y. Shehu, Explicit averaging cyclic algorithm for common fixed points of asymptotically strictly pseudocontractive maps, Appl. Math. Comput. 213 (2009) 548–553.

[23] S. Plubtieng, K. Ungchittrakool, Strong convergence of modified Ishikawa iteration for two asymptotically nonexpansive mappings and semigroups, Nonlinear Anal. 67 (2007) 2306–2315.

[24] X.L. Qin, H.Y. Zhou, S.M. Kang, Strong convergence of Mann type implicit iterative process for demi-continuous pseudo-contractions, J. Appl. Math. Comput. 29 (2009) 217–228.

[25] X.L. Qin, Y.J. Cho, S.M. Kang, M.J. Shang, A hybrid iterative scheme for asymptotically k-strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 70 (2009) 1902–1911.

[26] B.S. Thakur, Convergence of strictly asymptotically pseudo-contractions, Thai J. Math. 5 (2007) 41–52.

[27] H.Y. Zhou, Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 343 (2008) 546–556.

[28] W. Takahashi, K. Zembayashi, Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, Fixed Point Theory Appl., Vol. 2008, Article ID 528476, 11 pages.

[29] W. Takahashi, K. Zembayashi, Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, Nonlinear Anal. 70 (2009) 45–57.

[30] P. Kumam, N. Petrot, R. Wangkeeree, A hybrid iterative scheme for equilibrium problems and fixed point problems of asymptotically k-strict pseudo-contractions, J. Comput. Appl. Math. 233 (8) (2010) 2013–2026.

[31] P. Kumam, K. Wattanawitoon, Convergence theorems of a hybrid algorithm for equilibrium problems, Nonlinear Analysis: Hybrid Systems 3 (2009) 386–394.

[32] X.L. Qin, H.Y. Zhou, S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math. 225 (1) (2009) 20–30.

[33] S. Plubtieng, K. Ungchittrakool, Approximation of common fixed points for a countable family of relatively nonexpansive mappings in a Banach space and applications, Nonlinear Anal. 72 (6) (2010) 2896–2908.

[34] K. Ungchittrakool, A strong convergence theorem for a common fixed point of two sequences of strictly pseudo contractive mappings in Hilbert spaces and applications, Abst. Appl. Anal., Vol. 2010, Article ID 876819, 17 pages.

[35] Y. Yao, Y.-C. Liou, G. Marino, A hybrid algorithm for pseudo-contractive mappings, Nonlinear Anal. 71 (2009) 4997–5002.

[36] Q.B. Zhang, C.Z. Cheng, Strong convergence theorem for a family of Lipschitz pseudocontractive mappings in a Hilbert space, Math. Comput. Modelling 48 (2008) 480–485.

[37] C. Matinez-Yanes, H.K. Xu, Strong convergence of the CQ method for fixed point process, Nonlinear Anal. 64 (2006) 2400–2411.